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COMMON FIXED POINT THEOREM FOR SEMI - COMPATIBLE AND SUB-SEQUENTIALLY CONTINUOUS MAPPINGS IN FUZZY METRIC SPACE

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ABSTRACT

The purpose of this paper is to prove some common fixed point theorems in semi-compatible and sub-sequentially continuous mappings in fuzzy metric space.

Keywords: *Compatible, Semi- compatible, Sontinuous maps, Sub-sequentially continuous maps.*

INTRODUCTION

The concept of fuzzy sets was initially investigated by Zadeh [16] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined fuzzy metric space in various ways. In this chapter we deal with the fuzzy metric space defined by Kramosil and Michalek [10] and modified by George and Veeramani [4]. Recently, Grebiec [5] has proved fixed points result for Fuzzy metric space. In the sequel, Singh and Chouhan [14] introduced the compatible mappings of Fuzzy metric space and proved the common fixed point theorem.

Using the concept of R-weak commutativity of mappings, Vasuki [15] proved the fixed point theorem for Fuzzy metric space. Recently in 2009, using the concept of sub compatible maps, Bouhadjera *et.al* [1] proved common fixed point theorems. Using the concept of compatible maps of type (A), Jain *et.al.* [8] proved a fixed point theorem for six self maps in a fuzzy metric space. Using the concept compatible maps of type (β) Jain *et.al* [8] proved a fixed point theorem in fuzzy metric space and menger space using the concept of semi-compatibility, weak compatibility and compatibility of type (β) respectively.In this section, we introduced the new concept of semi- compatible and sub-sequentially continuous mapping in fuzzy metric space.

For the sake of completeness, we recall some definition and known results in fuzzy metric space.

Definition 2.1: [12] A triangular norm $*$ (shortly t- norm) is a binary operation on the unit interval [0, 1] to [0, 1] such that for all a, b, c, d $\in$ [0, 1] the following conditions are satisfied.

(a) $a * 1 = a$,
(b) $a * b = b * a$,
(c) $a * b \le c * d$ whenever $a \le c$ and $b \le d$,
(d) $(a * b) * c = a * (b * c)$.

Example of t-norms are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2: [10] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions for all x, y, z $\in$ X, t, s > 0,

(FM-1) $M(x, y, t) = 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$
(FM-5) $M(x, y, \cdot): [0, \infty) \to [0, 1]$ is continuous.
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

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In the definition of George and Veeramani [4],M is a fuzzy set on $X^2 \times [0, \infty)$ and (FM-1), (FM-2), (FM-3) are replaced respectively, with (GV-1), (GV-2), (GV-3) below the axiom (GV-2) is reformulated as in [7, Remark 1]:

(GV-1) $M(x, y, t) > 0$, for all $t > 0$;

(GV-2) $M(x, x, t) = 1$ for all $t > 0$ and $x \neq y \Rightarrow M(x, y, t) < 1$. for all $t > 0$;

(GV-5) $M(x, y, \cdot) = (0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y \in X$.

Then M is called a fuzzy metric on X. Then M(x, y, t) denotes the degree of nearness between x and y with respect to t. In the following example, it is showed that every metric induces a fuzzy metric:

Example 2.1: [4] Let (X, d) be a metric space. Define $a*b = ab$ (or $a*b = \min\{a, b\}$) for all $a, b \in X$ and let M be fuzzy set on $X^2 \times [0, \infty)$ defined as follows: $M(x, y, t) = \frac{t}{t+d(x,y)}$

Then (X, M, $*$) is a fuzzy metric space and the fuzzy metric M induced by the metric d is the standard fuzzy metric.

Example 2.2: [3] Let X = R, the set of all real numbers. Define $a*b = ab$ and

$M(x, y, t) = \left[exp\left(\frac{x-y}{t}\right)\right]^{-1}$ for all x, y in R and $t \in [0, \infty)$. then (X, M, *) is a fuzzy metric space.

Lemma 2.1: [5] For all $x, y \in X$, $(X, M, \cdot)$ is non decreasing function.

Definition 2.3: [4] Let (X, M, *) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be Convergent to a point $x \,\varepsilon\!\in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and for all p.

The space is said to be Complete if every Cauchy sequence in it converges to a point on it.

Proposition 2.1: [4] In a fuzzy metric space (X, M, *) the limit of a sequence is unique.

Definition 2.4: [9] A pair (A, S) of self mappings defined on a fuzzy metric space (X, M, $*$) is said to be compatible if and only if $M(ASx_n, SAx_n, t) \rightarrow 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Definition 2.5: [13] A pair (A, S) of self mappings defined on a fuzzy metric space (X, M, $*$) is said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence $\{x_n\} \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some $x \in X$.

Definition 2.6: [6] A pair (A, S) of self mappings defined on a fuzzy metric space (X, M, $*$) is said to be Sub-compatible if and only if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all $t > 0$.

Definition 2.7: [2] A pair (A,S) of self mappings defined on a fuzzy metric space (X, M, $*$) is said to be reciprocally continuous if for a sequence $\{x_n\}$ in X,

$\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$, whenever

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.8: [11] Two self maps A and B of a fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, i.e. $Ax = Bx \Rightarrow ABx = BAx$.

In this section, we define the concept of sub-sequentially continuous and semi-compatible mapping and we introduced the common fixed point of pairs, coincident point of pair and unique common fixed point.

Definition 2.9: [6] A pair (A, S) of self mappings defined on a fuzzy metric space (X, M, $*$) is said to be sub-sequentially continuous if and only if there exists sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$

Theorem 2.1: Let A, B, S, T, P and Q be self mappings of fuzzy metric space (X, M, *) with $a*b = \min\{a, b\}$ for all $a, b \,\varepsilon\, [0, 1]$. If the pairs (A, PQ) and (B, ST) are semi-compatible and sub-sequentially continuous mappings, then

(a) The pair (A, PQ) has a coincidence point.

(b) The pair (B, ST) has a coincidence point.

(c) Further, the mapping A, B, S, T, P and Q have a unique common fixed point in X provided the involved maps satisfy the inequality

$$M^2(Ax, By, t)*[M(PQx, Ax, t).M(STy, By, t)] \geq [pM(PQx, Ax, t) + qM(PQx, STy, t)]M(PQx, By, t) \qquad (1.1)$$

for all $x, y \in X$ and $t > 0$, where $0 < p, q < 1$ and $p+q=1$.

Proof: The pairs (A, PQ) and (B,ST) are Semi-compatible and Sub-sequentially continuous mappings, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} PQx_n = z$ for some $z \in X$

and $\lim_{n\to\infty} M(A(PQ)x_n, (PQ)Ax_n, t) = 1$, for all $t < 0$.

and $\lim_{n\to\infty} M(Az, PQz, t) = 1$ (1.2)

then we have Az=PQz,

Similarly, $\lim_{n\to\infty} By_n = \lim_{n\to\infty} STy_n = w \in X$

$\lim_{n\to\infty} M(B(ST)y_n, (ST)By_n, t) = 1$, for all $t < 0$

and $\lim_{n\to\infty} M(Bw, STw, t) = 1$ (1.3)

Hence z is coincidence point of A, PQ and w is coincidence point of B, ST

then we get Az=PQz. (1.4)

Bw=STw. (1.5)

Step-1:- First we prove that z = w. Putting $x = x_n$, $y = y_n$ in inequality (1.1) we have

$M^2(Ax_n, By_n, t)*[M(PQx_n, Ax_n, t).M(STy_n, By_n, t)] \geq [pM(PQx_n, Ax, t) + qM(PQx_n, STy_n, t)]M(PQx_n, By_n, t)$ (1.6)

Now

$M^2(z, w, t)*[M(z, z, t).M(w, w, t) \geq pM[(z, z, t) + qM(z, w, t)]M(z, w, t)$

$M^2(z, w, t) \geq [p + qM(z, w, t)]M(z, w, t)$

$M(z, w, t) \geq \frac{p}{1-q}$

$M(z, w, t) = 1$ (1.7)

Thus we have z = w

Step-2:-Now we prove that Az = z, Putting x = z and $y = y_n$ in (1.1)

$M^2(Az, By_n, t)*[M(PQz, Az, t).M(STy_n, By_n, t)] \geq [pM(PQz, Az, t) + qM(PQz, STy_n, t)]M(PQz, By_n, t)$

$M^2(Az, w, t)*[M(PQz, Az, t).M(w, w, t)] \geq [pM(PQz, Az, t) + qM(PQz, w, t)]M(PQz, w, t)$

$M^2(Az, w, t)*[M(Az, Az, t).M(w, w, t)] \geq [pM(Az, Az, t) + qM(Az, w, t)]M(Az, w, t)$

$M^2(Az, w, t) \geq [p + qM(Az, w, t)]M(Az, w, t)$

$M(Az, w, t) \geq \frac{p}{1-q}$

$M(Az, w, t) = 1$

Hence Az = w = z

Step-3:-Again we prove that Bz = z

Then we have $x = x_n$, y = z in (1.1)

$M^2(Ax_n, Bz, t)*[M(PQx_n, Ax_n, t)M(STz, Bz, t)] \geq [pM(PQx_n, Ax_n, t) + qM(PQx_n, STz, t)] M(PQx_n, Bz, t)$

$M^2(z, Bz, t)*[M(Az, Az, t).M(z, z, t)] \geq [pM(Az, Az, t) + qM(z, Bz, t)]M(z, Bz, t)$

$M^2(z, Bz, t) \geq [p + qM(z, Bz, t)] M(z, Bz, t)$

$M(z, Bz, t) \geq \frac{p}{1-q}$

$M(z, Bz, t) = 1$

we get z = Bz (1.8)

Step-4:-Again we claim that Tz = z,

putting x = Tz and y = z in (1.1)

$M^2(ATz, Bz, t)*[M(PQTz, ATz, t).M(STz, Bz, t) \geq [pM(PQTz, ATz, t) + qM(PQTz, STz, t)] M(PQTz, Bz, t)$

$M^2(TAz, z, t)*[M(TPQz, TAz, t).M(Bz, Bz, t)] \geq [pM(TPQz, ATz, t) + qM(TPQz, Bz, t)] M(TPQz, Bz, t)$

$M^2(Tz, z, t)*[M(Tz, Tz, t).M(z, z, t)] \geq [pM(Tz, Tz, t) + qM(Tz, z, t)] M(Tz, z, t)$

$M^2(Tz, z, t) \geq [p + qM(Tz, z, t)] M(Tz, z, t)$

$M(Tz, z, t) \geq \frac{p}{1-q}$

$M(Tz, z, t) = 1$

we get Tz = z (1.9)

Step-5:- Again we show that Sz = z,

putting x = Sz and y = z in (1.1)

$M^2(ASz, Bz, t)*[M(PQSz, ASz, t).M(STz, Bz, t)] \geq [pM(PQSz, ASz, t) + qM(PQSz, STz, t)] M(PQSz, Bz, t)$

$M^2(SAz, z, t)*[M(SPQz, SAz, t).M(Bz, Bz, t)] \geq [pM(SPQz, ASz, t) + qM(SPQz, Bz, t)] M(SPQz, Bz, t)$

$M^2(Sz, z, t)*[M(Sz, Sz, t).M(z, z, t)] \geq [pM(Sz, Sz, t) + qM(Sz, z, t)] M(Sz, z, t)$

$M^2(Sz, z, t) \geq [p + qM(Sz, z, t)] M(Sz, z, t)$

$M(Sz, z, t) \geq \frac{p}{1-q}$

$M(Sz, z, t) = 1$

we get Sz = z (1.10)

Step-6:- Again we prove that Qz=z,

putting x=Qz and y=z in (1.1)

$M^2(AQz, Bz,t)_*[M(PQQz, AQz, t).M(SQz, Bz, t)] \geq [pM(PQQz, AQz,t) + qM(PQQz, SQz, t)]\ M(PQQz, Bz, t)$

$M^2(QAz, z, t)_*[M(QPQz,TAz, t).M(Bz, Bz, t)] \geq [pM(QPQz, ATz,t) + qM(QPQz, Bz, t)]\ M(QPQz, Bz, t)$

$M^2(Qz, z, t)_*[M(Qz, Qz, t).M(z, z, t)] \geq pM(Qz, Qz, t) + qM(Qz, z, t)]\ M(Qz, z, t)$

$M^2(Qz, z, t) \geq [p+qM(Qz, z, t)]\ M(Qz, z, t)$

$M(Qz, z, t) \geq \frac{p}{1-q}$

$M(Qz, z, t) =1$

we get Qz = z (1.11)

Step-7:- Again we prove that Pz = z, putting x = Pz and y = z in (1.1)

$M^2(APz, Bz, t)_*[M(PQPz, ATz, t).M(SPz, Bz, t)]$

$[pM(PQPz, APz, t) + qM(PQPz, SPz, t)]\ M(PQPz, Bz, t)$

$M^2(PAz, z, t)_*[M(PPQz, PAz, t).M(Bz, Bz, t)] \geq [pM(PPQz, APz, t) + qM(PPQz, Bz, t)]\ M(PPQz, Bz, t)$

$M^2(Pz, z, t)_*[M(Pz, Pz, t).M(z, z, t)] \geq [pM(Pz, Pz, t) + qM(Pz, z, t)]\ M(Pz, z, t)$

$M^2(Pz, z, t) \geq [p+qM(Pz, z, t)]\ M(Pz, z, t)$

$M(Pz, z, t) \geq \frac{p}{1-q}$

$M(Pz, z, t) =1$

we get Pz = z (1.12)

i.e. Az = Bz = Tz = Sz = Pz = Qz = z

Hence z is a common fixed point of A, B, S, T, P, and Q.

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