International Journal of Mathematical Archive-12(5), 2021, 23-30
Available online through www.ijma.info ISSN 2229 – 5046

SOME CHARACTERIZATIONS OF ORDERED INVOLUTION Γ-SEMIHYPERGROUPS BY WEAKLY PRIME Γ-HYPERIDEALS

ABUL BASAR[1]*, NAVEED YAQOOB[2], M. YAHYA ABBASI[3] AND S. ALI KHAN[4]

[1]Department of Natural and Applied Sciences,
Mirzapur, Saharanpur, Uttar Pradesh-247 121, India.

[2]Department of Mathematics and Statistics,
Riphah International University, I-14, Islamabad, Pakistan.

[3,4]Department of Mathematics,
Jamia Millia Islamia, New Delhi-110 025, India.

(Received On: 17-04-21; Revised & Accepted On: 02-05-21)

ABSTRACT

In this paper, we introduce ordered Γ-semihypergroups with involution and weakly prime Γ- hyperideal, then we investigate some properties of prime, semiprime and weakly prime Γ-hyperideals in ordered Γ-semihypergroup with involution. Also, we study intra-regular ordered Γ-semihypergroups with involution.

***2010 AMS Classification**: 06F05, 16D25.*

Keywords: *Γ-semigroup, ordered Γ-semihypergroup, Γ-hyperideal, involution, weakly prime Γ-hyperideal.*

1. INTRODUCTION AND PRELIMINARIES

The notion of Γ-semigroup was introduced by Sen [19]. The concept of prime and weakly prime ideal in semigroups has been given by Szasz [21], and then Petrich [18] studied these notions for semigroups. Furthermore, Kehayopulu [10], [11], [12] introduced prime, weakly prime ideals in ordered semigroups (partially ordered semigroups) by extending the analogous concepts of ring theory that was given by McCoy [16] and Steinfeld [20]. Khan et al [24] studied derivations of σ-prime rings.

The concept of algebraic hyperstructures was given by Marty [15]. Algebraic hyperstructures are a standard generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The first association between binary relations and hyperstructures appeared in Nieminem [17]. For comprehensive study on semihypergroup by different algebraists, we refer [8], [4], [6], [3] and [1]. Kondo and Lekkoksung [13] studied intra-regular ordered Γ-semihypergroups. Later, Tang *et al.* [22] studied (fuzzy) quasi-Γ-hyperideals in ordered Γ-semihypergroups.

Foulis [7] introduced the concept of involution semigroups. Later, Baxter [2] studied rings with proper involution, and Drazin [5] studied regular semigroups with involution. Herstein [9] studied ring with involution, and Wu [23] studied intra-regular ordered semigroups with involution.

In this paper, the notion of a weakly prime Γ-hyperideal of a Γ-semihypergroup with involution is introduced. A weakly prime Γ-hyperideal of a Γ-semihypergroup is a generalization of a weakly prime ideal of a semigroup, a generalization of a weakly prime hyperideal of a semihypergroup and a generalization of a Γ-ideal of a Γ-semigroup.

Corresponding Author: Abul Basar[1],*
[1]Department of Natural and Applied Sciences,
Mirzapur, Saharanpur, Uttar Pradesh-247 121, India.

The notion of ordered Γ-semigroup was introduced by Kwon and Lee [14]. An ordered Γ-semigroup is an ordered set $(S, \leq)$ at the same time a Γ-semigroup (S, Γ) such that $a \leq b \Rightarrow a\alpha x \leq b\alpha x$ and $x\beta a \leq x\beta b$ forall a, $b, x \in S$ and $\alpha, \beta \in \Gamma$.

Let S be a non-empty set and let $P^*(S)$ be the set of all non-empty subsets of S. A hyperoperation on S is a map $\circ : S \times S \rightarrow P^*(S)$ and the couple $(S, \circ)$ is called a hypergroupoid. We denote by $x \circ y$, the hyperproduct of elements x, y of S.

Let A and B be two non-empty subsets of S, then the hyperproduct of A and B is defined as:

$$A \circ B = \cup_{a \in A, b \in B}\ a \circ b,\ x \circ A = \{x\} \circ A,\ A \circ x = A \circ \{x\}.$$

Also, $A\Gamma B = \cup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$.

Definition 1.1: *[13] A hyperstructure $(S, \Gamma, \leq)$ is called an ordered Γ-semihypergroup if (S, Γ) is Γ-semihypergroupand $\leq$ is a partial order relation on S such that the following condition hold:*
$x \leq y \Rightarrow a\gamma x \leq a\gamma y$ and $x\gamma a \leq y\gamma a$, for all $x, y, a \in S$ and $\gamma \in \Gamma$.

If A and B are non-empty subsets of S, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ suchthat $a \leq b$. Clearly, every ordered Γ-semigroup is an ordered Γ-semihypergroup. A non-empty subset A of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called a Γ-subsemihypergroup of S if $A\Gamma A \subseteq A$.

2. ORDERED INVOLUTION Γ- SEMIHYPERGROUPS

Here in this section we define ordered involution Γ-semihypergroup and provided some related properties.

Definition 2.1: *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ with a unary operation $* : S \rightarrow S$ is called an ordered Γ-semihypergroup with involution if*
(i) $(x^*)^* = x$
(ii) $(x\alpha y)^* = y^*\alpha x^*$
*for all $x, y \in S$ and $\alpha \in \Gamma$. The unary operation * is called an involution. Furthermore, if for all $a, b \in S$ with $a \leq b \Rightarrow a^* \leq b^*$, then we call * an order preserving involution.*

Example 2.2: *Consider a set $S = \{a, b, c\}$ with the set of binary hyperoperations $\Gamma = \{\alpha, \beta, \gamma\}$ and the order "$\leq$" :*

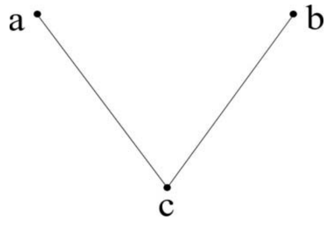
α	a	b	c
a	{a, b}	S	c
b	S	{a, b}	c
c	c	c	c

β	a	b	c
a	{a, c}	S	c
b	S	{b, c}	c
c	c	c	c

γ	a	b	c
a	S	{a, b}	c
b	{a, b}	S	c
c	c	c	c

$$\leq := \{(a, a), (b, b), (c, a), (c, b), (c, c)\}$$

We give the covering relation $\prec$ and the figure of S as follows:
$\prec = \{(c, a), (c, b)\}$



*Then $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. Now we define the involution * by $a^* = b$ (hence $b^* = a$) and $c^* = c$. It is easy to check that S is an ordered Γ-semihypergroup with order preserving involution *.*

Throughout the paper, we shall denote ordered involution Γ-semihypergroup $(S, \Gamma, \leq, *)$ by S.

Definition 2.3: *A non-empty subset A of an ordered involution Γ-semihypergroup S is called a sub Γ-semihypergroup of S if $A\Gamma A \subseteq A$ and $A^* \subseteq A$.*

Definition 2.4: *A non-empty subset I of an ordered involution Γ-semihypergroup S is called a left (resp., right) Γ-hyperideal of S if the following conditions hold:*
(i) $I\Gamma S \subseteq I$ (*resp.,* $S\Gamma I \subseteq I$),
(ii) $I^* \subseteq I$,
(iii) $a \in I$, $b \leq a$ *for* $b \in S \Rightarrow b \in I$.

A hyperideal I of S is both a right and left Γ-hyperideal of an ordered involution Γ-semihypergroup S. A right, left or Γ-hyperideal I of S is called proper if $I \neq S$. We denote by $L(s), R(s)$ and $I(s)$ the left Γ-hyperideal, right Γ-hyperideal and the Γ-hyperideal generated by s. Obviously, $L(s) = (s \cup S\Gamma s]$, $R(s) = (s \cup s\Gamma S]$, $I(s) = (s \cup S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S]$.

If $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup and $A \subseteq S$, then $(A]$ is the subset of S defined as follows:
$(A] = \{s \in S : s \leq a$, for some $a \in A\}$.

Definition 2.5: *Let S be an ordered involution Γ-semihypergroup and $P \subseteq S$. Then P is called prime if $A, B \subseteq S$, $A\Gamma B \subseteq P$ implies $A^* \subseteq P$ or $B^* \subseteq P$.*

Example 2.6: *Consider a set $H = \{a, b, c, d, e\}$ with the set of binary hyperoperations $\Gamma = \{\beta, \gamma\}$ and the order "$\leq$" :*

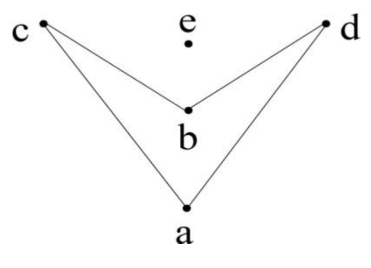
β	a	b	c	d	e
a	a	a	a	a	e
b	a	a	a	a	e
c	a	a	a	{a, b}	e
d	a	a	{a, b}	a	e
e	e	e	e	e	e

γ	a	b	c	d	e
a	a	a	a	a	e
b	a	a	a	a	e
c	a	a	{a, b}	a	e
d	a	a	a	{a, b}	e
e	e	e	e	e	e

$$\leq := \{(a,a),(a,c),(a,d),(b,c),(b,b),(b,d),(c,c),(d,d),(e,e)\}$$

We give the covering relation $\prec$ and the figure of H as follows:

$$\prec = \{(a,c),(a,d),(b,c),(b,d)\}$$



*Then $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. Now we define the involution * by $a^* = a$, $b^* = b$, $c^* = d$ (hence $d^* = c$) and $e^* = e$. It is easy to check that H is an ordered involution Γ-semihypergroup with order preserving involution *. Here $\{e\}$ and $\{a, b, c, d, e\}$ are prime.*

Definition 2.7: *Let S be an ordered involution Γ-semihypergroup and $P \subseteq S$. Then P is called semiprime if for any subset A of S, $A\Gamma A \subseteq P$ implies $A^* \subseteq P$.*

Definition 2.8: *Let S be an ordered involution Γ-semihypergroup and $P \subseteq S$. Then P is called weakly prime if for Γ-hyperideals A, B of S such that $A\Gamma B \subseteq P$ implies $A^* \subseteq P$ or $B^* \subseteq P$.*

We start with the following Lemma which is trivial and is essential for proving subsequent results.

Lemma 2.9: *Suppose that S is an ordered involution Γ-semihypergroup. Then we have the following:*

(i) $A \subseteq (A]$ *for any* $A \subseteq S$.
(ii) $(A] \subseteq (B]$ *for any* $A \subseteq B \subseteq S$. *(iii)* $(A]\Gamma(B] \subseteq (A\Gamma B]$ *for all* $A, B \subseteq S$.
(iv) $((A]] \subseteq (A]$ *for all* $A \subseteq S$.
(v) For any right (left, two-sided) Γ*-hyperideal* I *of* S, $(I] = I$.
(vi) If I *and* J *are* Γ*-hyperideals of* S, *then* $(I\Gamma J]$ *and* $I \cap J$ *are also* Γ*-hyperideals of* S.
(vii) For any $s \in S$, $(S\Gamma s\Gamma S]$ *is a* Γ*-hyperideal of* S.

Lemma 2.10: *Suppose that S is an ordered involution Γ-semihypergroup such that the involution * admits order. Then we have:*

(i) $(b\Gamma S\Gamma a]^* = (a^*\Gamma S\Gamma b^*]$ *for any* $a, b \in S$.
(ii) $(S\Gamma a\Gamma S]^* = (S\Gamma a^*\Gamma S]$ *for any* $a \in S$.
(iii) I^* *is a* Γ*-hyperideal of* S *for any* Γ*-hyperideal* I *of* S.

Proof:
(i) Suppose that $x \in (b\Gamma S\Gamma a]^*$. As $x^* \in (b\Gamma S\Gamma a]$, $x^* \le b\alpha s\beta a$ for $s \in S$ and $\alpha, \beta \in \Gamma$. Then $x \le (b\alpha s\beta a)^* = a^*\beta s^*\alpha b^* \subseteq a^*\Gamma S\Gamma b^*$ since * is an order preserving involution. So, $x \in (a^*\Gamma S\Gamma b^*]$ and therefore, we obtain $(b\Gamma S\Gamma a]^* \subseteq (a^*\Gamma S\Gamma b^*]$. Furthermore, if $x \in (a^*\Gamma S\Gamma b^*]$, then $x \le a^*\alpha s\beta b^*$ for some $s \in S$ and $\alpha, \beta \in \Gamma$. So, $x^* \le b\alpha s^*\beta a \subseteq b\Gamma S\Gamma a$ since $a^*\alpha s\beta b^* = (b\gamma s^*\delta a)^*$ for $\alpha, \beta, \gamma, \delta \in \Gamma$. This shows that $x^* \in (b\Gamma S\Gamma a]$ and $x \in (b\Gamma S\Gamma a]^*$. So, $(a^*\Gamma S\Gamma b^*] \subseteq (b\Gamma S\Gamma a]^*$. Hence, $(b\Gamma S\Gamma a]^* = (a^*\Gamma S\Gamma b^*]$.
(ii) The proof is similar to (i).
(iii) Suppose that I is a Γ-hyperideal of S. As $S\Gamma I \subseteq I$, we obtain $(S\Gamma I)^* \subseteq I^*$. So, $I^*\Gamma S^* \subseteq I^*$. As * is an involution on S, $(s^*)^* = s$ for every $s \in S$, and so $S^* = S$. Therefore, $I^*\Gamma S \subseteq I^*$. In the same way as $I\Gamma S \subseteq I$, we obtain $S\Gamma I^* \subseteq I^*$. Suppose that $a \in I^*$, and $b \le a$, then $b^* \le a^*$. Since $a^* \in I$ and I is a Γ-hyperideal. Therefore, $b^* \in I$, and so $b \in I^*$ and hence I^* is a Γ-hyperideal of S.

Theorem 2.11: *Suppose that S is an ordered Γ-semihypergroup such that S admits an order preserving involution*
**. A Γ-hyperideal of S is prime if and only if it is both weakly prime and semiprime. Furthermore, if S is commutative, then the prime and weakly prime Γ-hyperideals coincide.*

Proof: Let I be a prime hyperideal of S. Then it is obviously weakly prime and semiprime.
Conversely, let P be an ideal of S which is weakly prime and semiprime. Suppose $a\alpha b \subseteq P$ for $\alpha \in \Gamma$, we need to prove that $a^* \in P$ or $b^* \in P$. By Lemma 2.9, $(b\Gamma S\Gamma a]\Gamma(b\Gamma S\Gamma a] \subseteq (S\Gamma a\Gamma b\Gamma S] \subseteq (S\Gamma P\Gamma S] \subseteq (P] = P$. So, P is semiprime and it follows that $(b\Gamma S\Gamma a]^* \subseteq P$. Now we have

$$\begin{aligned}(S\Gamma a^*\Gamma S]\Gamma(S\Gamma b^*\Gamma S] &\subseteq (S\Gamma a^*\Gamma S\Gamma S\Gamma b^*\Gamma S]\\ &\subseteq (S\Gamma(a^*\Gamma S\Gamma b^*)\Gamma S]\\ &= (S\Gamma((S\Gamma b^*)^*\Gamma a)^*\Gamma S]\\ &= (S\Gamma(b\Gamma S\Gamma a)^*\Gamma S]\\ &\subseteq (S\Gamma(b\Gamma S\Gamma a]^*\Gamma S]\\ &\subseteq (S\Gamma P\Gamma S]\\ &\subseteq P.\end{aligned}$$

We note that $(S\Gamma a^*\Gamma S]$, $(S\Gamma b^*\Gamma S]$ are Γ-hyperideals, and P is weakly prime. So $(S\Gamma a^*\Gamma S]^* \subseteq P$ or $(S\Gamma b^*\Gamma S]^* \subseteq P$. Hence, by Lemma 2.10, $(S\Gamma a\Gamma S] \subseteq P$ or $(S\Gamma b\Gamma S] \subseteq P$. Now to show that P is prime, we simply need to prove that if $(S\Gamma a\Gamma S] \subseteq P$ then $a^* \in P$. The other statement can be proved similarly. If $(S\Gamma a\Gamma S] \subseteq P$, *then* we have
$I(a)\Gamma I(a)\Gamma I(a) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]^3 \subseteq ((a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S)^3] \subseteq (S\Gamma(a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S)\Gamma S] \subseteq (S\Gamma a\Gamma S] \subseteq P$. So, $I(a)\Gamma(I(a)\Gamma I(a)] = (I(a)]\Gamma(I(a)\Gamma I(a)] \subseteq ((I(a))^3] \subseteq (P] = P$ by Lemma 2.10. We know that P is weakly prime and $I(a)$, $(I(a)\Gamma I(a)]$ are hyperideals. This implies that $(I(a))^* \subseteq P$ or $(I(a)\Gamma I(a)]^* \subseteq P$. Let $(I(a))^* \subseteq P$. Therefore, $a^* \in (I(a))^* \subseteq P$.

Again, let $(I(a)\Gamma I(a)]^* \subseteq P$. So $a^*\gamma a^* \subseteq (I(a)\Gamma I(a))^* \subseteq (I(a)\Gamma I(a)]^* \subseteq P$ for $\gamma \in \Gamma$ since $a\gamma a \subseteq I(a)\Gamma I(a)$ and so $a = (a^*)^* \in P$ since P is semiprime. Now P is a hyperideal shows that $a\gamma a \subseteq P$, therefore, $a^* \in P$ as P is semiprime. Now we prove the last statement. Suppose P is a hyperideal of S. If P is prime then clearly P is weakly prime.

Conversely, Suppose that P is weakly prime. Let $a\gamma b \subseteq P$ or $\gamma \in \Gamma$. As S is commutative, we obtain $\boldsymbol{I}(a)\Gamma\boldsymbol{I}(b) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]\Gamma(b \cup S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S] \subseteq ((a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]\Gamma(b \cup S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S)] \subseteq (a\alpha b \cup S\Gamma a\beta b]$ for $\alpha, \beta \in \Gamma$. We note that $(a\alpha b \cup S\Gamma a\beta b] \subseteq (P] = P$ for $\alpha, \beta \in \Gamma$. Therefore, $\boldsymbol{I}(a)\Gamma\boldsymbol{I}(b) \subseteq P$, and so we obtain $(\boldsymbol{I}(a))^* \subseteq P$ or $(\boldsymbol{I}(b))^* \subseteq P$ since P is weakly prime. Hence $a^* \in P$ or $b^* \in P$ and it follows that P is prime.

Proposition 2.12: *Suppose that S is an ordered* Γ*-semihypergroup with order preserving involution *. Then the following statements are equivalent.*

(i) $(A^*\Gamma A^*] = A$ *for any* Γ*-hyperideal* A *of* S.
(ii) $A^* \cap B^* = (A\Gamma B]$ *for any* Γ*-hyperideals* A, B *of* S.
(iii) $\boldsymbol{I}(a) \cap \boldsymbol{I}(b) = ((\boldsymbol{I}(a))^*\Gamma(\boldsymbol{I}(b))^*]$ *for any* $a, b \in S$.
(iv) $I(a) = (I(a^*)\Gamma I(a^*)]$ *for any* $a \in S$.
(v) $a \in (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]$ *for any* $a \in S$.

Proof: $(i) \Rightarrow (ii)$. As A^*, B^* are Γ-hyperideals, by our assumption and Lemma 2.9, we obtain $(A\Gamma B] \subseteq (A\Gamma S] \subseteq (A] = ((A^*\Gamma A^*]] = (A^*\Gamma A^*] \subseteq (A^*] = A^*$. In a similar fashion, we have $(A\Gamma B] \subseteq (S\Gamma B] \subseteq (B] = ((B^*\Gamma B^*]] = (B^*\Gamma B^*] \subseteq (B^*] = B^*$. So $(A\Gamma B] \subseteq A^* \cap B^*$. Moreover, $A^* \cap B^*$ is a hyperideal shows that $A^* \cap B^* = ((A^* \cap B^*)^*\Gamma(A^* \cap B^*)^*] = ((A \cap B)\Gamma(A \cap B)] \subseteq (A\Gamma B]$. Thus we obtain $(A\Gamma B] \subseteq A^* \cap B^*$ and $A^* \cap B^* \subseteq (A\Gamma B]$. Hence $A^* \cap B^* = (A\Gamma B]$.

(ii) $\Rightarrow (iii)$. By Lemma 2.10, we have $(\boldsymbol{I}(a))^*$ and $(\boldsymbol{I}(b))^*$ are Γ-hyperideals. Hence follows the result.

(iii) $\Rightarrow (iv)$. As $I(a) = ((I(a))^*\Gamma(I(a))^*]$ by our assumption, we simply need to show that $(I(a))^* = I(a^*)$. Obviously $a^* \in (I(a))^*$. Therefore, $I(a^*) \subseteq (I(a))^*$ since $(I(a))^*$ is a Γ-hyperideal. Now suppose that $x \in (I(a))^*$. We have $x^* \in I(a) = (a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]$. This shows that $x^* \leq a$ or $x^* \leq a\alpha v$ or $x^* \leq v\alpha a$ or $x^* \leq v\alpha a\beta w$ for some $v, w \in S$ and $\alpha, \beta \in \Gamma$. So, $x \leq a^*$ or $x \leq v^*\alpha a^* \subseteq S\Gamma a^*$ or $x \leq a^*\alpha v^* \subseteq a^*\Gamma S$ or $x^* \leq w^*\alpha a^*\beta v^* \subseteq S\Gamma a^*\Gamma S$ for some $v^*, w^* \in S$ and $\alpha, \beta \in \Gamma$, and so $x \in (a^*]$ or $x \in (S\Gamma a^*]$ or $x \in (a^*\Gamma S]$ or $x \in (S\Gamma a^*\Gamma S]$. So, $x \in (a^*] \cup (S\Gamma a^*] \cup (a^*\Gamma S] \cup (S\Gamma a^*\Gamma S] \subseteq (a^* \cup S\Gamma a^* \cup a^*\Gamma S \cup S\Gamma a^*\Gamma S] = I(a^*)$. Thisimplies $(I(a))^* \subseteq I(a^*)$. Hence $(I(a))^* = I(a^*)$.

(iv) $\Rightarrow (v)$. For this, we show (1) $I(a) = ((I(a^*)^6\Gamma I(a)]$, and (2) $((I(a^*))^6\ \Gamma I(a)] \subseteq (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]$. This will imply that $a \in I(a) \subseteq (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]$.

(1) By Lemma 2.9, and our assumption, we obtain $I(a) = (I(a^*)\Gamma I(a^*)] = ((I(a)\Gamma I(a)]\Gamma(I(a)\Gamma I(a)]] \subseteq ((I(a)\Gamma I(a)\Gamma I(a)\Gamma I(a)]] = (I(a)\Gamma I(a)\Gamma I(a)\Gamma I(a)]$.

Moreover,

$$(I(a)\Gamma I(a)\Gamma I(a)\Gamma I(a)] = ((I(a^*)\Gamma I(a^*)]\Gamma(I(a^*)\Gamma I(a^*)]\Gamma(I(a^*)\Gamma I(a^*)]\Gamma(I(a)]$$
$$\subseteq ((I(a^*))^6\Gamma I(a)]$$
$$\subseteq (S\Gamma I(a)]\Gamma I(a) \subseteq (I(a)]$$
$$= I(a) \text{ such that } I(a) \subseteq ((I(a^*))^6\Gamma I(a)] \subseteq I(a). \text{ So, } I(a) = ((I(a^*))^6\Gamma I(a)].$$

(2) As $(I(a))^3 \subseteq (S\Gamma a\Gamma]$ by Theorem 2.11, we obtain $(I(a))^5 = (I(a))^3\Gamma I(a)\Gamma I(a) \subseteq (S\Gamma a\Gamma S]\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]\Gamma(S] \subseteq (S\Gamma a\Gamma S\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S)\Gamma S]$. Obviously, $S\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S)\Gamma S \subseteq S\Gamma a\Gamma S$, and so, $(S\Gamma a\Gamma S\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S)\Gamma S] \subseteq (S\Gamma a\Gamma S\Gamma S\Gamma a\Gamma S] \subseteq (S\Gamma a\Gamma S\Gamma a\Gamma S]$. So, $(I(a))^5 \subseteq (S\Gamma a\Gamma S\Gamma a\Gamma S]$ and therefore, $(I(a^*))^5 \subseteq (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]$. ‘

We have

$((I(a^*))^6\Gamma I(a)] \subseteq ((S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]\Gamma I(a^*)\Gamma I(a)]$

$\subseteq ((S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]\Gamma(S]]$

$\subseteq (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S\Gamma S]$

$\subseteq (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]$

Therefore, $((I(a^*))^6\Gamma I(a)] \subseteq (S\Gamma a^*\Gamma S\Gamma a^*\Gamma S]$.

(v) $\Rightarrow$ (i). Let $x \in (A^*\Gamma A^*]$. Then $x \leq y\alpha z$ for some $y, z \in A^*$ and $\alpha \in \Gamma$. By our assumption, $y \in (S\Gamma y^*\Gamma S\Gamma y^*\Gamma S]$, then $y \leq u_1\alpha y^*\beta u_2\gamma y^*\delta u_3$ for some $u_i \in S$, $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. In a similar fashion, $z \leq v_1\alpha z^*\beta v_2\gamma z^*\delta v_3$ for some $v_i \in S$, $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Therefore, $y\alpha z \leq u_1\beta y^*\gamma u_2\delta y^*\theta u_3\lambda v_1\mu z^*\nu v_2\gamma_2 z^*\gamma_1 v_3 \subseteq S\Gamma y^*\Gamma S \subseteq S\Gamma A\Gamma S \subseteq A$ for $\alpha, \beta, \gamma, \delta, \theta, \lambda, \mu, \nu, \gamma_1, \gamma_2 \in \Gamma$. So, $x \in (A]$ since $x \leq y\alpha z$, and so $(A^*\Gamma A^*] \subseteq (A] = A$. If $x \in A$, then we obtain $x \leq w_1\alpha x^*\beta w_2\gamma x^*\delta w_3$ for some $w_i \in S$, $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ since $x \in (S\Gamma x^*\Gamma S\Gamma x^*\Gamma S]$. It is now obvious that $w_1\alpha x^*\beta w_2 \in A^*$ and $x^*\alpha w_3 \in A^*$ as A^* is an ordered Γ-hyperideal of S by Lemma 2.10. So $x \leq w_1\alpha x^*\beta w_2\gamma x^*\lambda w_3 \subseteq A^*\Gamma A^*$ for $\alpha, \beta, \gamma, \lambda \in \Gamma$ and so $A \subseteq (A\Gamma A^*]$. Hence $A = (A^*\Gamma A^*]$.

Theorem 2.13: *Suppose that S is an ordered Γ-semihypergroup having order preserving involution *. The Γ- hyperideals of S are weakly prime if and only if* $A^{\wedge} = (A\Gamma A]$ *for any Γ-hyperideal A of S and any two Γ-hyperideals are comparable under the inclusion relation.*

Proof: Let the Γ-hyperideals of S be weakly prime. Suppose that A, B are any Γ-hyperideals of S. As B^* is a Γ-hyperideal and $(A\Gamma B^*]$ is weakly prime. Thus $A\Gamma B^* \subseteq (A\Gamma B^*]$ shows that $A^* \subseteq (A\Gamma B^*]$ or $B \subseteq (A\Gamma B^*]$. If $A^* \subseteq (A\Gamma B^*]$, then $A^* \subseteq (S\Gamma B^*] \subseteq (B^*] = B^*$ and so $(A^*)^* \subseteq (B^*)^*$. This means $A \subseteq B$. If $B \subseteq (A\Gamma B^*]$, then $B \subseteq (A\Gamma S] \subseteq (A] = A$. It now follows that A and B are comparable. We claim $A^* = (A\Gamma A]$. As $(A\Gamma A]$ is weakly prime and $A\Gamma A \subseteq (A\Gamma A]$, we obtain $A^* \subseteq (A\Gamma A]$. Also, suppose that $x \in (A\Gamma A]$. Then $x \leq a_1\alpha a_2 \subseteq A\Gamma A$ for some $a_1, a_2 \in A$ and $\alpha \in \Gamma$. As $A^* \subseteq (A\Gamma A]$, we obtain $a_1^* \leq u_1\alpha v_1 \subseteq A\Gamma A$ and $a_2^* \leq u_2\beta v_2 \subseteq A\Gamma A$ for some $u_1, u_2, v_1, v_2 \in A$ and $\alpha, \beta \in \Gamma$. Thus $a_1 \leq (u_1\alpha v_1)^*$ and $a_2 \leq (u_2\beta v_2)^*$.

This shows that $x \leq a_1\alpha a_2 \leq (u_1\beta v_1)^*\gamma(v_1\delta v_2)^* \subseteq (A\Gamma A)^*\Gamma(A\Gamma A)^* = A^*\Gamma A^*\Gamma A^*\Gamma A^* \subseteq A^*$ since A^* is a hyperideal for $\alpha, \beta, \gamma, \delta \in \Gamma$. It follows that $x \in (A^*] = A^*$. So, $(A\Gamma A] \subseteq A^*$.

Conversely, assume A, B *and* P *are* hyperideals of S *such* that $A\Gamma B \subseteq P$. As $A^* = (A\Gamma A]$, we obtain $A^* \cap B^* = (A\Gamma B]$ by Proposition 2.12. As A and B are comparable, two cases arise. If $A \subseteq B$, then $A^* \subseteq B^*$, and so, $A^* = A^* \cap B^* = (A\Gamma B] \subseteq (P] = P$ by Proposition 2.12. Also if $B \subseteq A$, then $B^* \subseteq A^*$, and so $B^* = A^* \cap B^* = (A\Gamma B] \subseteq (P] = P$. Hence P *is* weakly prime.

Proposition 2.14: *Suppose that S is an ordered involution Γ-semihypergroup. Then S is intra-regular if andonly if the Γ-hyperideals of S are semiprime.*

Proof: Let I be a Γ-hyperideal of S having $s\alpha s \subseteq I$ for some $s \in S$ and $\alpha \in \Gamma$. As S is intra regular, we obtain $s^* \in (S\Gamma s\gamma s\Gamma S] \subseteq (S\Gamma I\Gamma S] \subseteq (I] = I$ for $\gamma \in \Gamma$ and therefore I is semiprime.

Conversely, let $s \in S$. It is now obvious that $(S\Gamma s^*\gamma s^*\Gamma S]$ is a Γ-hyperideal. Therefore, $(s\Gamma s^*\gamma s^*\Gamma S]$ is semiprime by our assumption. This shows that $s\gamma s = (s^*\gamma s^*)^* \subseteq (S\Gamma s^*\beta s^*\Gamma S]$ since $(s^*\alpha s^*)\beta(s^*\gamma s^*) \subseteq S\Gamma s^*\delta s^*\Gamma S \subseteq (S\Gamma s^*\lambda s^*\Gamma S]$ for $\alpha, \beta, \gamma, \delta, \lambda \in \Gamma$. So, $s^* \in (S\Gamma s^*\alpha s^*\Gamma S]$ and so $s^*\alpha s^* \subseteq (S\Gamma s^*\beta s^*\Gamma S]$ for $\alpha, \beta \in \Gamma$. Hence $s \in (S\Gamma s^*\alpha s^*\Gamma S]$ and it follows that S is intra-regular.

Proposition 2.15: *Suppose that S is an ordered involution Γ-semihypergroup. If S is intra-regular, then* $(S\Gamma x\alpha y\Gamma S] = (S\Gamma x^*\beta y^*\Gamma S]$ *for some* $x, y \in S$ *and* $\alpha, \beta \in \Gamma$.

Proof: Suppose that $x, y \in S$. As S is intra-regular, it follows that $x\alpha y \subseteq (S\Gamma(x\beta y)^*\gamma(x\delta y)^*\Gamma S] = (S\Gamma y^*\gamma_1 x^*\gamma_2 y^*\gamma_3 x^*\Gamma S] \subseteq (S\Gamma x^*\alpha y^*\Gamma S]$ for $\alpha, \beta, \gamma, \gamma_1, \gamma_2, \gamma_3 \in \Gamma$. Therefore, $x\alpha y \leq u_1\beta x^*\gamma y^*\delta u_2$ for some $u_1, u_2 \in S$. Therefore, $u_3\alpha x\beta y\gamma u_4 \leq u_3\delta u_1\theta x^*\lambda y^*\mu u_2 \nu u_4 \subseteq S\Gamma x^*\alpha y^*\Gamma S \subseteq (S\Gamma x^*\alpha y^*\Gamma S]$ for any $u_3, u_4 \in S$ and $\alpha, \beta, \gamma, \delta, \theta, \lambda \in \Gamma$. This shows that $S\Gamma x\alpha y\Gamma S \subseteq (S\Gamma x^*\alpha y^*\Gamma]$, therefore, $(S\Gamma x\alpha y\Gamma S] \subseteq ((S\Gamma x^*\alpha y^*\Gamma S]] = (S\Gamma x^*\alpha\Gamma S]$ by Lemma 2.9. We obtain $(S\Gamma x^*\alpha y^*\Gamma S] \subseteq (S\Gamma x\beta y\Gamma S]$. Hence, $(S\Gamma x\alpha y\Gamma S] = (S\Gamma x^*\beta y^*\Gamma S]$ for $\alpha, \beta \in \Gamma$.

Proposition 2.16: *Suppose that S is an ordered Γ-semihypergroup with order preserving involution *. If the Γ-hyperideals of S are semiprime, then*

(i) $I(s) = (S\Gamma s\Gamma S]$ *for any* $s \in S$, *and*

(ii) $I(x\alpha y) = I(x) \cap I(y)$ *for any* $x, y \in S$ *and* $\alpha \in \Gamma$.

Proof: (i) Suppose that $s \in S$. Recall that $(S\Gamma s\Gamma S]$ is a Γ-hyperideal and so is semiprime. Since $(s\alpha s)\alpha(s\alpha s) = (s\alpha s)^2 = s^4 \subseteq (S\Gamma s\Gamma S]$ gives $s^*\alpha s^* = (s\alpha s)^* \subseteq (S\Gamma s\Gamma S]$ for $\alpha \in \Gamma$. In a similar fashion, $s \in (S\Gamma s\Gamma S]$ so that $I(s) \subseteq (S\Gamma s\Gamma S]$. Moreover, $(S\Gamma s\Gamma S] \subseteq (s \cup s\Gamma S \cup S\Gamma s \cup S\Gamma s\Gamma S] = I(s)$. Hence, $I(x) = (S\Gamma x\Gamma S]$. (ii)As $x\alpha y \subseteq I(x)\Gamma S \subseteq I(x)$, we obtain $I(x\alpha y) \subseteq I(x)$. Also $I(x\alpha y) \subseteq I(y)$ since $x\alpha y \subseteq S\Gamma I(y) \subseteq I(y)$. So, $I(x\alpha y) \subseteq I(x) \cap I(y)$. If $z \in I(x) \cap I(y)$, then $z \in (S\Gamma x\Gamma S] \cap (S\Gamma y\Gamma S]$ by (i), and so $z \leq u_1\alpha x\beta u_2$ and $z \leq v_1\alpha y\beta v_2$ for some $u_1, u_2, v_1, v_2 \in S$ and for $\alpha, \beta \in \Gamma$.

Recall $(y\alpha_1 v_2\alpha_2 u_1\alpha_3 x)^2 = (y\alpha_4 v_2\alpha_5 u_1\alpha_6 x)\alpha_7(y\alpha_8 v_2\alpha_9 u_1\alpha_{10} x) \subseteq (S\Gamma x\alpha_{11} y\Gamma S] = I(x\alpha_{12} y)$ for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12} \in \Gamma$ and that $I(x\alpha y)$ is semiprime. So, $(y\alpha v_2\beta u_1\gamma x)^* \subseteq I(x\alpha y)$. So, $z^*\alpha z^* \leq (u_1\alpha x\beta u_2)^*\gamma(v_1\alpha y\beta v_2)^* = u^*\alpha(y\beta v_2\gamma u_1\delta x)^*\theta v^* \subseteq I(x\alpha y)$, and so $z^*\alpha z^* \subseteq (I(x\alpha y)] = I(x\alpha y)$ for $\alpha, \beta, \gamma, \delta, \theta \in \Gamma$. This implies that $z \in I(x\alpha y)$, then $I(x) \cap I(y) \subseteq I(x\alpha y)$.

Theorem 2.17: *Suppose that S is an ordered involution Γ-semihypergroup such that the involution admits the order. Then the Γ-hyperideals of S are prime if and only if S is intra-regular and any two Γ-hyperideals are comparable under the inclusion relation.*

Proof: If the Γ-hyperideals are prime, then they are weakly prime and hence they are comparable by Theorem 2.13. Suppose that $s \in S$. Recall that $(S\Gamma s^*\alpha s^*\Gamma S]$ is a Γ-hyperideal by Lemma 2.9 and hence prime. So, $(s\alpha s)\alpha(s\alpha s) = s^4 \subseteq (S\Gamma s^*\alpha s^*\Gamma S]$ since $(s^*)^4\alpha(s^*)^4 \subseteq (S\Gamma s^*\beta s^*\Gamma S]$ for $\alpha, \beta \in \Gamma$. In a similar fashion, we have $(s^*\alpha s^*) = (s^*)^2 \subseteq (S\Gamma s^*\alpha s^*\Gamma S]$ and $s \in (S\Gamma s^*\alpha s^*\Gamma S]$. It follows that S is intra-regular.

Conversely, assume that S is intra-regular and any two Γ-hyperideals are comparable under the inclusion relation $\subseteq$. Suppose that T is any Γ-hyperideal of S and $a\alpha b \subseteq T$, where $a, b \in S$ and $\alpha \in \Gamma$. Claim $a^* \in T$ or $b^* \in T$. By Proposition 2.14, $I(a)$ is semiprime. Thus, we have $a\alpha a \subseteq I(a)$ implies $a^* \in I(a)$.

We can similarly prove $b^* \in \boldsymbol{I}(b)$. By our assumption, we obtain $\boldsymbol{I}(a) \subseteq \boldsymbol{I}(b)$ or $\boldsymbol{I}(b) \subseteq \boldsymbol{I}(a)$.

If $\boldsymbol{I}(a) \subseteq \boldsymbol{I}(b)$, then $a^* \in \boldsymbol{I}(a) = \boldsymbol{I}(a) \cap \boldsymbol{I}(b) = \boldsymbol{I}(a\alpha b) \subseteq T$ by Proposition 2.16. If $\boldsymbol{I}(b) \subseteq \boldsymbol{I}(a)$, then we obtain $b^* \in \boldsymbol{I}(b) = \boldsymbol{I}(a) \cap \boldsymbol{I}(b) = \boldsymbol{I}(a\alpha b) \subseteq T$.

REFERENCES

1. S.M. Anvariyeh, S. Mirvakili and B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, Carpathian J.Math., 26 (2010) 11-23.
2. W.E. Baxter, On rings with proper involution, Pacific J. Math., 27(1) (1968) 1-12.
3. P. Bonansinga and P. Corsini, On semihypergroup and hypergroup homomorphisms, Boll. Un. Mat. Ital., 6 (1982) 717-727.
4. B. Davvaz and N. S. Poursalavati, Semihypergroups and S-hypersystems, Pure Math. Appl., 11 (2000)43-49.
5. M.P. Drazin, Regular semigroups with involution, Proc. Symp. Regular Semigroups, (1979) 29-46.

6. D. Fasino and D. Freni, Existence of proper semihypergroups of type U on the right, Discrete Math.,307 (2007) 2826-2836.
7. D. J. Foulis, Involution semigroups, Ph.D. Thesis, Tulane University, New Orleans, (1958).
8. A. Hasankhani, Ideals in a semihypergroup and Green’s relations, Ratio Math., 13 (1999) 29-36.
9. I.N. Herstein, Ring with involution, University of Chicago Press, Chicago, (1976).
10. N. Kehayopulu, On weakly prime ideals of ordered semigroups, Math. Japon., 26(3) (1990) 1051-1056.
11. N. Kehayopulu, On prime, weakly prime ideals in ordered semigroups, Semigroup Forum, 44 (1992)341-346.
12. N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum, 46 (1993) 271-278.
13. M. Kondo and N. Lekkoksung, On intra-regular ordered Γ-semihypergroups, Int. J. Math. Analysis, 7(2013) 1379-1386.
14. Y.I. Kwon and S.K. Lee, Some special elements in ordered Γ-semigroups, Kyungpook Math J., 35 (1996)679-685.
15. F. Marty, Sur un generalisation de la notion de groupe, Huitieme Congres des mathematiciens Scand., Stockholm (1934) 45-49.
16. N. H. McCoy, Prime ideals of general rings, Amer. J. Math., 71 (1949) 823-833.
17. J. Nieminen, Join space graphs, J. Geom., 33 (1988) 99-103.
18. M. Petrich, Introduction to semigroups, Merill, Columbus, (1973).
19. M. K. Sen, On Γ-semigroups, Algebra and its applications, Int. Symp., New Delhi, (1981), Lecture Notesin Pure and Applied Mathematics 91, Decker, New York, (1984) 301-308.
20. O. Steinfeld, Remark on a paper of N. H. McCoy, Publ. Math. Debrecen, 3 (1953-54) 171-173.
21. G. Szasz, Eine charakteristik der primidealhalbgruppen, Publ. Math. Debrecen, 17 (1970) 209-213.
22. J. Tang, B. Davvaz and X. Xie, A study on (fuzzy) quasi-Γ-hyperideals in ordered Γ-semihypergroups,
23. J. Intell. Fuzzy Syst., doi: 10.3233/IFS-162117.
24. C.-Y. Wu, On intra-regular ordered ∗-semigroups, Thai J. Math., 12(1) (2014) 15-24.
25. M. R. Khan, D. Arora and M. A. Khan, Remarks on Derivations of σ-Prime Rings, International Journal of Algebra, 4(16)(2010) 761 - 767.

Source of support: Nil, Conflict of interest: None Declared.