# International Journal of Mathematical Archive-12(5), 2021, 8-10 <br> IMAAvailable online through www.ijma.info ISSN 2229-5046 

# MATRIX REPRESENTATIONS OF GROUP ALGEBRAS OF SPLIT METACYCLIC GROUPS 

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(Received On: 25-03-21; Revised \& Accepted On: 06-05-21)


#### Abstract

We compute in this paper the matrix representations of group algebras of split metacyclic groups. The representations are given in terms of circulant block matrices.


Key Words: Group Algebra, Metacyclic Group, Circulant Matrix.

## PRELIMINARIES

Let $F$ be a field. A ring $A$ is an algebra over $F$ (breifly $F$-algebra) if $A$ is a vector space over $F$ and the following compatibility condition holds $(s a) \cdot b=s(a \cdot b)=a \cdot(s b)$ for any $a, b \in A$ and anys $\in F$. $A$ is also called associative algebra (over F).

The dimension of the algebra $A$ is the dimension of $A$ as a vector space over $F$.
Theorem 1[1]: Let $A$ be a $n$-dimensional algebra over a field $F$. Then there is a one to one algebra homomorphism from $A$ into $M_{n}(F)$, the algebra of $n$-matrices over $F$.

Let $G=\left\{g_{1}=1, g_{2}, \ldots, g_{n}\right\}$ be a finite group of order $n$ and $F$ a field.
Define $F G=\left\{a_{1} g_{1}+a_{2} g_{2}+\cdots+a_{n} g_{n}: a_{i} \in F\right\} . F G$ is $n$-dimentional vector space over $F$ with basis $G$. Multiplication of $G$ can be extended linearly to $F G$ by using group operation of $G$. Thus $F G$ becomes an algebra over $F$ of dimention $n$. $F G$ is called group algebra. The following identifications should be realized.
i) $0_{F} g_{G}=0_{F G}=0$ for any $g \in G$.
ii) $1_{F} g_{G}=g_{F G}$ for any $g \in G$. In particular $1_{F} 1_{G}=1_{F G}=1$.
iii) $a_{F} 1_{G}=a_{F G}$ for any $a \in F$.

A group $G$ is metacyclic if it has a cyclic normal subgroup $N$ such that $G / N$ is cyclic. Eqivalently, $G$ has cyclic subgroups $H$ and $K$ such that $H \triangleleft G$ and $G=H K$ [2].If $H \cap K=\{1\}$ also, then $G$ is called a split metacyclic group. If $G$ is a split metacyclic group, then $G$ has a representation of the following form [3].

$$
G=\left\langle\alpha, \beta: \alpha^{n}=\beta^{m}=1, \beta \alpha=\alpha^{r} \beta\right\rangle \text { where } r^{m} \equiv 1(\bmod n) .|G|=n m .
$$

The general element of $G$ is of the form $\alpha^{u} \beta^{v}$, where $0 \leq u \leq n, 0 \leq v \leq m$.
By direct substitutions we have the following in $G$.
Lemma 2: i) $\beta^{v} \alpha^{u} \beta^{-v}=\alpha^{u r^{v}}$, ii) $\left(\alpha^{u_{1}} \beta^{v_{1}}\right)\left(\alpha^{u_{2}} \beta^{v_{2}}\right)=\alpha^{u_{1}+r^{v_{1}} u_{2}} \beta^{v_{1}+v_{2}}$, where $u, v, u_{1}, v_{1}, u_{2}, v_{2}$ are integers. A circulant matrix $M$ on parameters $a_{0}, a_{1}, \ldots, a_{n-1}$ is defined as follows

$$
M\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)=\left[\begin{array}{ccc}
a_{0} & a_{n-1} \cdots & a_{1} \\
a_{1} & a_{0} \cdots & a_{2} \\
\vdots & \vdots & \\
a_{n-1} & a_{n-2} \cdots & a_{0}
\end{array}\right]
$$

$M$ is said to be circulant block matrix if it is of the form $M\left(M_{1}, M_{2}, \cdots, M_{n}\right)$. i.e it is circulant blockwise on the blocks $M_{1}, M_{2}, \cdots, M_{n}$.

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Thus

$$
M=\left[\begin{array}{ccc}
M_{1} & M_{n} \cdots & M_{2} \\
M_{2} & M_{1} \cdots & M_{3} \\
\vdots & \vdots & \vdots \\
M_{n} & M_{n-1} \cdots & M_{1}
\end{array}\right] .
$$

## MAIN RESULTS

Theorem 3: Let $F$ be a field and $G=\left\langle\alpha: \alpha^{n}=1\right\rangle$ a cyclic group of order n. Then any element $a_{0} 1+a_{1} \alpha+\cdots+$ $a_{n-1} \alpha^{n-1}$ of $F G$ can be represented with respect to the ordered basis $\left\{1, \alpha, \cdots, \alpha^{n-1}\right\}$ by the circulant matrix $M\left(a_{0}, a_{1}, \cdots, a_{n-1}\right)$.

Proof: Let $w=a_{0} 1+a_{1} \alpha+\cdots+a_{n-1} \alpha^{n-1}$ be in $F G . w \alpha=a_{0} \alpha+a_{1} \alpha^{2}+\cdots+a_{n-1} 1=a_{n-1} 1+a_{0} \alpha+\cdots+$ $a_{n-2} \alpha^{n-1} \ldots w \alpha^{n-1}=a_{0} \alpha^{n-1}+a_{1} 1+\cdots+a_{n-1} \alpha^{n-2}=a_{1} 1+a_{2} \alpha+\cdots+a_{0} \alpha^{n-1}$ Then the matrix representation of $w$ with respect to the basis $\left\{1, \alpha, \cdots, \alpha^{n-1}\right\}$ is

$$
\left[\begin{array}{ccc}
a_{0} & a_{n-1} \cdots & a_{1} \\
a_{1} & a_{0} \cdots & a_{2} \\
\vdots & \vdots & \vdots \\
a_{n-1} & a_{n-2} \cdots & a_{0}
\end{array}\right]
$$

which is $M\left(a_{0}, a_{1}, \cdots, a_{n-1}\right)$.
Note that if the order of the basis elements is changed we obtain a different matrix of representation. The new matrix is obtained by suitable interchanging of the columns of the matrix $M\left(a_{0}, a_{1}, \cdots, a_{n-1}\right)$.

Now, let $G$ be a split metacyclic group. Then $G=\left\langle\alpha, \beta: \alpha^{n}=\beta^{m}=1, \beta \alpha=\alpha^{r} \beta\right\rangle$ where $r^{m} \equiv 1(\bmod n)$.
Consider the following natural basis of the group algebra $F G,\left\{1, \alpha, \cdots, \alpha^{n-1} ; \beta, \alpha \beta, \cdots, \alpha^{n-1} \beta ; \cdots ; \beta^{m-1}, \alpha \beta^{m-1}, \cdots\right.$, $\left.\alpha^{n-1} \beta^{m-1}\right\}$. This basis can be written as follows $\left\{1, \alpha, \cdots, \alpha^{n-1}\right\} \cup\left\{1, \alpha, \cdots, \alpha^{n-1}\right\} \beta \cup \cdots \cup\left\{1, \alpha, \cdots, \alpha^{n-1}\right\} \beta^{m-1}$.

By construction each part $\left\{1, \alpha, \cdots, \alpha^{n-1}\right\} \beta^{i} ; 0 \leq i \leq m-1$ induces the matrix with respect to the basis $\left\{1, \alpha, \cdots, \alpha^{n-1} \beta^{\beta^{i}} \equiv\left\{\beta^{i} 1 \beta^{-i}, \beta^{i} \alpha \beta^{-i}, \cdots, \beta^{i} \alpha^{n-1} \beta^{-i}\right\}\right.$. This basis can be simplified by using lemma 2 . Call the matrix obtained from the basis $\left\{1, \alpha, \cdots, \alpha^{n-1}\right\}^{\beta^{i}}$ by $M^{\beta^{i}}$. Thus we have the following theorem about matrix representation of the group algebra $F G$.

Theorem 4: Let $F$ be a field and $G$ a split metacyclic group as above. The representation of the general element $\sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{i j} \alpha^{i} \beta^{j}$ in $F G$ is given by the circulant block matrix

$$
M\left(M\left(a_{i 0}\right), M^{\beta}\left(a_{i 1}\right), \cdots, M^{\beta^{m-1}}\left(a_{i m-1}\right)\right) ; i=0,1, \cdots, n-1
$$

Corollary 5: Let $F$ be a field. Matrix representation of $F\left(C_{n} \times C_{m}\right)$, where $(m, n) \neq 1$ is given by

$$
M\left(M\left(a_{i 0}\right), M\left(a_{i 1}\right), \cdots, M\left(a_{i m-1}\right)\right) \text { for } i=0,1, \cdots, n-1 \text { and } a_{i j} \in F
$$

Corollary 6: Let $F$ be a field and $D_{n}=\left\langle\alpha, \beta: \alpha^{n}=\beta^{2}=1, \beta \alpha=\alpha^{n-1} \beta\right\rangle$ the dihedral group. Matrix representation of the general element $\sum_{i=0}^{n-1} a_{i} \alpha^{i}+\sum_{i=0}^{n-1} b_{i} \alpha^{i} \beta$ in $F D_{n}$ is given by $M\left(M\left(a_{0}, a_{1}, \cdots, a_{n-1}\right), M^{\beta}\left(b_{0}, b_{1}, \cdots, b_{n-1}\right)\right)$.

## APPLICATION

Consider the split metacyclic group $G=\left\langle\alpha, \beta: \alpha^{3}=\beta^{4}=1, \beta \alpha=\alpha^{2} \beta\right\rangle$ of order 12. The general element of $F G$, where $F$ is a field, is $a 1+b \alpha+c \alpha^{2}+d \beta+e \alpha \beta+f \alpha^{2} \beta+g \beta^{2}+h \alpha \beta^{2}+i \alpha^{2} \beta^{2}+j \beta^{3}+k \alpha \beta^{3}+l \alpha^{2} \beta^{3}$.

Let $B=\left\{1, \alpha, \alpha^{2}\right\}$ be the natural basis of $F G$. Then $B^{\beta}=\left\{1, \alpha^{2}, \alpha\right\}, B^{\beta^{2}}=\left\{1, \alpha, \alpha^{2}\right\}, B^{\beta^{3}}=\left\{1, \alpha^{2}, \alpha\right\}$ by lemma 2 . The basis $B, B^{\beta}, B^{\beta^{2}}, B^{\beta^{3}}$ induce by construction the following corresponding matrices.

$$
M=\left[\begin{array}{lll}
a & c & b \\
b & a & c \\
c & b & a
\end{array}\right], M^{\beta}=\left[\begin{array}{lll}
d & e & f \\
e & f & d \\
f & d & e
\end{array}\right], M^{\beta^{2}}=\left[\begin{array}{ccc}
g & i & h \\
h & g & i \\
i & h & g
\end{array}\right], M^{\beta^{3}}=\left[\begin{array}{lll}
j & k & l \\
k & l & j \\
l & j & k
\end{array}\right] .
$$

By theorem 4 the matrix representation of $F G$ is given as follows

$$
M\left(M, M^{\beta}, M^{\beta^{2}}, M^{\beta^{3}}\right)=\left[\begin{array}{cccc}
M & M^{\beta^{3}} & M^{\beta^{2}} & M^{\beta} \\
M^{\beta} & M & M^{\beta^{3}} & M^{\beta^{2}} \\
M^{\beta^{2}} & M^{\beta} & M & M^{\beta^{3}} \\
M^{\beta^{3}} & M^{\beta^{2}} & M^{\beta} & M
\end{array}\right]
$$

which is

$$
M=\left[\begin{array}{ccccccccccccccc}
a & c & b & \vdots & j & k & l & \vdots & g & i & h & \vdots & d & e & f \\
b & a & c & \vdots & k & l & j & \vdots & h & g & i & \vdots & e & f & d \\
c & b & a & \vdots & l & j & k & \vdots & i & h & g & \vdots & f & d & e \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
d & e & f & \vdots & a & c & b & \vdots & j & k & l & \vdots & g & i & h \\
e & f & d & \vdots & b & a & c & \vdots & k & l & j & \vdots & h & g & i \\
f & d & e & \vdots & c & b & a & \vdots & l & j & k & \vdots & i & h & g \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
g & i & h & \vdots & d & e & f & \vdots & a & c & b & \vdots & j & k & l \\
h & g & i & \vdots & e & f & d & \vdots & b & a & c & \vdots & k & l & j \\
i & h & g & \vdots & f & d & e & \vdots & c & b & a & \vdots & l & j & k \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
j & k & l & \vdots & g & i & h & \vdots & d & e & f & \vdots & a & c & b \\
k & l & j & \vdots & h & g & i & \vdots & e & f & d & \vdots & b & a & c \\
l & j & k & \vdots & i & h & g & \vdots & f & d & e & \vdots & c & b & a
\end{array}\right]
$$

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Source of support: Nil, Conflict of interest: None Declared.
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