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EXISTENTIAL AND UNIVERSAL QUANTIFIER OPERATORS ON BOOLEAN ALGEBRAS AND THEIR PROPERTIES

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ABSTRACT

In this paper, we investigate, explain and enlarged existential and universal quantifier operators on Boolean Algebras. Boolean algebra essentially introduced by George Boole in 1850's to express statement logic in algebraic. The existential and universal quantifier operators are introduced by Halmos, when he developed Monadic and Polyadic Algebra from Boolean Algebra.

***Keywords**: Boolean Algebras, Existential quantifier operator, Universal quantifier operator.*

1. INTRODUCTIONAND PRELIMINARIES

The purpose of this paper is to presents, explain and enlarge existential and universal quantifier operators on Boolean algebras. Boolean algebra essentially introduced George Boole in 1850's to express statement logic in algebraic form to making algebra out of logic. The existential and universal quantifier operators are introduced by Halmos [4, 5, 6] when he developed Monadic and Polyadic Algebra from Boolean Algebra. This paper divided into three sections. Section one contains the basic concepts about definitions and some theorems for properties of Boolean Algebras and emphasis on functional Boolean Algebra as the basic algebraic background of sections 2 and section 3.

Definition 1.1: A Boolean Algebra is an algebraic structure $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ consists of a set B, two binary operations $\vee$ (join) and $\wedge$ (meet), one unary operation '(complementation) and tow nullary operation 0 and 1 (fixed element) which satisfies the following axioms:
BA_1. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a, \forall\, a, b \in B$(commutative axiom).
BA_2. $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c), \forall\, a, b, c \in B$((associative axiom).
BA_3. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),\ \forall\, a, b, c \in B$(distributive axiom).
BA_4. $a \vee a = a$ and $a \wedge a = a$(idempotent axiom).
BA_5. $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a, \forall\, a, b \in B$(absorption axiom).
BA_6. $a \wedge 1 = a$ and $a \vee 0 = a, \forall\, a \in B$(existence of zero and unit elements axioms)and
BA_7. $\forall\, a \in B \Longrightarrow \exists\, a' \in B \ni a \vee a' = 1$ and $a \wedge a' = 0$(existence of complment axiom).
The following theorem give us the main properties of elements of Boolean Algebras.

Theorem 1.2: Let $\mathcal{B}$ be a Boolean Algebra and $a \in B$, Then:
1. $a \vee 1 = 1$ and $a \wedge 0 = 0$.
2. The complete element $a' \in B$ is a unique element. i.e. if $a \vee b = 1$ and $a \wedge b = 0$, then $b = a'$.
3. $a'' = a$.
4. $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$(De Morgan's Law).
5. $0' = 1$ and $1' = 0$.

Definition 1.3: Let $\mathcal{B}$ be a Boolean Algebra and $S \neq \emptyset \subset B$. Then S is called a Boolean sub algebra of B, if S is called itself is a Boolean algebra under the same operation of B.

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Theorem 1.4: Let $\mathcal{B}$ be a Boolean Algebra and $S \neq \emptyset \subset B$. Then S is called sub algebra of B, if for any a and $b \in S$, then $a \vee b \in S, a \wedge b \in S$ and $a' \in S$.

Remark: It is sufficient to shoe that S is closed either under $\wedge$ and ' or under $\vee$ and ', since $(a \vee b) = (a' \wedge b')'$ and $(a \wedge b) = (a' \vee b')'$ for all a and $b \in S$.

Definition 1.5: On A Boolean Algebra $\mathcal{B}$. We define an ordering relation $\preccurlyeq$ on set B as follows:$a \leq b \Leftrightarrow a \wedge b = a$ (or equivalently $a \vee b = b. \leq$ is reflexive, antisymmetric and transitive.

Note that $a \vee b = l.u.b\{a,b\} = sub\{a,b\}$ and $a \wedge b = g.l.b\{a,b\} = inf\{a,b\}$.

Lemma 1.6:
1. If $a_1 \leq b_1$ and $a_2 \leq b_2$, then $a_1 \vee a_2 \leq b_1 \vee b_2$ and$a_1 \wedge a_2 \leq b_1 \wedge b_2$.
2. If $a \leq b$, then $b' \leq a'$.

Definition 1.7: For any $a, b \in B$. Define:
1. Difference $a - b = a \wedge b'$;
2. The sum $a + b = (a - b) \vee (b - a) = (a \wedge b') \vee (b \wedge a')$;
3. Logical implication $a \longrightarrow b = a' \vee b$ and
4. Conditional $a \leftrightarrow b = (a \longrightarrow b) \wedge (b \longrightarrow a) = (a' \vee b) \wedge (b' \vee a)$.

Now, let$\mathcal{B}$ be a Boolean algebra, then for any statement A in B, the dual formula A^* obtained from A by exchange 0 and 1, and exchange $\wedge$ and $\vee$ will be true if and only if A is true. For infinite operation we assume the existence of infima and suprema, $\vee_i a_i = sup_i\{a_i\}$ and $\wedge_i a_i = inf_i\{a_i\}$.

Lemma 1.8:
1.If $\{a_i\}$ is a family of elements in a Boolean algebra, then:$(\vee_i a_i)' = \wedge_i a'_i$ and $(\wedge_i a_i)' = \vee_i a'_i$.
2.If $\{b_i\}$ is a family of elements in a Boolean algebra, then: $a \wedge (\vee_i b_i) = \vee_i (a \wedge b_i)$ and $a \vee (\wedge_i b_i) = \wedge_i (a \vee b_i)$.

Theorem 1.9: (Stone's Representation theorem) [8]: Every Boolean algebra is isomorphic to a field of sets.

The functional Boolean Algebra defined by the following definition

Definition 1.10: Let X be a nonempty set, B a Boolean Algebra. The set of all function from set X into set of B, given by $B^X = \{p | p: X \longrightarrow B \text{ is a function}\}$ for any $p, q \in B^X$.Define $, p \vee q, p \wedge q, p', 0$ and 1 respectively in B^X as follows:
1. $(p \vee q)(x) = p(x) \vee q(x), \forall x \in X$;
2. $(p \wedge q)(x) = p(x) \wedge q(x), \forall x \in X$;
3. $p'(x) = (p(x))', \forall x \in X$;
4. $0(x) = 0, \forall x \in X$ and
5. $1(x) = 1, \forall x \in X$.

Theorem 1.11: Consider p, q and $r \in B^X$, Then: $(B^X, \vee, \wedge, ', 0, 1)$ is a Boolean algebra.

Proof: Consider p, q and $r \in B^X$, Then:
BA_1. $(p \vee q)(x) = p(x) \vee q(x) = q(x) \vee p(x) = (q \vee p)(x), \ \forall x \in X$ and
$(p \wedge q)(x) = p(x) \wedge q(x) = q(x) \wedge p(x) = (q \wedge p)(x), \forall x \in X$;
$$BA_2. \big((p \vee q) \vee r\big)(x) = (p \vee q)(x) \vee r(x) = \big(p(x) \vee q(x)\big) \vee r(x) = p(x) \vee \big(q(x) \vee r(x)\big) = p(x) \vee (q \vee r)(x) = (p \vee (q \vee r))(x), \forall x \in X$$ and

$$\big((p \wedge q) \wedge r\big)(x) = (p \wedge q)(x) \wedge r(x) = \big(p(x) \wedge q(x)\big) \wedge r(x) = p(x) \wedge \big(q(x) \wedge r(x)\big) = p(x) \wedge (q \wedge r)(x) = \big(p \wedge (q \wedge r)\big)(x), \forall x \in X.$$

$$BA_3. \big(p \wedge (q \vee r)\big)(x) = p(x) \wedge (q \vee r)(x) = p(x) \wedge (q(x) \vee r(x)) = \big(p(x) \wedge q(x)\big) \vee \big(p(x) \wedge r(x)\big) = (p \wedge q)(x) \vee (p \wedge r)(x), \forall x \in X \text{ and}$$

$$\big(p \vee (q \wedge r)\big)(x) = p(x) \vee (q \wedge r)(x) = p(x) \vee (q(x) \wedge r(x)) = (p \vee q)(x) \wedge (p \vee r)(x), \forall x \in X.$$

$BA_4. (p \wedge p)(x) = p(x) \wedge p(x) = p(x), \forall x \in X$ and
$(p \vee p)(x) = p(x) \vee p(x) = p(x), \forall x \in X$

$BA_5. (p \wedge (p \vee q))(x) = p(x) \wedge (p \vee q)(x) = p(x) \wedge \big(p(x) \vee q(x)\big) = p(x), \forall x \in X$ and
$(p \vee (p \wedge q))(x) = p(x) \vee (p \wedge q)(x) = p(x) \vee \big(p(x) \wedge q(x)\big) = p(x), \forall x \in X.$
$BA_6. (p \wedge 1)(x) = p(x) \wedge 1(x) = p(x) \wedge 1 = p(x), \forall x \in X$ and
$(p \vee 0)(x) = p(x) \vee 0(x) = p(x) \vee 0 = p(x), \forall x \in X.$
$BA_7. (p \wedge p')(x) = p(x) \wedge p'(x) = p(x) \wedge p(x)' = 0, \forall x \in X$ and
$(p \vee p')(x) = p(x) \vee p'(x) = p(x) \vee p(x)' = 1, \forall x \in X.$

Example 1.12: Let $X = \{a, b, c\}$ and $B = \{0,1\}$, Then: $B^X = \{f \mid f: X \longrightarrow B, \text{is a function}\} = \{f_1, f_2, \cdots, f_8\}$,

Since $|B^X| = |B|^{|X|} = 2^3 = 8$. The following table illustrate all probability of functions.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
a	0	0	0	0	1	1	1	1
b	0	0	1	1	0	0	1	1
c	0	1	0	1	0	1	0	1

Table - 1.1.

The operations join, meet, and complementation given by the following tables respectively.

$\vee$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_2	f_2	f_2	f_4	f_4	f_6	f_6	f_8	f_8
f_3	f_3	f_4	f_3	f_4	f_7	f_8	f_7	f_8
f_4	f_4	f_4	f_4	f_4	f_8	f_8	f_8	f_8
f_5	f_5	f_6	f_7	f_8	f_5	f_6	f_7	f_8
f_6	f_6	f_6	f_8	f_8	f_6	f_6	f_8	f_8
f_7	f_7	f_8	f_7	f_8	f_7	f_8	f_7	f_8
f_8	f_8	f_8	f_8	f_8	f_8	f_8	f_8	f_8

Table - 1.2.

$\wedge$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1	f_1	f_1	f_1	f_1	f_1	f_1	f_1	f_1
f_2	f_1	f_2	f_4	f_4	f_6	f_6	f_8	f_8
f_3	f_1	f_4	f_3	f_4	f_7	f_8	f_7	f_8
f_4	f_1	f_4	f_4	f_4	f_8	f_8	f_8	f_8
f_5	f_1	f_6	f_7	f_8	f_5	f_6	f_7	f_8
f_6	f_1	f_6	f_8	f_8	f_6	f_6	f_8	f_8
f_7	f_1	f_8	f_7	f_8	f_7	f_8	f_7	f_8
f_8	f_1	f_8	f_8	f_8	f_8	f_8	f_8	f_8

Table - 1.3.

Complement: '	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
	f_8	f_7	f_6	f_5	f_4	f_3	f_2	f_1

Table - 1.4.

Definition 1.13: Let $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ be a Boolean Algebra. A closure operator C on $\mathcal{B}$ is a mapping$C: B \longrightarrow B$ such that the following axioms are satisfied:

1. $C(0) = 0$; " normalized'
2. $a \leq C(a), \forall\, a \in B$ and " increasing"
3. $C^2 = C \; or \; C(C(a)) = C(a), \forall\, a \in B$. "idempotent"
4. $C(a \vee b) = C(a) \vee C(b), \forall\, a, b \in B$. "additive".

2. EXISTENTIAL QUANTIFIER OPERATOR ON BOOLEAN ALGEBRAS

Definition 2.1: Let $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ be a Boolean Algebra. An existential operator $\exists$ on $\mathcal{B}$ is a mapping $\exists: B \longrightarrow B$ such that the following axioms are satisfied:

1. $\exists(0) = 0$; " normalized'
2. $a \leq \exists(a), \forall\, a \in B$ and " increasing"
3. $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b), \forall\, a, b \in B$. "quasi-multiplicative".

Example 2.2: Consider $\exists: B \longrightarrow B$ such that $\exists(a) = a, \forall\, a \in B$. Then $\exists$ is an existential quantifier operator on B called discrete. Because:

1. $\exists(0) = 0$;
2. $\exists(a) = a, \forall\, a \in B$. And
3. $\exists(a \wedge \exists(b)) = a \wedge \exists(b) = \exists(a) \wedge \exists(b), \forall\, a, b \in B$.

Example 2.3: Let $\exists: B \longrightarrow B$ such that $\exists(a) = \begin{cases} 0, & if\ a = 0 \\ 1, & if\ a \neq 0 \end{cases}$ for all $a \in B$. Then $\exists$ is an existential quantifier operator on B called simple. Suppose that $a, b \in B$.

Case (1): If$a = 0 \; and \; b = 0$, then:
1. If $a = 0 \Longrightarrow \exists(0) = 0$;
2. If $a = 0 \Rightarrow 0 \leq 0 = \exists(0)$; and
3. If $a = 0 \; and \; b = 0, then$: $\exists(a \wedge \exists(b)) = \exists(0 \wedge \exists(0)) = \exists(0 \wedge 0) = \exists(0) = 0 = 0 \wedge 0 = \exists(a) \wedge \exists(b)$.

Case (2): If $a \neq 0 \; and \; b \neq 0$, then:
1. If $a \neq 0 \Longrightarrow\; 1 \leq 1 = \exists(a)$;
2. If $a \neq 0 \Longrightarrow a \leq 1 = \exists(a)$. and
3. $\exists(a \wedge \exists(b)) = \exists(a \wedge 1) = \exists(a) = 1 = 1 \wedge 1 = \exists(a) \wedge \exists(b)$.

Case (3): If $a = 0 \; and \; b \neq 0$, then: $\exists(a \wedge \exists(b)) = \exists(0 \wedge 1) = \exists(0) = 0 \wedge 1 == \exists(a) \wedge = \exists(b)$.

Theorem 2.4: Let $\exists: B \longrightarrow B$ be an existential quantifier operator on a Boolean Algebra $\mathcal{B}$. Then:

1. $\exists(1) = 1$;
2. $\exists^2 = \exists$;
3. $a \in \exists(B)$ if and only if $\exists(a) = a$;
4. If $a \leq \exists(b)$, then $\exists(a) \leq \exists(b)$ and
5. If $a \leq b$, then $\exists(a) \leq \exists(b)$.

Proof:

1. $1 \leq \exists(1)$ (By axiom-2), since $\exists(1) \leq 1$, therefore $\exists(1) = 1$.
2. $\exists(1 \wedge \exists(b)) = \exists(1) \wedge \exists(b)$
   $\exists(\exists(b)) = 1 \wedge \exists(b)$
   $\exists(\exists(b)) = \exists(b) \Longrightarrow \exists^2 = \exists$.
3. Let $a \in \exists(B)$,then there exists $b \in B$ such that $a = \exists(b) \Longrightarrow \exists(a) = \exists\exists(b) = \exists(b) = a$.
   Conversely, Let $\exists(a) = a, a \in B$.Hence $a \in \exists(B)$.
4. Suppose that $a \leq \exists(b) \Longrightarrow a \wedge \exists(b) = a \Longrightarrow \exists(a) = \exists\big(a \wedge \exists(b)\big)$
   $\Longrightarrow \exists(a) = \exists(a) \wedge \exists(b)$
   $\Longrightarrow \exists(a) \leq \exists(b)$.
5. Suppose that $a \leq b$ and $b \leq \exists(b) \Longrightarrow a \leq \exists(b) \Longrightarrow \exists(a) \leq \exists(b)$.

Theorem 2.5: Let $\exists: B \longrightarrow B$ be an existential quantifier operator on a Boolean Algebra $\mathcal{B}$. Then:

1. $\exists\big(\exists(a)\big)' = \big(\exists(a)\big)'$;
2. $\exists(B)$ is a Boolean Subalgebra of B;
3. $\exists(a \vee b) = \exists(a) \vee \exists(\;\; b)$;
4. $\exists(a \wedge b) \leq \exists(a) \wedge \exists(b)$;

5. $\exists(a) - \exists(b) \leq \exists(a - b)$ and
6. $\exists(a) + \exists(b) \leq \exists(a + b)$.

Proof:

1. $0 = (\exists(a))' \wedge \exists(a) \Rightarrow \exists(0) = \exists((\exists(a))' \wedge \exists(a))$
$\Rightarrow 0 = \exists((\exists(a))' \wedge \exists(a))$
$\Rightarrow 0 = \exists\,(\exists(a))' \wedge \exists(a)$
$\Rightarrow \exists\,(\exists(a))' = \exists\,(\exists(a))' \wedge (\exists(a))'$
$\Rightarrow \exists\,(\exists(a))' \leq (\exists(a))'$, but $(\exists(a))' \leq \exists\,(\exists(a))'$.
$\Rightarrow \exists\,(\exists(a))' = (\exists(a))'$.
2. Suppose that a and $b \in \exists(B) \Rightarrow a = \exists(a)$and $b = \exists(b)$
$\Rightarrow a \wedge b = \exists(a) \wedge \exists(b)$
$\Rightarrow a \wedge b = \exists(\, a \wedge \exists(b))$
$\Rightarrow a \wedge b = \exists(\, a \wedge b)$
$\Rightarrow a \wedge b \in \exists(B)$.

Also Consider $a \in \exists(B) \Rightarrow a = \exists(a) \Rightarrow a' = (\exists(a))' \Rightarrow a' = \exists(\exists(a))' \Rightarrow a' = \exists(a') \Rightarrow a' \in \exists(B)$.

Hence $\exists(B)$ is a Boolean Subalgebra of B.

3. First to show that $\exists(a) \vee \exists(b) \leq \exists(a \vee b)$Since $a \leq a \vee b$ and $b \leq a \vee b \Rightarrow \exists(a) \leq \exists(a \vee b)$ and $\exists(b) \leq \exists(a \vee b) \Rightarrow \exists(a) \vee \exists(b) \leq \exists(a \vee b)$.

Now to show that $\exists(a \vee b) \leq \exists(a) \vee \exists(b)$.Since $\exists(a)$and $\exists(b) \in \exists(B)$.Therefore $(a) \vee \exists(b) \in \exists(B)$
$\Rightarrow \exists(\exists(a) \vee \exists(b)) = \exists(a) \vee \exists(b),$ Since $a \leq \exists(a)$and $b \leq \exists(b)$
Therefore,
$a \vee b \leq \exists(a) \vee \exists(b)$implie that $\exists(a \vee b) \leq \exists(\exists(a) \vee \exists(b)) = \exists(a) \vee \exists(b)$.Hence$\exists(a \vee b) = \exists(a) \vee \exists(b)$.

4. Since $a \wedge b \leq a$ and $a \wedge b \leq b \Rightarrow \exists(a \wedge b) \leq \exists(a)and\exists(a \wedge b) \leq \exists(b) \Rightarrow \exists(a \wedge b) \leq \exists(a) \wedge \exists(b)$.
5. Since $a \vee b = (a - b) \vee b \Rightarrow \exists(a \vee b) = \exists(a - b) \vee b)$
$\Rightarrow \exists(a) \vee \exists(b) = \exists(a - b) \vee \exists(b)$
$\Rightarrow (\exists(a) \vee \exists(b)) \wedge (\exists(b))' = (\exists(a - b) \vee \exists(b)) \wedge (\exists(b))'$.
$\Rightarrow \left(\exists(a) \wedge (\exists(b))'\right) \vee \left(\exists(b) \wedge (\exists(b))'\right) = (\exists(a - b) \wedge (\exists(b))' \vee (\exists(b)) \wedge (\exists(b))')$.
$\Rightarrow \left(\exists(a) \wedge (\exists(b))'\right) \vee 0 = (\exists(a - b) \wedge (\exists(b))' \vee 0$.
$\Rightarrow \left(\exists(a) \wedge (\exists(b))'\right) = (\exists(a - b) \wedge (\exists(b))'$.
$\Rightarrow \exists(a) - \exists(b) = \exists(a - b) - \exists(b) \leq (\exists(a - b) \Rightarrow \exists(a) - \exists(b) \leq \exists(a - b)$.
6. Since $\exists(a) - \exists(b) \leq \exists(a - b)$ also $\exists(b) - \exists(a) \leq \exists(b - a)$
$\Rightarrow (\exists(a) - \exists(b)) \vee (\exists(b) - \exists(a)) \leq \exists(a - b) \vee \exists(b - a)$
$\Rightarrow (\exists(a) - \exists(b)) \vee (\exists(b) - \exists(a)) \leq \exists((a - b) \vee (b - a)$
$\Rightarrow \exists(a) + \exists(b) \leq \exists(a + b)$.

Theorem 2.6: Let $\exists: B \longrightarrow B$ be an existential quantifier operator on a Boolean Algebra $\mathcal{B}$. Then $\exists$ is ac closure operator.

Proof: Up to definition (1.13) we have,

1. $\exists(0) = 0$;
2. $a \leq \exists(a)$;also;
3. $\exists^2 = \exists$ and
4. $\exists(a \vee b) = \exists(a) \vee \exists(b)$.Therefore, $\exists$ is ac closure operator.

Theorem 2.7: If $\exists$ is a closure operator on a Boolean algebra $\mathcal{B}$. Then the following condition are equivalents:

1. $\exists$ is a quantifier;
2. $\exists(B)$ is a Boolean subalgebra of B and
3. $\exists\,(\exists(a))' = (\exists(a))'$ for all $a \in B$.

Proof: Let $\exists: B \longrightarrow B$ be a closure operator on a Boolean Algebra $\mathcal{B}$. Frist to prove (1) implies (2).

Suppose that $\exists$ is a quantifier, therefore $\exists(B)$ is a Boolean subalgebra of B " by theorem (3.6)". Suppose that $\exists(B)$ is a Boolean subalgebra of B.

Since $\exists(a) = a$ whenrver $a \in \exists(B) \Rightarrow (\exists(a))' = a'$, whenever $a' \in \exists(B)$
$\Rightarrow (\exists(a))' \in \exists(B) \Rightarrow \exists(\exists(a))' = (\exists(a))'$. Finally, to prove (3) implies (1).

Suppose that $\exists(\exists(a))' = (\exists(a))'$ for all $a \in B$.Since $a \wedge \exists(b) \leq a \Rightarrow \exists(a \wedge \exists(b)) \leq \exists(a)$Also $a \wedge \exists(b) \leq \exists(b) \Rightarrow \exists(a \wedge \exists(b)) \leq \exists(\exists(b)) \Rightarrow \exists(a \wedge \exists(b)) \leq \exists(b) \Rightarrow \exists(a \wedge \exists(b)) \leq \exists(a) \wedge \exists(b)$.

On the other hand, $a = (a \wedge \exists(b)) \vee \left(a \wedge (\exists(b))'\right) \leq (a \wedge \exists(b)) \vee (\exists(b))'$

$$\Rightarrow \exists(a) \leq \exists((a \wedge \exists(b)) \vee (\exists(b))' \leq \exists(a \wedge \exists(b)) \vee \exists(\exists(b))' \leq \exists(a \wedge \exists(b)) \vee (\exists(b))'.$$

Therefore,

$$\exists(a) \wedge \exists(b) \leq \exists(a \wedge \exists(b)) \vee (\exists(b))' \wedge \exists(b) \Rightarrow \exists(a) \wedge \exists(b) \leq \exists(a \wedge \exists(b)) \vee 0$$
$$\Rightarrow \exists(a) \wedge \exists(b) \leq \exists(a \wedge \exists(b)) \Rightarrow \exists(a \vee b) = \exists(a) \vee \exists(b).$$

Definition 2.8: Let S be a Boolean subalgerbra of a Boolean algebra B. S is called relatively complete, if for any $a \in B$, the set $S(a)$ defined by $S(a) = \{b \in S: a \leq b\}$ has a least element.

Theorem 2.9: Let $\exists: B \longrightarrow B$ be an existential quantifier operator on a Boolean Algebra B. And $S = \exists(B)$. Then S is relatively complete Subalgebra of B and $\exists(a) = \inf S(a)$.

Proof: Let $\exists: B \longrightarrow B$ be an existential quantifier operator on a Boolean Algebra B.
Therefore, $S = \exists(B) = \{\exists(b): b \in B\}$ is a Boolean subalgebra.
Let $b \in S(a)$,then $a \leq b$. Therefore $\exists(a) \leq \exists(b)$. Since $\exists(a) \in S(a)$,therefore $S(a)$ has a least element $\exists(a)$. Thus $\exists(a) = \inf S(a)$.

Definition 2.10: If S a relatively complete Subalgebra of a Boolean Algebra B, Then there exists a unique existential quantifier operator on B such that $\exists(B) = S$.

Proof: Let S be a relatively complete Subalgebra of a Boolean algebra B. Define $\exists: B \longrightarrow B$ as follows:$\exists(a) = \inf S(a)$ for all $a \in B$. To prove $\exists$ is an existential quantifier operator on B and $\exists(B) = S$.

1. $\exists(0) = \inf S(0) = inf\{b \in S: 0 \leq b\} = \inf S = 0$.
2. Let $b \in S$,then $a \leq b$ for all $b \in S$,hence $S(a)$ has a least element a. Therefore, $a \leq \inf S(a)$and implies $a \leq \exists(a)$.
3. Assume that $a \in S \Rightarrow a \in S(a) \Rightarrow \exists(a) = \inf S(a) \leq a$,since $\exists(a) \leq a$ and $a \leq \exists(a)$, we deduce that $\exists(a) = a \Rightarrow \exists(a) \in S, a \in B$, hence $\exists(\exists(a)) = \exists(a), a \in B$. Assumethat$a, b \in B$, then $\exists(a), \exists(b) \in S$, implies that $\exists(a) \vee \exists(b) \in S$. Since$a \vee b \leq \exists(a) \vee \exists(b)$, then $\exists(a) \vee \exists(b) \in S(a \vee b)$, therefore $\exists(a \vee b) = \inf S(a \vee b) \leq \exists(a) \vee \exists(b)$. From the other hand, $\exists(a \vee b) \in S(a \vee b), a \vee b \leq \exists(a \vee b)$. Therefore, $a \leq \exists(a \vee b)$and $b \leq \exists(a \vee b)$. Hence $\exists(a) \leq \exists(a \vee b)$and $\exists(b) \leq \exists(a \vee b)So\exists(a) \vee \exists(b) \leq \exists(a \vee b)$, implies that $\exists(a \vee b) = \exists(a) \vee \exists(b)$. Hence $\exists$ is a closure operator. Now, since $\exists(B) \subseteq S$ and $S \subseteq \exists(B)$also $\exists(a) = a$, whenever $a \in B$. Therefore $\exists(B) = S$, and consequently $\exists(B)$ is a Boolean Subalgebra of B**,** this mean$\exists$ is a quantifier. Finally, to prove that $\exists$ is a unique. Suppose that $\exists_1$ and $\exists_2$ are two quantifies on B. Therefore $\exists_1(a) = inf\ S(a), a \in B$ and $\exists_2(a) = \inf S(a), a \in B$. Since $inf\ S(a)$ is a unique. We get $\exists_1 = \exists_2$.

3. UNIVERSAL QUANTIFIER OPERATOR ON BOOLEAN ALGEBRAS

Definition 3.1: A universal quantifier Operator $\forall$ on a Boolean algebra B is a mapping
$\forall: B \longrightarrow B$ given by: $\forall(a) = (\exists(a'))'$ for all $a \in B$.

Example 3.2: $\forall: B \longrightarrow B$ given by $\forall(a) = a$ for any $a \in B$, Then $\forall$ is universal quantifier operator on B.
Because $\forall(a) = (\exists(a'))' = (a')' = a$.

Theorem 3.3: Let $\forall: B \longrightarrow B$ be a universal quantifier operator on a Boolean Algebra $\mathcal{B}$. Then:

1. $\forall(0) = 0$ and $\forall(1) = 1$;
2. $\forall(a) \leq a$, for all $a \in B$; "decreasing"
3. $\forall^2 = \forall$;
4. $\forall(a \wedge b) = \forall(a) \wedge \forall(b)$ and
5. $\forall(a) \vee \forall(b) \leq \forall(a \vee b)$.

Proof:

1. Since, $\exists(1) = 1$ and $\exists(0') = 0' \Rightarrow \left(\exists(0')\right)' = (0')' = 0 \Rightarrow \forall(0) = 0$. Also, $\exists(0) = 0$ and $\exists(1') = 1' \Rightarrow \left(\exists(1')\right)' = (1')' = 1 \Rightarrow \forall(1) = 1$.
2. Since,$a' \leq \exists(a') \Rightarrow \left(\exists(a')\right)' \leq (a')' \Rightarrow \forall(a) \leq a$.
3. We have $\forall\left(\forall(a)\right) = \forall\left(\left(\exists(a')\right)'\right) = \left(\exists\left(\exists(a')\right)'\right)' = \left(\exists(a')\right)' = \forall(a)$.
4. Since $\exists(a \vee b) = \exists(a) \vee \exists(b) \Rightarrow \exists(a' \vee b') = \exists(a') \vee \exists(b')$
   $\Rightarrow \exists(a \wedge b)' = \exists(a') \vee \exists(b')$
   $\Rightarrow (\exists(a \wedge b)')' = \left(\exists(a') \vee \exists(b')\right)'$
   $= \left(\exists(a')\right)' \wedge \left(\exists(b')\right)'$
   $\Rightarrow \forall(a \wedge b) = \forall(a) \wedge \forall(b)$.
5. Since $\exists(a' \wedge b') \leq \exists(a') \wedge \exists(b') \Rightarrow \exists(a \vee b)' \leq \exists(a') \wedge \exists(b')$
   $\Rightarrow \left(\exists(a') \wedge \exists(b')\right)' \leq (\exists(a \vee b)')'$
   $\Rightarrow \left(\exists(a')\right)' \vee \left(\exists(b')\right)' \leq (\exists(a \vee b)')'$
   $\Rightarrow \forall(a) \vee \forall(b) \leq \forall(a \vee b)$.

Theorem 3.4: Let $\forall: B \longrightarrow B$ be a universal quantifier operator on a Boolean Algebra $\mathcal{B}$. Then:

1. If $a \leq b$, then $\forall(a) \leq \forall(b)$;
2. If $a \in B$ if and only if $\forall(a) = a$;
3. If $\forall(a) \leq b$, then $\forall(a) \leq \forall(b)$;
4. $\forall\left(\forall(a)\right)' = \left(\forall(a)\right)'$;
5. $\forall(B)$ is a Boolean Subalgebra of B.

Proof:

1. Let $a \leq b \Rightarrow b' \leq a' \Rightarrow \exists(b') \leq \exists(a') \Rightarrow (\exists(a')') \leq \left(\exists(b')\right)' \Rightarrow \forall(a) \leq \forall(b)$.
2. Assume that $a \in B$, then there exist $b \in B$ such that $a = \forall(b)$. Therefore $\forall(a) = \forall(\forall(b))$. Hence$\forall(a) = a$.
   Conversely, Let $\forall(a) = a, a \in B$, therefore $a \in \forall(B)$.
3. Suppose that $\forall(a) \leq b \Rightarrow \forall\left(\forall(a)\right) \leq \forall(b) \Rightarrow \forall(a) \leq \forall(b)$.
4. Since $\forall\left(\forall(a)\right)' \leq \left(\forall(a)\right)'$,
   We have $0 = \left(\exists(a)\right)' \wedge \exists(a)$.
   $\Rightarrow \exists(0) = \exists\left(\left(\exists(a')\right)' \wedge \exists(a')\right)$.
   $\Rightarrow 0 = \exists\left(\left(\exists(a')\right)' \wedge \exists(a')\right)$.
   $\Rightarrow 0 = \exists\left(\exists(a')\right)' \wedge \exists(a')$.
   $\Rightarrow \exists(a') = \exists(a') \wedge \left(\exists\left(\exists(a')\right)'\right)'$.
   $\Rightarrow \left(\forall(a)\right)' = \left(\forall(a)\right)' \wedge \forall\left(\forall(a)\right)'$.
   $\Rightarrow \left(\forall(a)\right)' \leq \forall\left(\forall(a)\right)'$.
   $\Rightarrow \forall\left(\forall(a)\right)' = \left(\forall(a)\right)'$.
5. Suppose that $a, b \in \forall(B) \Rightarrow \forall(a) = a$ and $\forall(b) = b$
   $\Rightarrow a \wedge b = \forall(a \wedge b)$.
   $\Rightarrow a \wedge b \in \forall(B)$ .Also consider $c \in \forall(B) \Rightarrow \forall(c) = c$.
   $\Rightarrow c' = \left(\forall(c)\right)' = \forall\left(\forall(c)\right)'$
   $\Rightarrow c' = \forall(c') \Rightarrow c' \in \forall(B)$. $\forall(B)$ is a Boolean Subalgebra of B.

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