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ON THE CHANGE OF THE BASIS OF FINITE QUANTUM SYSTEMS

MUNA TUBANI[1]* AND AMNA GRESH[2]

[1,2]Department of Mathematics, University of Tripoli, Libya.

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ABSTRACT

The analytic functions of finite quantum systems are considered. The zeros of this functions and their evolution time are discussed. As the basis change the quantum state $|f\rangle$ transforms into different quantum state. A brief introduction to the Symplectic Transformationsis given. A map between two toruses where the domain and the range of this map are the analytic functions on toruses is defined.

Keywords: *Symplectic Transformations, change of the basis, unitary transformation, finit systems.*

1. INTRODUCTION

Analytic representations have been considered extensively [1, 2, 3] and used in various places in quantum mechanics. The analytic Bargmann function [4, 5, 6, 7, 8, 9, 10] is the most famous one. Ref [12] has considered analytic representations of finite quantum systems on a torus. The analytic function has exactly N zeros. Ref [13] has studied the motion of the zeros. In some cases, N of the zeros follow the same path and in other cases each zero follow its own path. A unitary transformation is equivalent to a change of basis. Symplectic Transformations play an important role in quantum optics, where they are related to the concept of squeezing, in superconductivity, in the theory of accelerated observers , etc. We try to discuss how to define a map from torus to another such that the domain of this mapis the zeros of an analytic function in the first torus and the range is the zeros of analytic function in the second torus. The open problem is how to define the map.

2. ZEROS OF ANALYTIC REPRESENTATION OF FINITE QUANTUM SYSTEMS

Let H be a Hilbert space with dimension N and let $|X_m\rangle, |P_m\rangle$ be the position states and momentum states respectively($m \in Z_N$) where:

$$|P_m\rangle = F|X_m\rangle = N^{-1/2}\sum_n\left(\exp\left[i\frac{2\pi m}{N}\right]\right)|X_m\rangle, \tag{1}$$

and F the Fourier operator:

$$F = N^{-1/2}\sum_{m,n}\left(\exp\left[i\frac{2\pi m}{N}\right]\right)|X_m\rangle\langle X_n|. \in \tag{2}$$

The position and momentum operators are defined as:

$$X=\sum_{n=0}^{N-1} n|X_n\rangle\langle X_n|; p = FXF^{\uparrow} = \sum_{n=0}^{N-1} n|P_n\rangle\langle P_n|, \tag{3}$$

respectively.

We study an arbitrary normalized state $|F\rangle$:

$$|F\rangle = \sum_m F_m|X_m\rangle; \sum_m |F_m|^2 = 1, \tag{4}$$

Corresponding Author: Muna Tubani*,
Department of Mathematics, University of Tripoli, Libya.

Following ref [12, 13] we introduce the analytic representations of finite quantum systems on a torus.We represent the state $|F\rangle$ of Eq.(4), with the analytic function f(z):

$$f(z)=\pi^{-1/4}\sum_{m=0}^{N-1}F_m\vartheta_3\left[\pi mN^{-1}-z\sqrt{\frac{\pi}{2N}};iN^{-1}\right] \tag{5}$$

which satisfy quasi-periodic condition:

$$\begin{aligned} &f\left[z+\sqrt{2\pi N}\right]=f(z)\\ &f\left[z+i\sqrt{2\pi N}\right]=f(z)\exp\left[\pi N-i(2\pi N)^{1/2}z\right] \end{aligned} \tag{6}$$

where ϑ_3 is Theta function and it is defined as:

$$\vartheta_3(u,\tau)=\sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2+i2nu). \tag{7}$$

The analytic function $f(z)$ is defined on a cell $[x_0,x_0+\sqrt{2\pi N})\times[x_1,x_1+\sqrt{2\pi N})$ (on a torus)

The sum of the zeros z_n of analytic function $f(z)$ is:

$$\sum_{n=1}^{N}Z_n=(2\pi)^{1/2}N^{3/2}(l+ir)+\left(\frac{\pi}{2}\right)^{1/2}N^{3/2}(1+i), \tag{8}$$

where l,r are integers.

Ref.[12, 13] has constructed the function $f(z)$ from its zeros Z_n as following:

Let Z_n be the zeros of the analytic function $f(z)$ and suppose that this zeros satisfy the relation(8) then the analytic function $f(z)$ is defined as:

$$\begin{aligned} &f(z)=q\exp\left[-i\left(\frac{2\pi}{N}\right)^{1/2}lz\right]\prod_{n=1}^{N}\vartheta_3\left[w_n(z);i\right]\\ &w_n(z)=\left(\frac{\pi}{2N}\right)^{1/2}(z-Z_n)+\frac{\pi(1+i)}{2} \end{aligned} \tag{9}$$

where l is the integer relationof Eq.(8) and q is a constant calculated from the normalization condition.

We consider the state $|F(0)\rangle=\sum F_m(0)|X;m\rangle$ at $t=0$. Using the Hamiltonian H, the state $|F(0)\rangle$ evolves at time t

$$|F(t)\rangle=\exp(itH)|F(0)\rangle=\sum_{m=0}^{N-1}F_m(t)|X_m\rangle \tag{10}$$

Numerically, we calculate the zeros z_n of $f(z)$.

Ref.[13] has discussed the Periodic finite quantum systems. In some cases d of the zeros follow the same path. We say that this path has multiplicity d.

3. THE CHANGE OF THE BASIS

The unitary transformation is one-to-one function between two Helbert spaces. Let A be a Hermitean matrix, and let U be a unitary transformation. It is will known that the matrix $UAU^{\dagger}$ is Hermation and has the same of eigenvalues of A. A unitary transformation is equivalent to a change of basis. It is a transformation that transforms one basis into another. As a unitary transformation we consider the Symplectic transformations U.

3.1 Symplectic transformations

3.1.1 Symplectic transformations in the harmonic oscillator phase-space

An important class of transformations in the harmonic oscillator case is the Bogoliubov Transformations:

$$X' = kX + \lambda P \qquad P' = \mu X + \nu P \qquad \kappa\nu - \lambda\eta = 1$$

which preserve the commutation relations:

$$[X', P'] = [X, P] = 1i$$

They are associated with the symplectic group which has three X^2, P^2, XP.

Following ref.[11] we introduce the Symplectic transformations U in the $Z_N \times Z_N$ phase-space of a finite quantum system. We consider the unitary transformations:

$$X' = UXU^{\dagger} = X^{\kappa} Z^{\lambda} \psi(2^{-1}\kappa\lambda),$$

$$Z' = UZU^{\dagger} = X^{\mu} Z^{\nu} \psi(2^{-1}\mu\nu), \tag{11}$$

Here:

$$X = \exp\left[\frac{-i2\pi p}{N}\right], Z = \exp\left[\frac{-i2\pi x}{N}\right], \psi(a) = \exp\left[i\frac{-i2\pi a}{N}\right] \tag{12}$$

where x, p are the position and momentum operators and $\lambda, \kappa, \mu, \nu$ are integers in Z_N obey the relation:

$$\kappa\nu - \lambda\mu = 1(mod(N)). \tag{13}$$

When N is a power of a prime p then $Z(p^n)$is a Galois field.

We call Galois quantum systems those with a dimension that is a power of a prime.

$GF(p^n) \times GF(p^n)$ is a finite geometry. This is a geometrical structure with strong mathematical properties. For example, transformations like dilations, contractions, discrete rotations, etc. are well defined and form groups.

In the case of non-Galois quantum systems (with a dimension that is not a power of a prime), the phase-space is a set of points with no geometrical structure. Consequently, the harmonic oscillator phase-space formalism, which is a set of very powerful techniques, can be transferred to other quantum systems, provided that the corresponding phase-space has some geometrical structure. In Galois quantum systems the phase-space has a geometrical structure.

By reference to ref.[11] we construct the unitary operator U. We consider a three-dimensional Hilbert space $(N = 3)$ and $U(1,-1-1)$, which leads (by definition in Eq.(11)) to the transformations:

$$X' = UXU^{\dagger} = XZ^{-1}\omega\left(-\frac{1}{2}\right),$$

$$Z' = UZU^{\dagger} = X^{-1}Z^{2}\omega(-1). \tag{14}$$

The operator U is given in a matrix $U(i,j)$ and the matrix elements $U(i,j)$ are given in table 1.

Table-1: The coefficients $U(i,j)$ for the transformations of Eq.(14).

	$i = 0$	$i = 1$	$i = 2$
$j = 0$	0.5774	0.2887 + 0.5i	0.5774
$j = 1$	-0.2887 + 0.5i	0.2887 - 0.5i	0.5774
$j = 2$	0.5774	0.2887 - 0.5i	-0.2887 + 0.5i

The transformation with operatore U on the analytic function $f(z)$:

$$f(z) = \pi^{-1/4} \sum_{n=0}^{N-1} F_m \vartheta_3 \left[\frac{\pi m}{N} - z\sqrt{\frac{\pi}{2N}}; \frac{i}{N}\right] \tag{15}$$

can be expressed as:

$$Uf(z) \to \pi^{-1/4} \sum_{n=0}^{N-1} U_{m1} F_1 \vartheta_3 \left[\frac{\pi m}{N} - z\sqrt{\frac{\pi}{2N}}; \frac{i}{N}\right] \tag{16}$$

We denote as z_n the zeros of function $f(z)$ in Eq.(15) and we denote as η_n the zeros of function:

$$g(z)=\pi^{-1/4}\sum_{n=0}^{N-1}U_{m1}F_1\vartheta_3\left[\frac{\pi m}{N}-z\sqrt{\frac{\pi}{2N}};\frac{i}{N}\right] \tag{17}$$

The paths of the zeros define completely a finite quantum system. Hence the study of paths of the zeros is equivalent to the study of the system. We consider the paths of the zeros of both functions $f(z)$ and $g(z)$.

Let $Z_0(t),Z_1(t),Z_2(t)$ be the paths of the three zeros of $f(z)$, and let $\eta_0(t),\eta_1(t),\eta_2(t)$ be the paths of the three zeros of $g(z)$. We consider the Hamiltonian:

$$H=\begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{18}$$

which has the eigenvalues $0,2,2$ with period $O=\pi.$ We calculate the Hamiltonian $UHU^{\dagger}$ which has the same eigenvalues of $H.$ We assume that the initial values the zeros of $f(z)$ are:

$$Z_0(0)=2+2i,Z_1(0)=2.2+2i,Z_2(0)=2.3+2.3i \tag{19}$$

The initial values the zeros of $g(z)$ are:

$$Z_0(0)=1+1i,Z_1(0)=2+3.3i,Z_2(0)=3.4+1i \tag{20}$$

In the case of Eq.(19) we get:

$$Z_0(O+t)=Z_1(t),Z_1(O+t)=Z_2(t),Z_2(O+t)=Z_0(t) \tag{21}$$

After period the three zeros follow the same path. In Fig.2 we present the paths of these zeros.

In the case of Eq.(20), after period we found numerically that:

$$Z_0(O+t)=Z_2(t),Z_2(O+t)=Z_0(t). \tag{22}$$

Therefore two of the zeros follow the same path and the third one follows a different path. In Fig.1. we present the paths of these zeros.

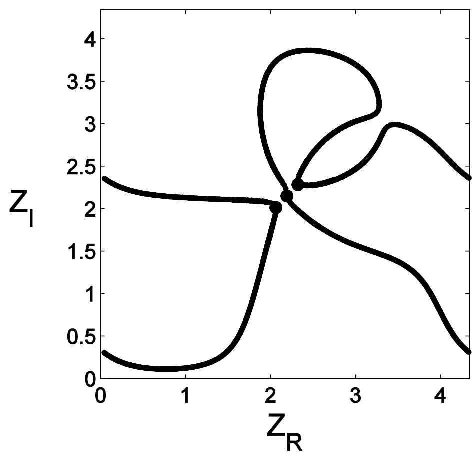

Figure-1: The path of the zeros z0(t), z _1(t), z _2(t) for the system of Eq.(19)with the Hamiltonian of Eq.(18).

Another example is the Hamiltonian of Eq.(18) and zeros with the initial values:

$$Z_0(0)=1.4+3.4i,Z_1(0)=1.7+2.5i,Z_2(0)=3.4+0.6i \tag{23}$$

and the initial values of zeros of $g(z)$ are:

$$Z_0(0)=0.8+3.9i,Z_1(0)=2+0.36i,Z_2(0)=3.7+2.3i \tag{24}$$

The period is $O = \pi$.

In the case of Eq.(23) after period the zeros obey the relation:

$$Z_0(O+t) = Z_2(t), Z_2(O+t) = Z_1(t), Z_1(O+t) = Z_0(t) \quad (25)$$

Here the three zeros follow the same path. In Fig.2. we present the paths of these zeros.

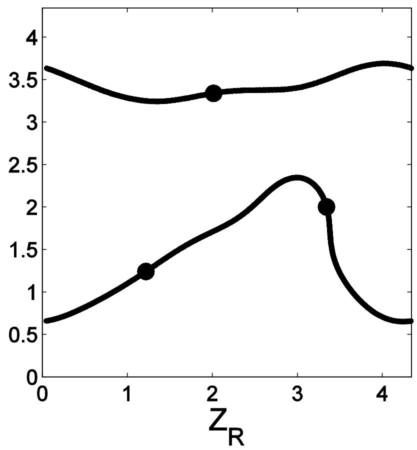

Figure-2: The path of the zeros of the function g(z) where the initial values of these zeros aregiven in Eq.(20) with the Hamiltonian $UHU^{\dagger}$ where H of Eq.(18).

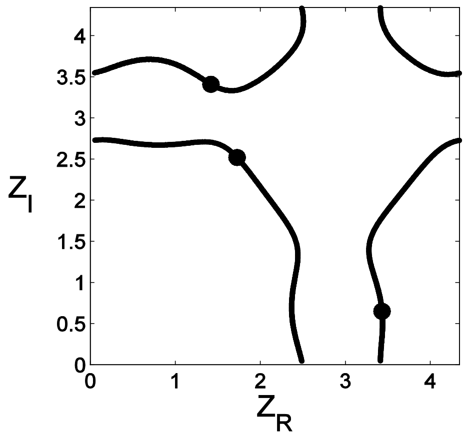

Figure-3: The path of the zeros z_0(t), z _1(t), z _2(t) for the system of Eq.(23) with the Hamiltonian of Eq.(18).

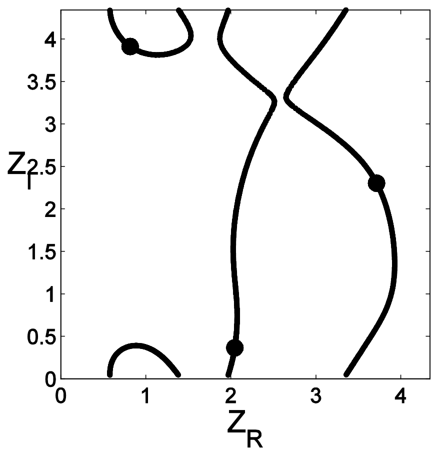

Figure-4: The path of the zeros of the function g(z) where the initial values of these zeros are given in Eq.(24)) with $UHU^{\dagger}$ where H of Eq.(18).

In the case of Eq.(24) we found numerically that each zero follows a different path.

In Fig.4. we present the paths of these zeros.

A unitary transformation is equivalent to a change of basis. As the basis changes, the quantum state $|f\rangle$ transforms into different quantum state.

Let U be an arbitrary unitary transformation. We can define a map from torus T_1 into another torus T_2

$$G : T_1 \to T_2 \tag{26}$$

as following:

$$G(f(z)) = g(z) = \pi^{-1/4} \sum_{n=0}^{N-1} U_{m1} F_1 \vartheta_3 \left[\frac{\pi m}{N} - z\sqrt{\frac{\pi}{2N}}; \frac{i}{N} \right] \tag{27}$$

where $f(z)$ is the analytic function in Eq.(15). It is seen that this map is one-to-one and on to.

Let us try define another map from torus T_1 into T_2

$$W : T_1 \to T_2 \tag{28}$$

such that:

$$W(Z_n) = \eta_n \tag{29}$$

Where Z_n are the zeros of $f(z)$ and η_n are the zeros of $g(z)$. The domain of this map is the zeros of function $f(z)$ and the range is the zeros of function $g(z)$. This map is not one-to-one, it is enough to give the following example to show that. Let $U = U(1,-1,-1)$ in Eq.(14). We assume that the initial zeros of $f(z)$ are:

$$Z_0(0) = Z_1(0) = Z_2(0) = 2.1708 + 2.1708i. \tag{30}$$

In this case the three zeros are identical, we can say that they are one zero.

The initial values of the zeros of $g(z)$ are:

$$Z_0(0) = 1 + 1i, Z_1(0) = 2 + 3.34i, Z_2(0) = 3.34 + 2i \tag{31}$$

In Fig.5 we present the zeros of $f(z)$ (circles), and the zeros of $g(z)$ (triangles). Let Z_n be the zeros of the analytic function $f(z)$ in Eq.(15) and η_n be the zeros of the analytic function $g(z)$ in Eq.(17).
Let

$$W : T_1 \to T_2 \tag{32}$$

be a map from torus T_1 into another T_2 such that:

$$W(Z_n) = \eta_n. \tag{33}$$

What is the definition of that map?

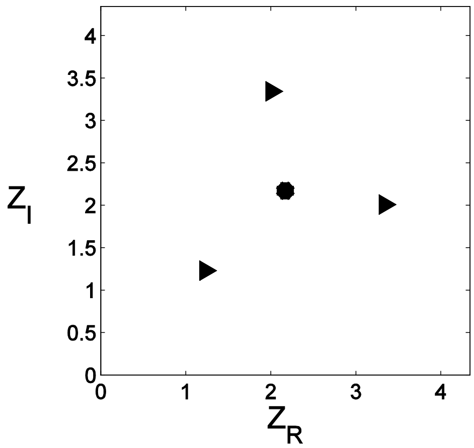

Figure-5: The zeros of the function f(z)) (circle)where initial values of these zeros are given in Eq.(30). The zeros of the function g(z)) (circle) where initial values of these zeros are given in Eq.(30))

4 CONCLUSION

We briefly discussed the analytic representation of finite quantum systems. We reviewed briefly the zeros of analytic theta function and their time evolution. Ref [11, 12] has studied analytic representations of finite quantum systems on a torus. The analytic function representing a quantum state has exactly N zeros which define uniquely the quantum state. Ref [13] has studied the motion of the N zeros on the torus.In some cases d of the zeros follow the same path and in other cases each zero follow its own path.

A unitary transformation is equivalent to a change of basis.As an example on the unitary transformation, we introduced the Symplectic transformations. We try to discuss how to define a map from torus to another such that the domain and the range is the zeros of analytic functions. The open problem is how to construct the map. We gave several examples to demonstrate these ideas.

REFERENCES

1. A.M. Perelomov, 'Generalized coherent states and their applications' (Springer, Berlin, 1986)
2. A. Vourdas, J. Phys. A39, R65 (2006)
3. A. Vourdas, R.F. Bishop, Phys. Rev. A50, 3331 (1994)
4. V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961)
5. V. Bargmann, Commun. Pure Appl. Math. 20, 1 (1967)
6. S. Schweber, J. Math. Phys. 3, 861 (1962)
7. S. Schweber, Ann. Phys.(NY) 41, 205 (1967)
8. J. Kurchan, P. Leboeuf, M. Saraceno, Phys. Rev. A40, 6800 (1989)
9. A. Voros, Phys. Rev. A40, 6814 (1989)
10. J.H. Hannay, J. Phys. A 29, L101 (1996)
11. J.H. Hannay, J. Phys. A 31, L755 (1998)
12. N.L. Balazs, A. Voros, Phys. Rep. C143, 109 (1986)
13. P. Leboeuf, A. Voros, J. Phys. A23, 1765 (1990)
14. P. Leboeuf, J. Phys. A24, 4575 (1991)
15. M.B. Cibils, Y. Cuche, P. Leboeuf, W.F. Wreszinski, Phys. Rev A46, 4560 (1992)
16. J.M. Tualle, A. Voros, Chaos, Solitons and Fractals, 5, 1085 (1995)
17. S. Nonnenmacher, A. Voros, J. Phys. A30, L677 (1997)
18. H.J. Korsch, C. Múller, H. Wiescher, J. Phys. A30, L677 (1997)
19. F. Toscano, A.M.O. de Almeida, J. Phys. A32, 6321 (1999)
20. D. Biswas, S. Sinha, Phys. Rev. E60, 408 (1999)
21. A. Vourdas, Rep. Prog. Phys. 67, 267 (2004)
22. S. Zhang, A. Vourdas, J. Phys. A37, 8349 (2004)
23. S. Zhang, A. Vourdas, J. Phys. A38, 1197 (2005) (corrigendum)
24. M. Tabuni, A. Vourdas and S. Zhang, 'Zeros in analytic representations of fnite quantum systems on a torus'(Physica Scripta, 2010)

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