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MATRIX REPRESENTATIONS OF GROUP ALGEBRAS OF SPLIT METACYCLIC GROUPS

KAHTAN H. ALZUBAIDY & NABILA M. BENNOUR*

University of Benghazi, Faculty of Science, Department of Mathematics

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ABSTRACT

We compute in this paper the matrix representations of group algebras of split metacyclic groups. The representations are given in terms of circulant block matrices.

Key Words: *Group Algebra, Metacyclic Group, Circulant Matrix.*

PRELIMINARIES

Let F be a field. A ring A is an algebra over F (breifly F-algebra) if A is a vector space over F and the following compatibility condition holds $(sa)\cdot b = s(a\cdot b) = a\cdot(sb)$ for any $a, b \in A$ and any$s \in F$. A is also called associative algebra (over F).

The dimension of the algebra A is the dimension of A as a vector space over F.

Theorem 1[1]: Let A be a n-dimensional algebra over a field F. Then there is a one to one algebra homomorphism from A into $M_n(F)$, the algebra of n-matrices over F.

Let $G = \{g_1 = 1, g_2, \dots, g_n\}$ be a finite group of order n and F a field.

Define $FG = \{a_1g_1 + a_2g_2 + \cdots + a_ng_n : a_i \in F\}$. FG is n-dimentional vector space over F with basis G. Multiplication of G can be extended linearly to FG by using group operation of G. Thus FG becomes an algebra over F of dimention n. FG is called group algebra. The following identifications should be realized.

i) $0_F g_G = 0_{FG} = 0$ for any $g \in G$.
ii) $1_F g_G = g_{FG}$ for any $g \in G$. In particular $1_F 1_G = 1_{FG} = 1$.
iii) $a_F 1_G = a_{FG}$for any $a \in F$.

A group G is metacyclic if it has a cyclic normal subgroup N such that ${}^{G}/_{N}$ is cyclic. Eqivalently, G has cyclic subgroups H and K such that $H \lhd G$ and $G = HK$ [2].If $H \cap K = \{1\}$ also, then G is called a split metacyclic group. If G is a split metacyclic group, then G has a representation of the following form [3].

$$G = \langle \alpha, \beta : \alpha^n = \beta^m = 1, \beta\alpha = \alpha^r\beta \rangle \text{ where } r^m \equiv 1 \ (mod\ n).\ |G| = nm.$$

The general element of G is of the form$\alpha^u\beta^v$, where $0 \le u \le n,\ 0 \le v \le m$.

By direct substitutions we have the following in G.

Lemma 2: i) $\beta^v\alpha^u\beta^{-v} = \alpha^{ur^v}$, ii) $(\alpha^{u_1}\beta^{v_1})(\alpha^{u_2}\beta^{v_2}) = \alpha^{u_1 + r^{v_1}u_2}\beta^{v_1+v_2}$, where u, v, u_1, v_1, u_2, v_2 are integers.
A circulant matrix M on parameters $a_0, a_1, \dots, a_{n-1}$ is defined as follows

$$M(a_0, a_1, \dots, a_{n-1}) = \begin{bmatrix} a_0 & a_{n-1}\cdots & a_1 \\ a_1 & a_0\ \cdots & a_2 \\ \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2}\cdots & a_0 \end{bmatrix}$$

M is said to be circulant block matrix if it is of the form $M(M_1, M_2, \cdots, M_n)$. *i.e* it is circulant blockwise on the blocks $M_1, M_2, \cdots, M_n$.

Corresponding Author: Nabila M. Bennour*
University of Benghazi, Faculty of Science, Department of Mathematics

Thus $$M=\begin{bmatrix} M_1 & M_n\cdots & M_2 \\ M_2 & M_1\cdots & M_3 \\ \vdots & \vdots & \vdots \\ M_n & M_{n-1}\cdots & M_1 \end{bmatrix}.$$

MAIN RESULTS

Theorem 3: Let F be a field and $G=\langle\alpha:\alpha^n=1\rangle$ a cyclic group of order n. Then any element $a_0 1+a_1\alpha+\cdots+a_{n-1}\alpha^{n-1}$ of FG can be represented with respect to the ordered basis $\{1,\alpha,\cdots,\alpha^{n-1}\}$ by the circulant matrix $M(a_0,a_1,\cdots,a_{n-1})$.

Proof: Let $w=a_0 1+a_1\alpha+\cdots+a_{n-1}\alpha^{n-1}$ be in FG. $w\alpha=a_0\alpha+a_1\alpha^2+\cdots+a_{n-1}1=a_{n-1}1+a_0\alpha+\cdots+a_{n-2}\alpha^{n-1}$ ... $w\alpha^{n-1}=a_0\alpha^{n-1}+a_1 1+\cdots+a_{n-1}\alpha^{n-2}=a_1 1+a_2\alpha+\cdots+a_0\alpha^{n-1}$ Then the matrix representation of w with respect to the basis $\{1,\alpha,\cdots,\alpha^{n-1}\}$ is
$$\begin{bmatrix} a_0 & a_{n-1}\cdots & a_1 \\ a_1 & a_0\ \cdots & a_2 \\ \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2}\cdots & a_0 \end{bmatrix}$$
which is $M(a_0,a_1,\cdots,a_{n-1})$.

Note that if the order of the basis elements is changed we obtain a different matrix of representation. The new matrix is obtained by suitable interchanging of the columns of the matrix $M(a_0,a_1,\cdots,a_{n-1})$.

Now, let G be a split metacyclic group. Then $G=\langle\alpha,\beta:\alpha^n=\beta^m=1,\beta\alpha=\alpha^r\beta\rangle$ where $r^m\equiv 1\ (mod\ n)$.

Consider the following natural basis of the group algebra FG, $\{1,\alpha,\cdots,\alpha^{n-1};\ \beta,\alpha\beta,\cdots,\alpha^{n-1}\beta;\ \cdots;\beta^{m-1},\alpha\beta^{m-1},\ \cdots,\ \alpha^{n-1}\beta^{m-1}\}$. This basis can be written as follows $\{1,\alpha,\cdots,\alpha^{n-1}\}\cup\{1,\alpha,\cdots,\alpha^{n-1}\}\beta\cup\ \cdots\ \cup\{1,\alpha,\cdots,\alpha^{n-1}\}\beta^{m-1}$.

By construction each part $\{1,\alpha,\cdots,\alpha^{n-1}\}\beta^i$; $0\le i\le m-1$ induces the matrix with respect to the basis $\{1,\alpha,\cdots,\alpha^{n-1}\}^{\beta^i}\equiv\{\beta^i 1\beta^{-i},\beta^i\alpha\beta^{-i},\ \cdots,\beta^i\alpha^{n-1}\beta^{-i}\}$. This basis can be simplified by using lemma 2. Call the matrix obtained from the basis $\{1,\alpha,\cdots,\alpha^{n-1}\}^{\beta^i}$ by M^{β^i}. Thus we have the following theorem about matrix representation of the group algebra FG.

Theorem 4: Let F be a field and G a split metacyclic group as above. The representation of the general element $\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}a_{ij}\,\alpha^i\beta^j$ in FG is given by the circulant block matrix
$$M\left(M(a_{i0}),M^{\beta}(a_{i1}),\cdots,M^{\beta^{m-1}}(a_{im-1})\right);i=0,1,\cdots,n-1.$$

Corollary 5: Let F be a field. Matrix representation of $F(C_n\times C_m)$, where $(m,n)\neq 1$ is given by
$M\big(M(a_{i0}),M(a_{i1}),\cdots,M(a_{i\,m-1})\big)$ for $i=0,1,\cdots,n-1$ and $a_{ij}\in F$.

Corollary 6: Let F be a field and $D_n=\langle\alpha,\beta:\alpha^n=\beta^2=1,\ \beta\alpha=\alpha^{n-1}\beta\rangle$ the dihedral group. Matrix representation of the general element $\sum_{i=0}^{n-1}a_i\alpha^i+\sum_{i=0}^{n-1}b_i\alpha^i\beta$ in FD_n is given by $M\left(M(a_0,a_1,\cdots,a_{n-1}),M^{\beta}(b_0,b_1,\cdots,b_{n-1})\right)$.

APPLICATION

Consider the split metacyclic group $G=\langle\alpha,\beta:\alpha^3=\beta^4=1,\beta\alpha=\alpha^2\beta\rangle$ of order 12. The general element of FG, where F is a field, is $a1+b\alpha+c\alpha^2+d\beta+e\alpha\beta+f\alpha^2\beta+g\beta^2+h\alpha\beta^2+i\alpha^2\beta^2+j\beta^3+k\alpha\beta^3+l\alpha^2\beta^3$.

Let $B=\{1,\alpha,\alpha^2\}$ be the natural basis of FG. Then $B^{\beta}=\{1,\alpha^2,\alpha\},B^{\beta^2}=\{1,\alpha,\alpha^2\},B^{\beta^3}=\{1,\alpha^2,\alpha\}$ by lemma 2. The basis $B,B^{\beta},B^{\beta^2},B^{\beta^3}$ induce by construction the following corresponding matrices.
$$M=\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix},M^{\beta}=\begin{bmatrix} d & e & f \\ e & f & d \\ f & d & e \end{bmatrix},M^{\beta^2}=\begin{bmatrix} g & i & h \\ h & g & i \\ i & h & g \end{bmatrix},M^{\beta^3}=\begin{bmatrix} j & k & l \\ k & l & j \\ l & j & k \end{bmatrix}.$$

By theorem 4 the matrix representation of FG is given as follows
$$M\big(M,M^{\beta},M^{\beta^2},M^{\beta^3}\big)=\begin{bmatrix} M & M^{\beta^3} & M^{\beta^2} & M^{\beta} \\ M^{\beta} & M & M^{\beta^3} & M^{\beta^2} \\ M^{\beta^2} & M^{\beta} & M & M^{\beta^3} \\ M^{\beta^3} & M^{\beta^2} & M^{\beta} & M \end{bmatrix}$$

which is

$$M = \begin{bmatrix} a & c & b & \vdots & j & k & l & \vdots & g & i & h & \vdots & d & e & f \\ b & a & c & \vdots & k & l & j & \vdots & h & g & i & \vdots & e & f & d \\ c & b & a & \vdots & l & j & k & \vdots & i & h & g & \vdots & f & d & e \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ d & e & f & \vdots & a & c & b & \vdots & j & k & l & \vdots & g & i & h \\ e & f & d & \vdots & b & a & c & \vdots & k & l & j & \vdots & h & g & i \\ f & d & e & \vdots & c & b & a & \vdots & l & j & k & \vdots & i & h & g \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ g & i & h & \vdots & d & e & f & \vdots & a & c & b & \vdots & j & k & l \\ h & g & i & \vdots & e & f & d & \vdots & b & a & c & \vdots & k & l & j \\ i & h & g & \vdots & f & d & e & \vdots & c & b & a & \vdots & l & j & k \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ j & k & l & \vdots & g & i & h & \vdots & d & e & f & \vdots & a & c & b \\ k & l & j & \vdots & h & g & i & \vdots & e & f & d & \vdots & b & a & c \\ l & j & k & \vdots & i & h & g & \vdots & f & d & e & \vdots & c & b & a \end{bmatrix}$$

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