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Double Full Subsets By m Of Z

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ABSTRACT

Let A be a subset of Z such that $A= A^+ \cup A^-$ *, where* $A^+ =\{a_1,\ldots, a_k\}$ *,* $A^- =\{-a_k,\ldots, -a_1\}=\{b_k,\ldots, b_1\}$ *and* $a_i \geq 0$*,* $a_1<\ldots <a_k$ *. We say that A is double full by m if* $\sum A^+= [m]$ *and* $\sum A^- =[-m]$ *for a positive integer m, where* $\sum A^+$ *is the set of all positive integers and* $\sum A^-$ *is the set of all negative integers . We show that a set* A^+ *is full if and only if* $a_1=1$ *and* $a_i\leq + \ldots + a_{i-1} +1$ *for each i ,* $2 \leq i \leq k$ *and* A^- *is full if and only if* $b_1= -1$ *and* $b_i\geq b_1 + \ldots + b_{i-1}-1$ *for each i ,* $2 \leq i \leq k$ *.*

We also prove that for each integer $m\notin \{ \pm 2 , \pm 4 , \pm 5 , \pm 8 , \pm 9 \}$ *there is an double full by m set. We also give formula for F(m), the number of m full sets of* Z^+ *and F(-m), the number of -m full sets of* Z^-*.*

Keywords: *Double Full, Double Full By m, Partition Of Integer*

1. INTRODUCTION:

Let n be a positive integer and denote by D (n) and σ(n) the set of its positive divisors and the sum of its positive divisors, respectively.

Let A be a subset of Z. Define the sum set of A, dented by $\sum A^{+,-}$:

$$\sum A^+ = \{ a_{i_1} + \cdots + a_{i_r} : a_{i_1} < \cdots < a_{i_r} , 1 \leq r \leq$$
$$\sum A^- = \{ b_{i_1} + \cdots + b_{i_r} : b_{i_1} > \cdots > b_{i_r} , 1 \leq r \leq k\}$$

For what positive integer m does there exist a set $A= A^+ \cup A^-$ with $\sum A^+=[m]$ and $\sum A^-= [-m]$, where $[m] =\{1, \ldots , m\}$ and $[-m] = \{-1\ldots-m\}$?

We show that each integer $m\notin \{\pm 2, \pm 4, \pm 5, \pm 8, \pm 9\}$ has this property and determine the numbers:

$\alpha (m) = \min\{|A| : \sum A^+ =[m]\}$.

$\beta (m) = \max\{|A| : \sum A^+ =[m]\}$.

$L (m) = \min \{\max A^+ : \sum A^+ = [m]\}$.

$U (m) =\max \{\max A^+ : \sum A^+ = [m]\}$.

We define α (-m), β (-m), L (-m) and U (-m) similar as above.

Example 1: If m=1, then A= {1} is 1 full subset of Z^+.

Example 2: If m=3, then A= {1, 2} is 3 full subset of Z^+, because (i) $a_1=1$ and $\sum A= [3]$, (ii) $a_2=2 \leq a_1+1=2$

Example 3: If m=-6 then A= {-1,-2,-3} is -6 full subset of Z^- , because (i) $a_1=-1$ and $\sum A= [-6]$, (ii) $a_2=-2\geq a_1-1=-1-1$ and $a_3=-3 \geq a_1+a_2-1=-4$.

Example 4: If A= {±1, ±2}, then A is double full subset of Z, because $A^-= \{-2,-1\}$ is -3 full subset of Z^- and $A^+= \{1, 2\}$ is 3 full subset of Z^+ , this means that A is double full by 3.

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2. THE RESULTS:

Definition 1: Let m be a positive integer. A subset $A=A^+ \cup A^-$ of Z is called double full by m if $\sum A^+ = [m]$ and $\sum A^- = [-m]$. A is called double full if it is double full by m for some positive integer m.

Theorem 1:.A subset $A=A^+\cup A^-$ of Z where $A^+=\{a_1,\dots,a_k\}$, $A^-=\{-a_k,..,a_1\}=\{b_k,\dots,b_1\}$ with $a_1<\dots<a_k$.

(i) A^+ is full if and only if $a_1=1$ and $a_i \le a_1+\dots+a_{i-1}+1$ for each i, $2\le i\le k$ and

(ii) A^- is full if and only if $b_1=-1$ and $b_i \ge b_1+\dots+b_{i-1}-1$ for each i, $2\le i\le k$.

Proof: Let $A=A^+ \cup A^-$ be double full and $\sum A^+ = [m]$, $\sum A^- = [-m]$ for a positive integer m.

(i) Its shown in [3].

(ii) Clearly $b_1=1$. If $b_j > b_1+\dots+b_{j-1}-1$ for some j, $2\le j\le k$, then $b_1+\dots+b_{j-1}-1$ is not of a sum of distinct elements of A^-.

But $-m = b_1+\dots+b_k \le b_1+\dots+b_{j-1}-1 \le -1$. This contradicts to the fact that $\sum A^- = [-m]$.

Conversely, suppose that $b_1=-1$ and $b_i \ge b_1+\dots+b_{i-1}-1$ for each i, $2\le i\le k$. We claim that $\sum A^- = [b_1+\dots+b_k]$.

We prove this by induction on k. For k=1 the result is obvious. Suppose that the result is true for k-1.
Then $\sum A^- \setminus \{b_k\} = [b_1+\dots+b_{k-1}]$.

Now suppose that $b_1+\dots+b_k \le L \le b_1+\dots+b_{k-1}-1$ and write $L=b_k+b$. If $b = 0$, then $L=b_k \in \sum A^-$ and if $b \ne 0$, then $b \in [b_1+\dots+b_{k-1}] = \sum A^- \setminus \{b_k\}$. Thus $L \in \sum A^-$.

Proposition 1(i): Let $n = p_1^{\alpha_1}\dots p_r^{\alpha_r}$, with $p_1<\dots<p_r$ primes, be a positive integer. Then $D(n) = \{d \in Z^+ : d|n\}$ is full if and only if $p_1=2$ and $p_i \le \sigma(p_1^{\alpha_1}\dots p_r^{\alpha_r})+1$ for each i, $2\le i\le r$, and

(ii) Let $n=-p_1\dots p_r$, with $p_1<\dots<p_r$ primes, be a positive integer. Then

$D(n) = \{d \in Z^- : d|n\}$ is full if and only if $p_1=2$ and $p_i \le \sigma(p_1^{\alpha_1}\dots p_r^{\alpha_r})+1$ for each i, $2\le i\le r$.

Proof (i): If D (n) is m full, then $m=\sigma(n)$. since $p_1^{\alpha_1}\dots p_r^{\alpha_r} | n$ and $p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}} \ne n$.

We have $\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}}) < \sigma(n)$.

Hence $\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ is a member of $[\sigma(n)]$. Thus if

$p_i > \sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ for some i , then the member $\sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ is a member of $[\sigma(n)]$ which is not a sum of distinct elements of D(n). On the other hand , if the condition $p_i < \sigma(p_1^{\alpha_1}\dots p_{i-1}^{\alpha_{i-1}})+1$ for each i , $2\le i\le r$, is satisfied, then using an argument similar to the one used in theorem 1 , we can inductively prove that each element of $[\sigma(n)]$ can be written as a sum of distinct elements of D(n).

(ii) This part proved just like as part (i).

Definition 2: Define F (m) is number of m full sets of Z^+ and F (-m) is the number of -m full sets of Z^-.

Theorem 2: Let m be a positive integer. There is a set A such that $\sum A^+ = [m]$ and $\sum A^- = [-m]$ if and only if
$m \ne \{2,4,5,8,9\}$.

Proof: Its shown in [3].

Example 5 D: (6) = {1, 2, 3, 6} is 12 full subset of Z^+, because (i) $P_1=2$ and (ii) $3\le \sigma(2)+1$.

Example 6 D: (-6) = {-6,-3,-2,-1} is -12 full subset of Z^- , because (i) $p_1=2$ and (ii) $3\le \sigma(2)+1$ or $-3\ge -\sigma(2)-1$.

Example 7: For m=12 and m=-12 note that $A^+=D(6)=\{1,2,3,6\}$ and $A^- = D(-6) = \{-6,-3,-2,-1\}$ are 12 full and -12 full subsets of Z, respectively . Therefore, $A=A^+ \cup A^-$ is double full by 12 set of Z.

Theorem 3 (i): If $\alpha(m) = \min\{|A| : \sum A^+ = [m]\}$, $\beta(m) = \max\{|A| : \sum A^+ = [m]\}$. Then

$\alpha(m) = [\log_2(m+1)]$,

$\beta(m) = \max\{l: \frac{l(l+1)}{2} \leq m\}$.

(ii) If $m \neq \{2,4,5,8,9,14\}$and $L(m) = \min\{\max A : \sum A^+ = [m]\}$.and $m = \frac{n(n+1)}{2} + r$,where r = 0,1,...,n, then

$$L(m) = \begin{cases} n & r = 0 \\ n+1 & 1 \leq r \leq n-2 \\ n+2 & 1 \leq r = n-1 \text{ or } n \end{cases}$$

(iii) If $n \geq 20$ and $U(m) = \max\{\max A: \text{then } \sum A^+ = [m]\}$, then $U(m) = [\frac{m}{2}]$.

Proof: Its shown in [2].

Theorem 4:.Let m be a positive integer and F(m, i) denote the number of m full sets A with max A=i, where $L(m) \leq i \leq U(m)$, then

$$F(m, i) = \sum_{j=L(m-i)}^{\min\{U(m-i), i-1\}} F(m-i, j)$$

Proof: Its shown in [3].

Example 8:.By definition for L (m) and U (m), in [1]. We have

m	1	3	6	7	10	11	12	13	14	15	16	17	18	19	20
L(m)	1	2	3	4	4	5	5	6	7	5	6	6	6	7	7
U(m)	1	2	3	4	4	5	6	7	7	8	6	7	8	9	10

Theorem 5 (i): Let me be a positive integer and denote the number of m full sets A by F (m). Then

$$F(m, i) = \sum_{j=L(m)}^{U(m)} F(m, i)$$

(ii) F (m) =F (-m)

Proof (i): Its shown in [3].

(ii) Its obvious by definition.

Corollary 1: By theorem 5, the number of double full by m sets of Z are $2\sum_{j=L(m)}^{U(m)} F(m, i)$.

Example 9: By definition of F (m), the first few values of F (m) are

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
F(m)	1	0	1	0	0	1	1	0	0	1	1	2	2	1	2	1	2	3	4	5

Example 10: For evaluate F(21) by using theorem 4 and theorem 5 , we have

$$F(21) = \sum_{j=L(21)}^{U(21)} F(21, i) = \sum_{i=6}^{12} F(21, i) = F(21,6) + F(21,7) + \cdots + F(21,12)$$
$$= F(15,5)+F(13,6)+F(13,7)+F(12,5)+F(12,6)+F(11,5)+F(10,4)=7.$$

This means that, there are the seven 21 full sets are

{1,2,3,4,5,6} , {1,2,4,6,8} , {1,2,3,7,8} , {1,2,4,5,9} , {1,2,3,6,9}, {1,2,3,5,10} , {1,2,3,4,11}

Example 11: For evaluate F (-6), by using theorem 4 and theorem 5 , we have

F (-6) =F (6), so by theorem 5, we have

$$F(6) = \sum_{j=L(6)}^{U(6)} F(6,i) = \sum_{i=3}^{3} F(6,i)$$

$$= \sum_{j=L(6-3)}^{\min\{U(6-3),3-1\}} F(6-3,j) = \sum_{j=2}^{2} F(3,j) = F(3,2) = \sum_{j=L(3-2)}^{\min\{U(1),1\}} F(1,j) = \sum_{1}^{1} F(1,1) = 1$$

This means that, there is a -6 full set such that define as follows; A= {-1,-2,-3}

Example 12: The number of double full by 6 sets of Z is one and define as follows; A= {±1, ±2, ±3}.

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