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TRADITIONAL OPERATORS FOR EXPLODED NUMBERS
In memory of Brahmagupta (c.598 – c.668)

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ABSTRACT

H*aving the ordered set* $\left(\overset{\frown}{\mathbb{R}}, \leq, \mathcal{A}, \mathcal{M}\right)$ *of exploded numbers we extend the traditional addition, multiplication subtraction and division for exploded numbers and investigate the surprising properties of algebraic structure* $\left(\overset{\frown}{\mathbb{R}}, \leq, +, \cdot\right)$*. We solve the general first-order equation without using the associativity of addition and multiplication and distributivity.*

INTRODUCTION

The geometric explosion of real numbers is detailed in [1]. Here we repeat the most important facts. Denoting the ordered field of real numbers by $(\mathbb{R}, \leq, +, \cdot)$ for any real number x the pair

(1.1) $$\overset{\frown}{x} =^{def} \left((sgn\,x)\frac{\{|x|\}}{1-\{|x|\}}, (sgn\,x)[|x|]\right), \quad x \in \mathbb{R}.$$

is called exploded of x and the set of exploded numbers is denoted by $\overset{\frown}{\mathbb{R}}$.

The equality for exploded numbers means that

$$\overset{\frown}{x} =^{in\ \overset{\frown}{\mathbb{R}}} \overset{\frown}{y} \text{ if and only if } (sgn\,x)\frac{\{|x|\}}{1-\{|x|\}} = (sgn\,y)\frac{\{|y|\}}{1-\{|y|\}} \text{ and } (sgn\,x)[|x|] = (sgn\,y)[|y|].$$

If $\overset{\frown}{x}$ and $\overset{\frown}{y}$ are real numbers then $\overset{\frown}{x} =^{in\ \overset{\frown}{\mathbb{R}}} \overset{\frown}{y} \Leftrightarrow \overset{\frown}{x} = \overset{\frown}{y}$. So, instead of „$=^{in\ \overset{\frown}{\mathbb{R}}}$" we can write the familiar „=". It is proved that $\overset{\frown}{x} = \overset{\frown}{y}$ if and only if $x = y$. (See [1], Theorem 1.2.)

By definition (1.1) the exploded of x is an element of two – dymensional space, that is $\overset{\frown}{x} \in \mathbb{R}^2$, such that its second coordinate is an integer and product of the coordinates is non negative. On the other hand, if a point $u = (x, y) \in \mathbb{R}^2$ has these properties then

(1.2) $$\overgroup{y + \frac{x}{1+|x|}} = u.$$

is fulfilled. (See [1], Theorem 1.7.) So, we obtain

$$\overset{\frown}{\mathbb{R}} = \left\{u = (x, y) \in \mathbb{R}^2 \,\middle|\, \begin{Bmatrix} y \in \mathbb{Z} \\ x \cdot y \geq 0 \end{Bmatrix}\right\}. \quad (\mathbb{Z} \text{ is the set of integer numbers.})$$

Considering a real number x as pair $x = (x, 0)$, $x \in \overset{\frown}{\mathbb{R}}$ is obvious. Moreover, it is easy to see, that if $x \in]-1,1[$then $\overset{\frown}{x} = \left(\frac{x}{1-|x|}, 0\right)$ and $\frac{x}{1-|x|}$ overruns the set of real numbers. Set $\overset{\frown}{\mathbb{R}}$ is seen the Fig. 1.3.

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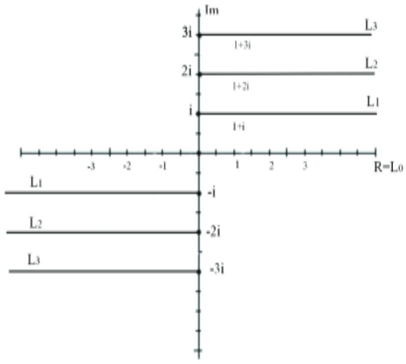

Fig.-1.3

With respect to (1.2) for the exploded number $u=(x,y)\in\overbrace{\mathbb{R}}$ we introduce the geometric compressor formula

(1.4) $$\underbrace{u}=y+\frac{x}{1+|x|}\qquad ,u\in\overbrace{\mathbb{R}},$$

Of course, the real number $\underbrace{u}$ is unambiguously determined. Moreover, $\underbrace{u}$ is called the compressed of exploded number u and we have the first inversion formula

(1.5) $$\overbrace{\left(\underbrace{u}\right)}=u\qquad ,\quad u\in\overbrace{\mathbb{R}}.$$

Denoting $x=\underbrace{u}$ by (1.5) we have $\overbrace{x}=u$. So, the second inversion formula

(1.6) $$\underbrace{\left(\overbrace{x}\right)}=x\qquad x\in\mathbb{R}$$

is obtained.

Using (1.4) and (1.1.) we define the super – addition and super multiplication

(1.7) $$u\,\mathcal{A}\,v=\overbrace{\underbrace{u}+\underbrace{v}}\qquad ,u,v\in\overbrace{\mathbb{R}}$$

and

(1.8) $$u\,\mathcal{M}\,v=\overbrace{\underbrace{u}\cdot\underbrace{v}}\qquad ,u,v\in\overbrace{\mathbb{R}},$$

respectively.

Writing $\underbrace{u}=x\,,u=\overbrace{x}$ and $\underbrace{v}=y\,,v=\overbrace{y}$ with (1.7) and (1.8), the mutually unambiguous mapping $x\leftrightarrow\overbrace{x}\,,x\in\mathbb{R}$ is izomorphism between the algebraic structures $(\mathbb{R},+,\cdot)$ and $\left(\overbrace{\mathbb{R}},\mathcal{A},\mathcal{M}\right)$. As $(\mathbb{R},+,\cdot)$ is a field, $\left(\overbrace{\mathbb{R}},\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{M}}\right)$ **is a field**, too. Important properties: If $x,y\ and\ z\in\mathbb{R}$ then

Associative laws: $\left(\overbrace{x}\ \mathcal{A}\ \overbrace{y}\right)\mathcal{A}\ \overbrace{z}=\overbrace{x}\ \mathcal{A}\left(\overbrace{y}\,\mathcal{A}\ \overbrace{z}\right)$ and $\left(\overbrace{x}\ \mathcal{M}\ \overbrace{y}\right)\mathcal{M}\ \overbrace{z}=\overbrace{x}\ \mathcal{M}\left(\overbrace{y}\,\mathcal{M}\ \overbrace{z}\right)$

Commutative laws: $\overbrace{x}\ \mathcal{A}\ \overbrace{y}=\overbrace{y}\ \mathcal{A}\ \overbrace{x}$ and $\overbrace{x}\ \mathcal{M}\ \overbrace{y}=\overbrace{y}\ \mathcal{M}\ \overbrace{x}$

Distributive law: $\left(\overbrace{x}\ \mathcal{A}\overbrace{y}\right)\mathcal{M}\ \overbrace{z}=\left(\overbrace{x}\ \mathcal{M}\ \overbrace{z}\right)\mathcal{A}\left(\overbrace{y}\,\mathcal{M}\ \overbrace{z}\right)$.

Unit element is for the super- addition is 0, that is, $\overbrace{x}\,\mathcal{A}\ 0=\overbrace{x}$, uniqueness can be proved.

Super - additive inverse element: $\overbrace{x}\,\mathcal{A}\ \overbrace{(-x)}=0$, uniqueness can be proved.

Unit element is for the super – multiplication is $\overbrace{1}$, that is, $\overbrace{x}\,\mathcal{M}\ \overbrace{1}=\overbrace{x}$, uniqueness can be proved Moreover, $\overbrace{1}=(0,1)$. (See (1.1).)

Super - multiplicative inverse element: If $x\neq 0\ then\ \overbrace{x}\,\mathcal{M}\overbrace{\left(\frac{1}{x}\right)}=\overbrace{1}$, uniqueness can be proved.Moreover, $\overbrace{\left(\frac{1}{x}\right)}=\left((sgn\,x)\left(\frac{\left\{\frac{1}{|x|}\right\}}{1-\left\{\frac{1}{|x|}\right\}}\right),(sgn\,x)\left[\frac{1}{|x|}\right]\right)$.

***Definition* 1.9:** (Ordering of exploded numbers.)

For any pair $x\,,y\ \in\mathbb{R}$ we say, that $\overbrace{x}<^{in\ \overbrace{\mathbb{R}}}\overbrace{y}$ if

$$(sgnx)\cdot[|x|]<(sgnx)\cdot[|y|]$$

or

if $(sgnx)\cdot[|x|]=(sgnx)\cdot[|y|]$ then $(sgn\,x)\frac{\{|x|\}}{1-\{|x|\}}<(sgn\,y)\frac{\{|y|\}}{1-\{|y|\}}$. ■

If $\overbrace{x}$ and $\overbrace{y}$ are real numbers then $\overbrace{x} <^{in\, \overbrace{\mathbb{R}}} \overbrace{y} \Leftrightarrow \overbrace{x} < \overbrace{y}$. (See [1], Lemma 1.16.) So, instead of „$<^{in\, \overbrace{\mathbb{R}}}$" we can write the familiar „<". For any pair $x , y \in \mathbb{R}$, $\overbrace{x} < \overbrace{y}$ if and only if $x < y$. (See [1], Theorem 1.17.) If x is running the open interval $]-1,1[$ then $\overbrace{x}$ is running the real number line. In this case we say that $\overbrace{x}$ is a ***visible exploded number*** . If $x \in \mathbb{R} \setminus]-1,1[$ then $\overbrace{x}$ is outside the number line. (See. Fig. 1.3.) In this case the number line is considered as an one dimensional space and is $\overbrace{x}$ called ***invisible exploded number***. The greatest invisible exploded number which is smaller than all of real numbers is $\overbrace{-1}$ is called ***negative discriminator.*** The smallest invisible exploded number which is greater than all of real numbers is $\overbrace{1}$ called ***positive discriminator***.

***Lemma* 1.10:** (The monotonity property of explosion.) Let $x \neq 0$ an arbitrary real number.
a/ If
(1.11) $\qquad 0 < x$
then
(1.12) $\qquad x < \overbrace{x}$
b/ If
(1.13) $\qquad x < 0$
then
(1.14) $\qquad \overbrace{x} < x.$
Proof.
Ad a/

Having that $x = (x, 0)$, by (1.11) and (1.1) we have $\overbrace{x} = \left(\frac{\{x\}}{1-\{x\}} , [x]\right)$ and Definition 1.9 yields that $0 < \overbrace{x}$. If $\overbrace{x} < \overbrace{1}$ then $0 < x < 1$ and we obtain that $\overbrace{x} = \left(\frac{x}{1-x}, 0\right)$. As the function $f(x) = \frac{x}{1-x}$ is strictly convex on the interval]0,1[, the inequality (1.12) is obtained. If $\overbrace{1} \leq \overbrace{x}$, then the mentioned property of positive discriminator gives (1.12) because *x* is a real number..
Ad b/

Having that $x = (x, 0)$, by (1.13) and (1.1) we have $\overbrace{x} = \left(-\frac{\{-x\}}{1-\{-x\}} , -[(-x)]\right)$ and Definition 1.9 yields that $\overbrace{x} < 0$. If $\overbrace{-1} < \overbrace{x}$ then $-1 < x < 0$ and $\overbrace{x} = \left(\frac{x}{1+x}, 0\right)$. As the function $f(x) = \frac{x}{1+x}$ is strictly concave on the interval $]-1,0[$, the inequality (1.12) is obtained. If $\overbrace{x} \leq \overbrace{-1}$, then the mentioned property of negative discriminator gives (1.12) because *x* is a real number.■

***Lemma* 1.15:** (The monotonity property of compression.) Let $u \neq 0$ an arbitrary exploded number.
a/ If
(1.16) $\qquad 0 < u$
then
(1.17) $\qquad 0 < \underbrace{u} < u$
b/ If
(1.18) $\qquad u < 0$
then
(1.19) $\qquad u < \underbrace{u} < 0.$
Proof.
Ad a/
By (1.5) and (1.16) we have that $\overbrace{\left(\underbrace{0}\right)} < \overbrace{\left(\underbrace{u}\right)}$. Hence, the inequaity $\underbrace{0} < \underbrace{u}$ is obvious. As $\underbrace{0} = 0$, the left hand side of (1.17) is obtained. For the right hand side we use the part a/ of Lemma1.10, which says that $\underbrace{u} < \overbrace{\left(\underbrace{u}\right)}$. Using the inversion formula (1.5) again, we have the right hand side of (1.17).
Ad b/

By (1.5) and (1.18) we have that $\overbrace{\left(\underbrace{u}\right)} < \overbrace{\left(\underbrace{0}\right)}$. Hence, the inequaity $\underbrace{u} < \underbrace{0}$ is obvious. As $\underbrace{0} = 0$, the right hand side of (1.19) is obtained. For the left hand side we use the part b/ of Lemma 1.10, which says that $\overbrace{\left(\underbrace{u}\right)} < \underbrace{u}$. Using the inversion formula (1.5) again, we have the left hand side of (1.19).■

***Definition* 1.20:** The exploded number *u* is called negative if $u < 0$ and positive if $u > 0$, respectively.■

The exploded number *u* is negative or positive if and only if $\underbrace{u} < 0$ or $\underbrace{u} > 0$,
respectively. For example $\overbrace{-1}\ is\ negative\ \ and\ \ \overbrace{1}\ is\ positive.$

If u and v are arbitrary exploded numbers and $u < v$, then for any exploded number w

$$u\,\mathcal{A}\,w < v\,\mathcal{A}\,w$$

is fulfilled. (See [1], Lemma 1.22.) If u and v are arbitrary exploded numbers and $u < v$, then for any positive exploded number w

$$\mathcal{M}\,w < v\,\mathcal{M}\,w.$$

Moreover, for any negative exploded number w

$$u\,\mathcal{M}\,w > v\,\mathcal{M}\,w$$

holds. (See [1], Lemma 1.24.)

Now we already have that $\left(\overbrace{\mathbb{R}},\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{M}},<\right)$ **is an ordered field**. Moreover, we introduce the super – subtraction and super - division

(1.21) $$u\,\mathcal{S}\,v = \overbrace{\underbrace{u}-\underbrace{v}} \quad ,u,v \in \overbrace{\mathbb{R}},$$

(1.22) $$u\,\mathcal{D}\,v = \overbrace{\left(\frac{\underbrace{u}}{\underbrace{v}}\right)} \quad ,u,v(\neq 0) \in \overbrace{\mathbb{R}},$$

respectively.

Of course, for any pair $u,v \in \overbrace{\mathbb{R}}$ indentities $v\,\mathcal{A}\,(u\,\mathcal{S}\,v) = u$ and $v\,\mathcal{M}\,(u\,\mathcal{D}\,v) = u$, where $v \neq 0$, are valid.

Finally, we give some useful signs and symbols.

(1.23) $$-\overbrace{x} =^{def} \overbrace{(-x)} \quad , x \in \mathbb{R}.$$ (See the additive inverse element.)

This definition is the extension of the sign „minus” because (1.23) is valid for real numbers: If $\overbrace{x} \in \mathbb{R}$ then $-1 < x < 1$, $so[|x|] = 0$. Hence, $\overbrace{(-x)} = \left(\overbrace{(-x)},0\right)$ real number. Moreover, (1.1) yields that

$$\overbrace{(-x)} = \left(sgn(-x)\right)\frac{|-x|}{1-|-x|} = -(sgn\,x)\frac{|x|}{1-|x|} = -\overbrace{x}.$$

Denoting $\overbrace{x} = u$ by (1.23) and (1.6) we have the equivalent definition

(1.24) $$-u =^{def} \overbrace{\left(-\underbrace{u}\right)} \quad , u \in \overbrace{\mathbb{R}}.$$

Hence, (1.5) yields

(1.25) $$\underbrace{(-u)} = -\underbrace{u} \quad , u \in \overbrace{\mathbb{R}}.$$

Using (1.24) , (1.7) and (1.8) we have that $u\,\mathcal{A}(-u) = 0$ and $-u = \overbrace{-1}\,\mathcal{M}\,u$. Moreover $-(-u) = u$.

The next lemma is independent of the theory of exploded numbers but it will useful for it.

***Lemma* 1.26:** If x and y are real numbers such that y is integer and the product $x \cdot y$ is non-negative numbers then

(1.27) $$x = \left(sgn\left(y + \frac{x}{1+|x|}\right)\right) \cdot \frac{\left\{\left|y+\frac{x}{1+|x|}\right|\right\}}{1-\left\{\left|y+\frac{x}{1+|x|}\right|\right\}}$$

and

(1.28) $$y = \left(sgn\left(y + \frac{x}{1+|x|}\right)\right) \cdot \left[\left|y + \frac{x}{1+|x|}\right|\right]$$

are valid.

Proof: First of all we remark if $x = y = 0$, then $sgn\left(y + \frac{x}{1+|x|}\right) = 0$, so, (1.27) and (1.28) are true.

Assuming that $y \in \mathbb{Z}$ and $x \in \mathbb{R}$, our conditions of lemma are satisfied if and only if

Case I: $(y = 0\ and\ x > 0)\ or\ (y > 0\ and\ x \geq 0)$

or

Case II.: $(y = 0\ and\ x < 0)\ or\ (y < 0\ and\ x \leq 0)$

are fulfilled.

Ad case I.

Clearly, $sgn\left(y + \frac{x}{1+|x|}\right) = 1$. As $\left|y + \frac{x}{1+|x|}\right| = y + \frac{x}{1+x}$ and $0 \leq \frac{x}{1+x} < 1$,

$$\left[\left|y + \frac{x}{1+|x|}\right|\right] = y \quad \text{and} \quad \frac{\left\{\left|y+\frac{x}{1+|x|}\right|\right\}}{1-\left\{\left|y+\frac{x}{1+|x|}\right|\right\}} = \frac{\frac{x}{1+x}}{1-\frac{x}{1+x}} = x,$$

(1.27) and (1.28) are obtained.

Ad case II.

Clearly, $sgn\left(y + \frac{x}{1+|x|}\right) = -1$. As $\left|y + \frac{x}{1+|x|}\right| = -y - \frac{x}{1-x}$ and $0 \leq \frac{-x}{1-x} < 1$,

$$\left[\left|y + \frac{x}{1+|x|}\right|\right] = -y \quad \text{and} \quad \frac{\left\{\left|y+\frac{x}{1+|x|}\right|\right\}}{1-\left\{\left|y+\frac{x}{1+|x|}\right|\right\}} = \frac{-\frac{x}{1-x}}{1+\frac{x}{1-x}} = -x,$$

(1.27) and (1.28) are obtained.■

Using (1.4) Lemma 1.26 yield the

***Corollary* 1.29:** If $u=(x,y)\in\overset{\frown\frown}{\mathbb{R}}$, then $x=\left(sgn\ \underset{\smile}{u}\right)\cdot\frac{\{|\underset{\smile}{u}|\}}{1-\{|\underset{\smile}{u}|\}}$ and $y=\left(sgn\ \underset{\smile}{u}\right)\cdot\left[\left|\underset{\smile}{u}\right|\right]$.■

The explosion formula (1.1) , definition (1.24) and Corollary 1.29 yield the

***Corollary* 1.30:** If $u=(x,y)\in\overset{\frown\frown}{\mathbb{R}}$, then the pair $(-x,-y)\in\overset{\frown\frown}{\mathbb{R}}$ and $(-x,-y)=-u$.

Proof: $(-x,-y)\in\overset{\frown\frown}{\mathbb{R}}$, because $-y$ is integer if and only if y is integer. Moreover $(-x)\cdot(-y)=x\cdot y$, so, $(-x)\cdot(-y)$ is non—negative, if and only if $x\cdot y$ is non – negative.

By definition (1.24) and (1.1) we can write

$$-u=\left(\left(sgn\,(-\underset{\smile}{u})\right)\cdot\frac{\left\{\left|-\underset{\smile}{u}\right|\right\}}{1-\left\{\left|-\underset{\smile}{u}\right|\right\}},\left(sgn\,(-\underset{\smile}{u})\right)\cdot\left[\left|-\underset{\smile}{u}\right|\right]\right)=\left(-\left(sgn\ \underset{\smile}{u}\right)\cdot\frac{\left\{\left|\underset{\smile}{u}\right|\right\}}{1-\left\{\left|\underset{\smile}{u}\right|\right\}},-\left(sgn\ \underset{\smile}{u}\right)\cdot\left[\left|\underset{\smile}{u}\right|\right]\right)$$

Moreover, considering $u=(x,y)$ with $x=\left(sgn\ \underset{\smile}{u}\right)\cdot\frac{\{|\underset{\smile}{u}|\}}{1-\{|\underset{\smile}{u}|\}}$ and $y=\left(sgn\ \underset{\smile}{u}\right)\cdot\left[\left|\underset{\smile}{u}\right|\right]$ by Corollary 1.29 we get $-u=(-x,-y)$.■

The extension of „absolute value” for any $u\in\overset{\frown\frown}{\mathbb{R}}$

$$|u| =^{def} \begin{cases} u\ if\ \ u>0\\ 0\ \ if\ \ u=0.\\ -u\,if\ \ u<0\end{cases}$$

***Lemma* 1.31:** For any $u\in\overset{\frown\frown}{\mathbb{R}}$

(1.32) $$|u|=\overset{\frown}{\left|\underset{\smile}{u}\right|}$$

and

(1.33) $$\underset{\smile}{|u|}=\left|\underset{\smile}{u}\right|$$

hold.

Proof: Having that $\left|\underset{\smile}{u}\right|$ is a non – negative real number we apply the explosion formula (1.1) and write

If $u>0$ then $\underset{\smile}{u}>0$, too. So, first inversion formula (1.5) we have $\overset{\frown}{\left|\underset{\smile}{u}\right|}=\overset{\frown}{\left(\underset{\smile}{u}\right)}=u$.

If $u=0$, then (1.32) is obvious.

If $u<0$ then $\underset{\smile}{u}<0$, too. Hence, by (1.24) we have $\overset{\frown}{\left|\underset{\smile}{u}\right|}=\overset{\frown}{\left(-\underset{\smile}{u}\right)}=-u$.

For (1.33) we use (1.32) and apply (1.5).■

The traditional triangular inequality by (1.7), (1.32), (1.6), (1.33) and (1.7) again, yields

(1.34) $$|u\ \mathcal{A}\ v|\le|u|\mathcal{A}|v|;\quad u\,,v\in\overset{\frown\frown}{\mathbb{R}}.$$

Moreover, by (1.8), (1.32), (1.6), (1.33) and (1.8) again,

(1.35) $$|u\ \mathcal{M}\ v|=|u|\mathcal{M}|v|\,;\quad u\,,v\in\overset{\frown\frown}{\mathbb{R}}$$

is obtained.

The definition of the super - function „$\overset{\frown}{sgn}$” for any $u\in\overset{\frown\frown}{\mathbb{R}}$,

$$\overset{\frown}{sgn}\ u=\begin{cases}\overset{\frown}{1}\ \ if\ \ u>0\\ 0\ \ \ if\ \ u=0\ .\\ -\overset{\frown}{1}\ if\ \ u<0\end{cases}$$

It is easy to see, that $\overset{\frown}{sgn}\ u=\overset{\frown}{sgn\ \underset{\smile}{u}}$. Moreover, if u is a real number, then $\overset{\frown}{sgn}\ u=\overset{\frown}{sgn\,u}$.

Using the definition of extended absolute value, for any $u\in\overset{\frown\frown}{\mathbb{R}}$

(1.36) $$u=\left(\overset{\frown}{sgn}\ u\right)\mathcal{M}|u|$$

is valid, because for $u\ge0$ the equality (1.36) is obvious, while for $u<0$ by (1.23), (1.24), (1.6), (1.8) and (1.5) we write

$\left(\overset{\frown}{sgn}\ u\right)\mathcal{M}|u|=\left(-\overset{\frown}{1}\right)\mathcal{M}(-u)=\overset{\frown}{-1}\,\mathcal{M}\,\overset{\frown}{\left(-\underset{\smile}{u}\right)}=\overset{\frown}{(-1)\cdot\left(-\underset{\smile}{u}\right)}=\overset{\frown}{\left(\underset{\smile}{u}\right)}=u$. Istennek Hála,

2019-12-15;5.38, Sz I.

2. TRADITIONAL ADDITION AND MULTIPLICATION FOR EXPLODED NUMBERS

In [1] we proved the identities

(2.1) $u+v=\left(u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)|\right)$

and

(2.2) $$u\cdot v=(u\mathcal{M}v)\mathcal{D}\left(\overbrace{1}\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\overbrace{2}\mathcal{M}|u\mathcal{M}v|\right)$$

for any pair u , v of real numbers. (See, [1], Theorems 2.1 and 2.3, respectively)

Our purpose to extend the identities (2.1) and (2.2) for exploded numbers as definitions.

Formula (1.22) shows that the denominator of a super – division cannot be 0, so we give

Definition 2.3. The exploded numbers u and v are called addition – incompetent partners if

(2.4) $$\overbrace{1}\ \mathcal{S}\ |u|\ \mathcal{S}\ |v|\ \mathcal{A}\ |u\ \mathcal{M}\ v|\ \mathcal{A}\ |u\ \mathcal{A}\ v\ \mathcal{S}\ (u\ \mathcal{M}\ |v|)\ \mathcal{S}\ (v\ \mathcal{M}\ |u|)|=0.$$

Otherwise we say that u and v are addition - competent partners.■

For example, real numbers are additionally competent partners.

Using the inversion formulas with (1.21), (1.7), (1.8), (1.35) and (1.33) we can write

$$\overbrace{1}\ \mathcal{S}\ |u|\ \mathcal{S}\ |v|\ \mathcal{A}\ |u\ \mathcal{M}\ v|\ \mathcal{A}\ |u\ \mathcal{A}\ v\ \mathcal{S}\ (u\ \mathcal{M}\ |v|)\ \mathcal{S}\ (v\ \mathcal{M}\ |u|)|=$$

$$=\overbrace{1-|\underbrace{u}|-|\underbrace{v}|+\underbrace{|u\ \mathcal{M}\ v|}+\underbrace{|u\ \mathcal{A}\ v\ \mathcal{S}\ (u\ \mathcal{M}\ |v|)\ \mathcal{S}\ (v\ \mathcal{M}\ |u|)|}}=$$

$$=\overbrace{1-|\underbrace{u}|-|\underbrace{v}|+\underbrace{|u|\mathcal{M}|v|}+\underbrace{\left|\overbrace{\underbrace{u}+\underbrace{v}-\underbrace{u\ \mathcal{M}\ |v|}-\underbrace{v\ \mathcal{M}\ |u|}}\right|}}=$$

$$=\overbrace{1-|\underbrace{u}|-|\underbrace{v}|+|\underbrace{u}|\cdot|\underbrace{v}|+\underbrace{\left|\overbrace{\underbrace{u}+\underbrace{v}-\underbrace{u}\cdot|\underbrace{v}|-\underbrace{v}\cdot|\underbrace{u}|}\right|}}=$$

$$=\overbrace{1-|\underbrace{u}|-|\underbrace{v}|+|\underbrace{u}|\cdot|\underbrace{v}|+\left|\underbrace{u}+\underbrace{v}-\underbrace{u}\cdot|\underbrace{v}|-\underbrace{v}\cdot|\underbrace{u}|\right|}=$$

$$=\overbrace{1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+\left|\underbrace{u}\cdot\underbrace{v}\right|+\left|\underbrace{u}+\underbrace{v}-\underbrace{u}\cdot\left|\underbrace{v}\right|-\underbrace{v}\cdot\left|\underbrace{u}\right|\right|}$$

In the following we need to explode the points $P=(x,y)$ of two-dimensional space $\mathbb{R}^2$. Namely, the exploded of P is $\overbrace{P}=(\overbrace{x},\overbrace{y})\in\overbrace{\mathbb{R}^2}$. Consequently, if $\mathcal{P}=(u,v)\in\overbrace{\mathbb{R}^2}$ then its compressed in coordinates is $\underbrace{\mathcal{P}}=\left(\underbrace{u},\underbrace{v}\right)\in\mathbb{R}^2$. The procedure is similar in the case of the sets. For example $\underbrace{\left(\overbrace{\mathbb{R}^2}\right)}=\mathbb{R}^2$. The set

(2.5) $$\mathbb{C}_{\mathcal{A}}=\left\{(u,v)\in\overbrace{\mathbb{R}^2}\middle|\,1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+\left|\underbrace{u}\cdot\underbrace{v}\right|+\left|\underbrace{u}+\underbrace{v}-\underbrace{u}\cdot\left|\underbrace{v}\right|-\underbrace{v}\cdot\left|\underbrace{u}\right|\right|=0\right\}$$

is called the super –curve of the addition – incompetence. The following should be for the simplicity $\underbrace{u}=x\ and\ \underbrace{v}=y$. Of course, we have that $-\infty<x<\infty$ and $-\infty<y<\infty$. Clearly,

(2.6) $$\underbrace{\mathbb{C}_{\mathcal{A}}}=\left\{(x,y)\in\mathbb{R}^2|1-|x|-|y|+|x\cdot y|+\left|x+y-x\cdot|y|-y\cdot|x|\right|=0\right\}$$

and it is shown on the next figure:

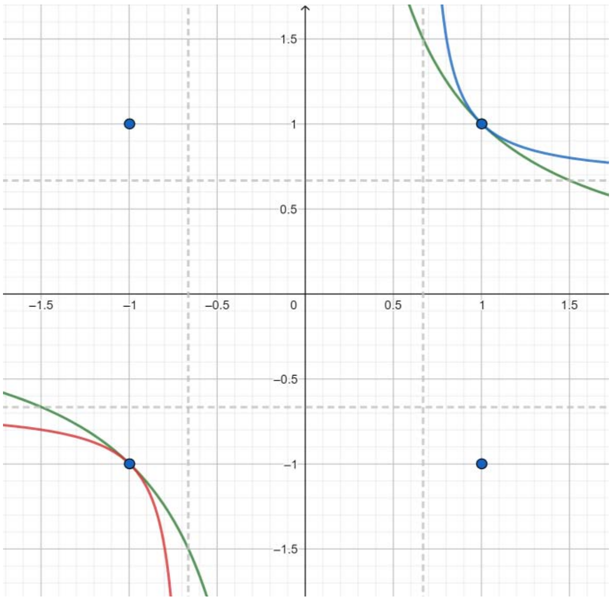

Fig.-2.7

Considering the compressed of $\mathbb{R}^2$

(2.8) $$\underbrace{\mathbb{R}^2} = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, {}^{-1<x<1}_{-1<y<1}\right\}$$

and observing Fig. 2.7 we have

***Remark* 2.9:**

1. $\underbrace{\mathbb{R}^2} \cap \underbrace{\mathbb{R}_{\mathcal{A}}} = \{\quad\}$, so for any real number each each real number is addition – competent partner.

2. 0 has no addition – incompetent partner.

3. If $0 < |u| \leq 2$ then u has unique (invisible) addition - incompetent parner v and $u\ \mathcal{M}\ v = \overbrace{1}$.

4. If $2 < |u| < \infty$ then u has two (invisible) addition- incompetent parners $v_1$and v_2.

5. The positive discriminator $\overbrace{1}$ is an addition – incompetent partner to himself.

6. The negative discriminator $-\overbrace{1}$ is an addition – incompetent partner to himself.

7. The discriminators are addition – incompetent partners.

8. If $\overbrace{1} < |u|$ then u has two (visible) addition - incompetent parners $v_1$and v_2.

***Definition* 2.10:**

Assuming that the exploded numbers u and v are addition – competent partners

$$u + v =^{def} \left(u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)|\right). \blacksquare$$

Definition 2.10 yields if the exploded numbers u and v are addition – competent partners then

(2.10)* $$-(u+v) = (-u) + (-v)$$

holds.

***Definition* 2.11:** The exploded numbers u and v are called multiplication – incompetent partners if

(2.12) $$\overbrace{1}\ \mathcal{S}\ |u|\ \mathcal{S}|v|\ \mathcal{A}\ \overbrace{2}\,\mathcal{M}|u\mathcal{M}v|\ = 0.$$

Otherwise we say that u and v are multiplication - competent partners.■

For example, real numbers are multiplication - competent partners.

Using the inversion formulas with (1.21), (1.7), (1.8) and (1.33) we can write

$$\overbrace{1}\,\mathcal{S}\ |u|\ \mathcal{S}\ |v|\ \mathcal{A}\ \overbrace{2}\,\mathcal{M}|u\ \mathcal{M}\ v| = \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + 2\cdot\underbrace{|u\ \mathcal{M}\ v|}} =$$

$$= \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + 2\cdot\underbrace{|u|\mathcal{M}|v|}} = \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + 2\cdot\underbrace{\left(\overbrace{\underbrace{|u|}\cdot\underbrace{|v|}}\right)}} =$$

$$= \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + 2\cdot\underbrace{|u|}\cdot\underbrace{|v|}} = \overbrace{1 - \left|\underbrace{u}\right| - \left|\underbrace{v}\right| + 2\cdot\left|\underbrace{u}\right|\cdot\left|\underbrace{v}\right|}.$$

The set

(2.13) $$\mathbb{C}_{\mathcal{M}} = \left\{(u,v) \in \overbrace{\mathbb{R}^2}\,\middle|\,1 - \left|\underbrace{u}\right| - \left|\underbrace{v}\right| + 2\cdot\left|\underbrace{u}\right|\cdot\left|\underbrace{v}\right| = 0\right\}$$

is called the super – curve of the multiplication– incompetence. The following should be for the simplicity $\underbrace{u} = x\ and\ \underbrace{v} = y$. Of course, we have that $-\infty < x < \infty$ and $-\infty < y < \infty$. Clearly,

(2.14) $$\underbrace{\mathbb{C}_{\mathcal{M}}} = \{(x,y) \in \mathbb{R}^2 | 1 - |x| - |y| + 2\cdot|x|\cdot|y| = 0\}$$

and it is shown on the next figure:

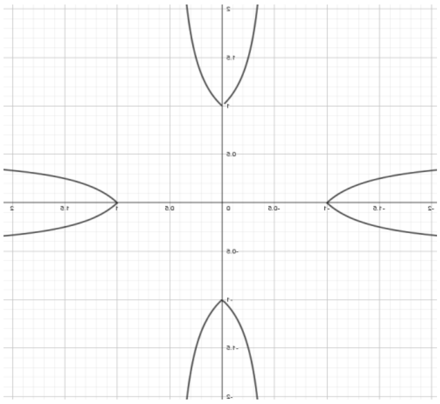

Fig.-2.15

Observing Fig. 2.15 we have

***Remark* 2.16:**

1. $\underbrace{\mathbb{R}^2} \cap \underbrace{\mathbb{R}_{\mathcal{M}}} = \{\quad\}$, so for any real number each each real number is multiplication– competent partner.

2. If $0 \le |u| < 1$ then *u* has two (invisible) multiplication - incompetent parners v_1 and v_2.
For example $u = \frac{1}{3}$, $v_1 = \overset{\frown\frown}{\left(\frac{3}{2}\right)}$ and $v_2 = -\overset{\frown\frown}{\left(\frac{3}{2}\right)}$.

3. If $1 \le |u| < \infty$ then *u* has no multiplication– incompetent partner

4. If $|u| = \overset{\frown\frown}{1}$ then *u* and 0 are multiplication - incompetent partners.

5. $|u| > \overset{\frown\frown}{1}$ then *u* has two (visible) multiplication - incompetent parners v_1 and v_2.
For example $u = -\overset{\frown\frown}{4}$, $v_1 = \frac{3}{4}$ and $v_2 = -\frac{3}{4}$.

***Definition* 2.17:**
Assuming that the exploded numbers *u* and *v* are multiplication – competent partners

$$u \cdot v =^{def} (u\mathcal{M}v)\mathcal{D}\left(\overset{\frown\frown}{1}\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\overset{\frown\frown}{2}\mathcal{M}|u\mathcal{M}v|\right) \blacksquare$$

3. IMPORTANT ELEMENTS OF ALGEBRA $\left(\overset{\frown\frown}{\mathbb{R}}, \le, +, \cdot\right)$

The 0 is the addition unit – element of algebra $(\mathbb{R}, \le, +, \cdot)$. What is its role in algebra $\left(\overset{\frown\frown}{\mathbb{R}}, \le, +, \cdot\right)$?

The point 2. of Remark 2.9 says that it is addition – competent partner for any exploded number. Moreover by Definition 2.10 we have for any $u \in \overset{\frown\frown}{\mathbb{R}}$ that $u + 0 = u\,\mathcal{D}\,\overset{\frown\frown}{1} = u$ and for any $v \in \overset{\frown\frown}{\mathbb{R}}$ that $0 + v = v\,\mathcal{D}\,\overset{\frown\frown}{1}$. So, we have

***Property* 3.1:** The 0 is the addition unit – element of algebra $\left(\overset{\frown\frown}{\mathbb{R}}, \le, +, \cdot\right)$.

Proof: Considering Definition 2.10 we write for any $u \in \overset{\frown\frown}{\mathbb{R}}$

$$u + 0 =$$
$$= \left(u\,\mathcal{A}\,0\,\mathcal{S}(u\mathcal{M}|0|)\,\mathcal{S}\,(0\mathcal{M}|u|)\right)\mathcal{D}\left(\overset{\frown\frown}{1}\mathcal{S}|u|\mathcal{S}|0|\,\mathcal{A}|u\mathcal{M}0|\,\mathcal{A}|u\,\mathcal{A}\,0\,\mathcal{S}(u\mathcal{M}|0|)\,\mathcal{S}\,(v\mathcal{M}|0|)|\right) =$$
$$= u\,\mathcal{D}\,\left(\overset{\frown\frown}{1}\,\mathcal{S}\,|u|\,\mathcal{A}\,|u|\right) = u. \blacksquare$$

By the commutativity of extended addition its unit element is unambiguously determined, but gives us a surprise the following

***Theorem* 3.2:** Let *u* be an arbitrarily given exploded number such that $|u| \ne \overset{\frown\frown}{1}$. The solution of the equation

(3.3) $$u + v = u$$

In the cases

A) $|u| \le \frac{1+\sqrt{3}}{2} \approx 1{,}366025404$
there exists only one solution $v = 0$.

B) $\frac{1+\sqrt{3}}{2} < |u| \left(\ne \overset{\frown\frown}{1}\right)$
the solutions are $v_1 = 0$ and $v_2 = \overset{\frown\frown}{\left(\left(sgn\ \underbrace{u}\right) \cdot \frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2 - 1}\right)}$.

Proof: By Definition 2.10 we have to solve the equation

$$\left(u\,\mathcal{A}\,v\,\mathcal{S}(u\mathcal{M}|v|)\,\mathcal{S}\,(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\,\mathcal{A}|u\mathcal{M}v|\,\mathcal{A}|u\,\mathcal{A}\,v\,\mathcal{S}(u\mathcal{M}|v|)\,\mathcal{S}\,(v\mathcal{M}|u|)|\right)=u.$$

Denoting $\underbrace{u} = x\ and\ \underbrace{v} = y$ and using the inversion formulas (1.5), (1.6), operations (1.7), (1.8), (1.21) and (1.22) , with (1.33) , equation (3.3) has the form

(3.4) $$\frac{x+y-x\cdot|y|-y\cdot|x|}{1-|x|-|y|+|xy|+|x+y-x\cdot|y|-y\cdot|x||}=x.$$

Of course, by (2.5) we have that $1-|x|-|y|+|xy|+\left|x+y-x\cdot|y|-y\cdot|x|\right|\neq 0$.

For any $x\in\mathbb{R}$ addition unit – element of algebra $(\mathbb{R}\,,\leq,+,\cdot)$ is a solution of (3.4), therefore $y=0$ is no longer sought. Moreover, if $x=0$, then (3.4) has solution $y=0$, only. So, we may assume, that $x\neq 0$ and $y\neq 0$.

Given the absolute values we will consider eight cases.

A) $-\infty < x(\neq -1) < 0$.

The transmuted (3.4) is

(3.5) $$\frac{x+y-x\cdot|y|+yx}{1+x-|y|-x\cdot|y|+|x+y-x\cdot|y|+yx|}=x.$$

a) $-\infty < y < 0$.

The transmuted (3.5) is

(3.6) $$\frac{x+y+2xy}{1+x+y+xy+|x+y+2xy|}=x.$$

i) $x+y+2xy\leq 0$.

The transmuted (3.6) is

(3.7) $$\frac{x+y+2xy}{1-xy}=x \Leftrightarrow y(x+1)^2=0$$

has no solution.

ii) $x+y+2xy\geq 0$.

The transmuted (3.6) is

(3.8) $$\frac{x+y+2xy}{1+2x+2y+3xy}=x$$

which has the solution

$$y=-\frac{2x^2}{3x^2-1},\text{ such that } x<-\frac{1}{\sqrt{3}}.$$

(See the reservations $-\infty < x(\neq -1) < 0$ and $-\infty < y < 0$.).

So, in the case A)a)ii) the equation (3.4) has a solution depending on u

(3.9) $$v=\overbrace{\left(-\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}\right)},\text{ where } u<\overbrace{\left(-\frac{1}{\sqrt{3}}\right)}=-\frac{1+\sqrt{3}}{2}.$$

b) $0 < y < \infty$.

The transmuted (3.5) is

(3.10) $$\frac{x+y+}{1+x-y-xy+|x+y|}=x.$$

i) $x+y\leq 0$.

The transmuted (3.10) is

(3.11) $$\frac{x+y}{1-2y-xy}=x \Leftrightarrow y(x+1)^2=0$$

has no solution.

ii) $x+y\geq 0$.

The transmuted (3.10) is

(3.12) $$\frac{x+y}{1+2x-xy}=x$$

which has the solution $y=\frac{2x^2}{x^2+1}$, but it is false because $x+y=\frac{x(x+1)^2}{x^2+1}<0$.

B) $0 < x(\neq 1) < \infty$.

The transmuted (3.4) is

(3.13) $$\frac{x+y-x\cdot|y|-yx}{1-x-|y|+x\cdot|y|+|x+y-x\cdot|y|-yx|}=x.$$

a) $-\infty<y<0$.

The transmuted (3.13) is

(3.14) $$\frac{x+y}{1-x+y-xy+|x+y|}=x.$$

i) $x+y\leq 0$.

The transmuted (3.14) is

(3.15) $$\frac{x+y}{1-2x-xy}=x$$

which has the solution $y=-\frac{2x^2}{x^2+1}$, but it is false because $x+y=\frac{x(x-1)^2}{x^2+1}>0$.

ii) $x+y\geq 0$.

The transmuted (3.14) is

(3.16) $$\frac{x+y}{1+2y-xy}=x\Leftrightarrow y(x-1)^2=0$$

has no solution.

b) $0<y<\infty$.

The transmuted (3.13) is

(3.17) $$\frac{x+y-2xy}{1-x-y+xy+|x+y-2xy|}=x.$$

i) $x+y-2xy\leq 0$.

The transmuted (3.17) is

(3.18) $$\frac{x+y-2xy}{1-2x-2y+3xy}=x$$

which has the solution

$$y=\frac{2x^2}{3x^2-1},\text{ such that } x>\frac{1}{\sqrt{3}}.$$

(See the reservations $0<x(\neq 1)<\infty$. and $0<y<\infty$.

So, in the case B)b)i) the equation (3.4) has a solution depending on *u*

(3.19) $$v=\overbrace{\left(\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}\right)},\text{ where } u>\overbrace{\left(\frac{1}{\sqrt{3}}\right)}=\frac{1+\sqrt{3}}{2}.$$

ii) $x+y-2xy\geq 0$.

The transmuted (3.17) is

(3.20) $$\frac{x+y-2xy}{1-xy}=x\Leftrightarrow y(x-1)^2=0$$

has no solution.

Considering (3.9) and (3.19) $v=\overbrace{\left(sgn\ \underbrace{u}\right)\cdot\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}}$ where $\frac{1+\sqrt{3}}{2}<|u|$ is obtained.

Finally, by Property 3.1 our theorem is proved.■

***Remark* 3.21:** Considering the part B of Theorem 3.2 and knowing that if

(3.22) $$\frac{1+\sqrt{3}}{2}<|u|<\infty$$

then *u* is a real number, a novelty that **there is a number** $\overbrace{\left(sgn\ \underbrace{u}\right)\cdot\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}}$ **other than zero** that is **added to *u*, the result is *u***. Using (1.4) by (3.22) $\underbrace{\left(\frac{1+\sqrt{3}}{2}\right)}=\frac{1}{\sqrt{3}}<\underbrace{|u|}<1\Leftrightarrow\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}>1$ is obtained. Hence, $\left|\overbrace{\left(sgn\ \underbrace{u}\right)\cdot\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}}\right|>\overbrace{1}$, that is $\overbrace{\left(sgn\ \underbrace{u}\right)\cdot\frac{2\left(\underbrace{u}\right)^2}{3\left(\underbrace{u}\right)^2-1}}$ not a real number.■

Definition* 3.23:** If *u* is an arbitrary exploded number then the exploded number *v* is called the ***addition – value partner for *u* if it that leaves *u* unchanged when it added up.■

Clearly, $v = 0$ is addition - value partner for any exploded number u.

Considering Definition 2.10 the esteemed reader can easily prove

***Property* 3.24:** (The behavior of discriminators for addition.) Assuming that $|v| \neq \overbrace{1}$, for any $v \in \overbrace{\mathbb{R}}$

(3.25)
$$\overbrace{1} + v = \begin{cases} \overbrace{1} & if\ |v| < \overbrace{1} \\ \overbrace{-1} & if\ |v| > \overbrace{1} \end{cases}$$

and

(3.26)
$$\overbrace{-1} + v = \begin{cases} \overbrace{-1} & if\ |v| < \overbrace{1} \\ \overbrace{1} & if\ |v| > \overbrace{1} \end{cases}$$

hold.
If $|v| = \overbrace{1}$, then $\overbrace{1} + v$ and $\overbrace{-1} + v$ are undetermined. (See (1.22).)■

We can see that every real number v, is addition – valuae partner for $\overbrace{1}$ and $\overbrace{-1}$, too.
Fig. 2.7 shows that except the pairs $\left(-\overbrace{1}, \overbrace{1}\right)$ and $\left(\overbrace{1}, -\overbrace{1}\right)$ the pair $(u\,, -u) \notin \mathbb{C}_{\mathcal{A}}$.

***Property* 3.27:** If $|u| \neq \overbrace{1}$ the for any $u \in \overbrace{\mathbb{R}}$ $u + (-u) = 0$.

Proof: Using Definition 2.10 we can write

$$u + (-u) =$$
$$= \left(u\,\mathcal{A}\,(-u)\,\mathcal{S}(u\mathcal{M}|-u|)\,\mathcal{S}\left((-u)\mathcal{M}|u|\right)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|-u|\,\mathcal{A}|u\mathcal{M}(-u)|\,\mathcal{A}\left|u\,\mathcal{A}\,(-u)\,\mathcal{S}(u\mathcal{M}|-u|)\,\mathcal{S}\left((-u)\mathcal{M}|u|\right)\right|\right) =$$
$$= \left(u\ \mathcal{S}\ u\ \ \mathcal{S}(u\mathcal{M}|u|)\,\mathcal{A}\,(u\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|u|\,\mathcal{A}(u\mathcal{M}u)\,\mathcal{A}|u\ \ \mathcal{S}\ u\ \ \mathcal{S}(u\mathcal{M}|u|)\,\mathcal{A}\,(u\mathcal{M}|u|)|\right) =$$
$$= 0\,\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|u|\,\mathcal{A}(u\mathcal{M}u)\right) = 0\,\mathcal{D}\,\left(\left(\overbrace{1}\ \mathcal{S}\ |u|\right)\,\mathcal{M}\left(\overbrace{1}\ \mathcal{S}\ |u|\right)\right) = 0 \blacksquare$$

The following theorem is important when researching the addition inverse element.

***Theorem* 3.28:** Let u be an arbitrarily given exploded number such that $|u| \neq \overbrace{1}$. The solution of the equation
(3.29)
$$u + v = 0$$
If $|u| \leq 1$ then there exists only one solution $v = -u$.
If $1 < |u|$ then solutions are $v_1 = -u$ and $v_2 = \overbrace{\left(\frac{\underbrace{u}}{2\left|\underbrace{u}\right| - 1}\right)}$.

Proof: Starting from (3.29) by Definition 2.10 we have to solve the equation
$$\left(u\,\mathcal{A}\ v\,\mathcal{S}(u\mathcal{M}|v|)\,\mathcal{S}\ (v\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\,\mathcal{A}|u\mathcal{M}v|\,\mathcal{A}|u\,\mathcal{A}\ v\,\mathcal{S}(u\mathcal{M}|v|)\,\mathcal{S}\ (v\mathcal{M}|u|)|\right) = 0.$$

Denoting $\underbrace{u} = x\ and\ \underbrace{v} = y$ and using the inversion formulas (1.5) , (1.6), operations (1.7) , (1.8) , (1.21) and (1.22) , with (1.33) , equation (3.3) has the form
$$\frac{x + y - x \cdot |y| - y \cdot |x|}{1 - |x| - |y| + |xy| + \left|x + y - x \cdot |y| - y \cdot |x|\right|} = 0.$$

Of course, by (2.5) we have that $1 - |x| - |y| + |xy| + \left|x + y - x \cdot |y| - y \cdot |x|\right| \neq 0$, so we have to solve the equation
(3.30)
$$x + y - x \cdot |y| - y \cdot |x| = 0.$$

For any $x \in \mathbb{R}$ addition inverse – element in algebra $(\mathbb{R}\,, \leq, +\,,\cdot)$ is a solution of (3.30), therefore $y = -x$ is no longer sought. Moreover, if $x = 0$, then (3.30) has solution $y = 0$, only. So, we may assume, that $x \neq 0$ and $y \neq 0$.

Given the absolute values we will consider four cases.
Case A) $-\infty < x(\neq -1) < 0$.

The transmuted (3.30) is
(3.31)
$$x + y - x \cdot |y| + xy = 0.$$
Part a) $-\infty < y < 0$.

The transmuted (3.31) is

(3.32) $$x + y + 2xy = 0$$

which has the solution

$$y = \frac{x}{2|x| - 1}\text{ , such that } x < -\frac{1}{2}.$$

(See the reservations $-\infty < x(\neq -1) < 0$ and $-\infty < y < 0$.)

So, in the case A)a) the equation (3.31) has a solution depending on u

(3.33) $$v = \overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)} \text{ where } u < \overbrace{\left(-\frac{1}{2}\right)} = -1\,.$$

Part b) $0 < y < \infty$.

The transmuted (3.31) is

(3.34) $$x + y = 0$$

has no solution.

Case B) $0 < x(\neq 1) < \infty$.

The transmuted (3.30) is

(3.35) $$x + y - x \cdot |y| - xy = 0.$$

Part a) $-\infty < y < 0$.

The transmuted (3.35) is

(3.36) $$x + y = 0$$

has no solution.

Part b) $0 < y < \infty$

The transmuted (3.35) is

(3.37) $$x + y - 2xy = 0$$

which has the solution

$$y = \frac{x}{2|x| - 1}\text{ , such that } x > \frac{1}{2}.$$

(See the reservations $0 < x(\neq 1) < \infty$. and $0 < y < \infty$.)

So, in the case B)b) the equation (3.35) has a solution depending on u

(3.38) $$v = \overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)} \text{ , where } u > \overbrace{\left(\frac{1}{2}\right)} = 1.$$

Considering (3.33) and (3.38) $v = \overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)}$, where $|u| > 1$ is obtained.

Finally, by Property 3.27 our theorem is proved.■

***Remark* 3.39:** Considering the part B of Theorem 3.28 and knowing that if

(3.40) $$1 < |u| < \infty$$

then u is a real number, a novelty that **there is a number** $v = \overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)}$ **other than (-*u*)** that is **added to *u*, the result is zero** . Using (1.4) by (3.30) $\frac{1}{2} < \left|\underset{\smile}{u}\right| < 1 \Leftrightarrow \left|\underset{\smile}{v}\right| = \frac{\left|\underset{\smile}{u}\right|}{2\left|\underset{\smile}{u}\right|-1} > 1$ is obtained. Hence, $\left|\overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)}\right| > \overset{\frown}{1}$, that is $\overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)}$ not a real number.■

Definition 3.41: If u is an arbitrary exploded number such that $|u| \neq \overset{\frown}{1}$ then the exploded number v is called the ***nullifying partner*** for u if adding it the result is 0.■

Clearly, if $|u| \neq \overset{\frown}{1}$, then $(-u)$ is nullifying partner for u. The discriminators have no nullifying partners. (See Remark 2.9, points 5-7 and Property 3.24.)

***Remark* 3.42:** Theorems 3.2 and 3.28 point out that even linear equations can have two solutions. For example in the case of Theorem 3.2 , B) the (3.3) equation and in Theorem 3.28 , B) the equation (3.29) have two nullifying or addition **-** value solutions.

The 1 is the multiplication unit – element of algebra $(\mathbb{R}, \leq, +, \cdot)$.
What is the situation in algebra $\left(\overbrace{\mathbb{R}}, \leq, +, \cdot\right)$?
The point 3. of Remark 2.16 says that 1 is multiplication – competent partner for any exploded number.
Moreover, we have

***Property* 3.43:** The 1 is the multiplication unit – element of algebra $\left(\overbrace{\mathbb{R}}, \leq, +, \cdot\right)$.

Proof: Considering Definition 2.17 we write for any $u \in \overbrace{\mathbb{R}}$

$$u \cdot 1 = (u\ \mathcal{M}\ 1)\ \mathcal{D}\left(\overbrace{1}\ \mathcal{S}|u|\mathcal{S}|1|\ \mathcal{A}\ \overbrace{2}\,|u\mathcal{M}1|\ \right) = \overbrace{\left(\frac{\underbrace{u} \cdot \frac{1}{2}}{1 - \left|\underbrace{u}\right| - \frac{1}{2} + 2 \cdot \left|\underbrace{u}\right| \cdot \frac{1}{2}}\right)} = u. \blacksquare$$

By the commutativity of extended multiplication its unit element is unambiguously determined, but gives us a surprise the following

***Theorem* 3.44:** Let u be an arbitrarily given exploded number such that $u \neq 0\ and |u| \neq \overbrace{1}$. The solution of the equation

(3.45) $$u \cdot v = u$$

In the cases

A) $0 \neq |u| < \overbrace{1}$

there exists only one solution $v = 1 = \overbrace{1}\,\mathcal{D}\,\overbrace{2}$.

B) $\overbrace{1} < |u|$

the solutions are $v_1 = 1$ and $v_2 = \left(\overbrace{1}\ \mathcal{S}\ |u|\right)\mathcal{D}\left(\overbrace{2}\ \mathcal{M}\ |u|\right) = \overbrace{\left(\frac{1}{2\left|\underbrace{u}\right|} - \frac{1}{2}\right)}$.

Proof: Considering (3. 45) by Definition 2.17 we have to solve the equation

(3.46) $$(u\ \mathcal{M}\ v)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}\ |u|\ \mathcal{S}\ |v|\ \mathcal{A}\ \overbrace{2}\,\mathcal{M}|u\mathcal{M}v|\right) = u$$

wehere by (2.13) $\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\,\overbrace{2}\,\mathcal{M}|u\mathcal{M}v| \neq 0$ is obtained. Definition 2.17 gives that

(3.47) $\boldsymbol{u \cdot 0 = 0}$ **, where $\boldsymbol{|u| \neq \overbrace{1}}$** (see Remark 2.16, point 4 and [1], (3.12))
holds, so, $v = 0$ is excluded.
Starting from (3.46) and using (1.35) by the algebra $\left(\overbrace{\mathbb{R}}, \leq, +, \cdot\right)$ we can write

$$v = \overbrace{1}\,\mathcal{S}\ |u|\ \mathcal{S}\ |v|\ \mathcal{A}\,\overbrace{2}\,\mathcal{M}\ |u|\ \mathcal{M}\ |v|$$

a) $|u| \neq \overbrace{1}\ and\ v > 0$.

Now, (3.46) has the forms $v = \overbrace{1}\ \mathcal{S}\ |u|\ \mathcal{S}\ v\ \mathcal{A}\ \overbrace{2}\,\mathcal{M}|u|\mathcal{M}v$ and $\overbrace{2}\,\mathcal{M}\ v\ \mathcal{S}\ \overbrace{2}\,\mathcal{M}|u|\mathcal{M}v = \overbrace{1}\ \mathcal{S}\ |u|$.
Hence, $\left(\overbrace{2}\ \mathcal{M}\ v\right)\ \mathcal{M}\left(\overbrace{1}\ \mathcal{S}\ |u|\right) = \overbrace{1}\ \mathcal{S}\ |u|$ and solution $v = \overbrace{1}\,\mathcal{D}\,\overbrace{2}$ is obtained.

b) $|u| \neq \overbrace{1}\ and\ v < 0$.

Now, (3.46) has the forms $v = \overbrace{1}\ \mathcal{S}\ |u|\ \mathcal{A}\ v\ \mathcal{S}\ \overbrace{2}\,\mathcal{M}|u|\mathcal{M}v$ and $0 = \overbrace{1}\ \mathcal{S}\ |u|\ \ \mathcal{S}\ \overbrace{2}\,\mathcal{M}|u|\mathcal{M}v$
Moreover, $\overbrace{2}\,\mathcal{M}|u|\mathcal{M}v = \overbrace{1}\ \mathcal{S}\ |u|$ and hence, $v = \left(\overbrace{1}\ \mathcal{S}\ |u|\right)\mathcal{D}\left(\overbrace{2}\ \mathcal{M}\ |u|\right)$,but the condition $v < 0$ reqires that $1 < |u|$.
Finally, Property 3.43 says that for any $u \in \overbrace{\mathbb{R}}$ the multiplicative unit – element 1 is a solution of (3.45). ■

Definition* 3.48:** If u is an arbitrary exploded number then the exploded number v is called the ***multiplication– value partner for u if it that leaves u unchanged when it multiplied up.■

Clearly, $v = 1$ is multiplication - value partner for any exploded number *u*. (See Property 3.43.) If $|u| > \overbrace{1}$ then u has two multiplication– value partners. (See Theorem 3.44,B.)
Considering Definition 2.17 the esteemed reader can easily prove

Property 3.49: (The behavior of discriminators for multiplication.) Assuming that $v \neq 0$, for any $v \in \overbrace{\mathbb{R}}$

(3.50) $$\overbrace{1} \cdot v = \begin{cases} \overbrace{1} & if\ v > 0 \\ \overbrace{-1} & if\ v < 0 \end{cases}$$,(see [1], (3.5)*), and

(3.51) $$\overset{\frown}{-1} \cdot v = \begin{cases} \overset{\frown}{-1} & if\ v > 0 \\ \overset{\frown}{1} & if\ v < 0 \end{cases}$$

hold.

If $v = 0$, then $\overset{\frown}{1} \cdot v$ and $\overset{\frown}{-1} \cdot v$ are undetermined. (See (1.22) and [1], (3.6)*.)■

We can see that every positive (exploded) number v, is multiplicative – valuation partner for $\overset{\frown}{1}$ and $\overset{\frown}{-1}$, respectively . Especially,

(3.52) $$\overset{\frown}{1} \cdot \overset{\frown}{1} = \overset{\frown}{1} \qquad \text{(see [1], (3.4) and (3.7)*)}$$

and

(3.53) $$\overset{\frown}{-1} \cdot \overset{\frown}{1} = \overset{\frown}{-1}$$

are valid. Moreover, we have

(3.54) $$\overset{\frown}{-1} \cdot \overset{\frown}{-1} = \overset{\frown}{1}\,.$$

The following theorem is important when researching the multiplication inverse element.

***Theorem* 3.55:** Let u be an arbitrarily given exploded number such that $u \neq 0$ and $|u| \neq \overset{\frown}{1}$. The solution of the equation

(3.56) $$u \cdot v = 1$$

Case A) If $0 < |u| < \frac{1}{3}$ then solutions are

$$v_1 == \frac{1}{u} \text{ and } v_2 = \overbrace{\left(\frac{\left(sgn\ \underset{\smile}{u}\right)\left(1-\left|\underset{\smile}{u}\right|\right)}{4\left|\underset{\smile}{u}\right|-1}\right)} = \left(\widetilde{sgn}\, u\right)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\,\mathcal{S}\,\overset{\frown}{1}\right)\right).$$

Case B) If $\frac{1}{3} \leq |u| < \overset{\frown}{1}$ then there exists only one solution $v = \left(\widetilde{sgn}\, u\right)\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right) = \frac{1}{u}$.

Case C) If $|u| > \overset{\frown}{1}$ then there exists only one solution $v = \left(\widetilde{sgn}\, u\right)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\,\mathcal{S}\,\overset{\frown}{1}\right)\right) = \left(sgn\ \underset{\smile}{u}\right)\frac{1-\left|\underset{\smile}{u}\right|}{3\left|\underset{\smile}{u}\right|-2}$.

Proof: Starting from (3.56) by Definition 2.17 we have to solve the equation

$$(u\mathcal{M}v)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\,\overset{\frown}{2}\,\mathcal{M}|u\mathcal{M}v|\right) = \overset{\frown}{1}\,\mathcal{D}\,\overset{\frown}{2}\,.$$

By (2.13) we may assume that $\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\,\overset{\frown}{2}\,\mathcal{M}|u\mathcal{M}v| \neq 0$ and using definition of super function $\widetilde{sgn}$ we have to solve the equation

(3.57) $$\overset{\frown}{2}\,\mathcal{M}\ u\ \mathcal{M}\ v = \overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|v|\ \mathcal{A}\ \overset{\frown}{2}\,\mathcal{M}|u|\mathcal{M}|v|. \qquad \text{(See (1.35).}$$

By (3.47) $v = 0$ is excluded, so we consider the following parts:

Part a) $\left(\overset{\frown}{1} \neq\right) u > 0\ and\ v > 0.$

Now, (3.57) is reduced to $0 = \overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}v$, so, we have

(3.58) $$v = \overset{\frown}{1}\,\mathcal{S}|u| = \left(\widetilde{sgn}\, u\right)\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right) = \frac{1}{u}$$

but the reservation $v > 0$ requires the condition

(3.59) $$|u| < \overset{\frown}{1}\,.$$

Part b) $\left(\overset{\frown}{1} \neq\right) u < 0\ and\ v > 0.$

Now, (3.57) is reduced to $\overset{\frown}{2}\,\mathcal{M}\ u\ \mathcal{M}\ v = \overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}\ v\ \mathcal{S}\ \overset{\frown}{2}\,\mathcal{M}u\ \mathcal{M}\ v$, so, we have

(3.60) $$v = \left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}u\ \mathcal{A}\ \overset{\frown}{1}\right) = \left(\widetilde{sgn}\, u\right)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\,\mathcal{S}\,\overset{\frown}{1}\right)\right) = \overbrace{\left(\frac{\left(sgn\ \underset{\smile}{u}\right)\left(1-\left|\underset{\smile}{u}\right|\right)}{4\left|\underset{\smile}{u}\right|-1}\right)}$$

but the reservation $v > 0$ requires one of conditions

(3.61) $$0 < |u| < \frac{1}{3}$$

or

(3.62) $$|u| > \overset{\frown}{1}.$$

Part c) $\left(\overset{\frown}{1} \neq\right) u < 0\ and\ v < 0.$

Now, (3.57) is reduced to $0 = \overset{\frown}{1}\,\mathcal{S}|u|\mathcal{A}v$, so, we have

(3.63) $$v = |u|\mathcal{S}\,\overset{\frown}{1} = \left(\widetilde{sgn}\, u\right)\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right) = \frac{1}{u}$$

but the reservation $v < 0$ requires the condition (3.59).

Part d) $\left(\overset{\frown}{1} \neq\right) u > 0\ and\ v < 0.$

Now, (3.57) is reduced to $\overset{\frown}{2}\,\mathcal{M}\;u\;\mathcal{M}\;v = \overset{\frown}{1}\,\mathcal{S}|u|\mathcal{A}v\;\mathcal{S}\;\overset{\frown}{2}\,\mathcal{M}\;u\mathcal{M}\;v$, so, we have

(3.64) $\quad v = \left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}\;u\;\mathcal{S}\;\overset{\frown}{1}\right) = (\overset{\frown}{sgn}\,u)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\;\mathcal{S}\,\overset{\frown}{1}\right)\right) = \overbrace{\left(\frac{(sgn\;\underset{\smile}{u})\left(1-\left|\underset{\smile}{u}\right|\right)}{4\left|\underset{\smile}{u}\right|-1}\right)}$

but the reservation $v < 0$ requires one of conditions (3.61) or (3.62).
Collecting our results, we can see

- Under condition (3.61) Parts a) and c) give $v_1 = (\overset{\frown}{sgn}\,u)\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)$ while Parts b) and d) give $v_2 = (\overset{\frown}{sgn}\,u)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\;\mathcal{S}\,\overset{\frown}{1}\right)\right)$. So, Case A) is obtained.
- Under condition (3.59) Part a) and c) give $v = (\overset{\frown}{sgn}\,u)\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)$, only. So, Case B) is obtained.
- Under condition (3.62) Part b) and d) give $v = (\overset{\frown}{sgn}\,u)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\;\mathcal{S}\,\overset{\frown}{1}\right)\right)$, only. So, Case C) is obtained. ■

***Remark* 3.65:** Considering the Case A of Theorem 3.55 and knowing that in this case *u* is a real number, a novelty that **there is a number** $(\overset{\frown}{sgn}\,u)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\;\mathcal{S}\,\overset{\frown}{1}\right)\right)$ **other than** $\boldsymbol{\frac{1}{u}}$ that is **multiplied to *u*, the result is 1**. However, this number not a real number because by using definition of super function $\overset{\frown}{sgn}$ and (1.35) we can write

$$\left|(\overset{\frown}{sgn}\,u)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\;\mathcal{S}\,\overset{\frown}{1}\right)\right)\right| = |(\overset{\frown}{sgn}\,u)|\mathcal{M}\left|\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\right|\mathcal{D}\left|\overset{\frown}{4}\,\mathcal{M}|u|\;\mathcal{S}\,\overset{\frown}{1}\right| == \left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{4}\,\mathcal{M}|u|\right) > \overset{\frown}{1}$$ ■

Definition* 3.66:** If $u(\neq 0)$ is an arbitrary exploded number such that $|u| \neq \overset{\frown}{1}$ then the exploded number *v* is called the ***unifying partner for *u* if multiplying it the result is 1. Provided that $v \neq \frac{1}{u}$ but it is unifying partner for *u*, use ***pseudo reciprocial value*** for *u*.■
Clearly,

- if $u \neq 0\;and\;|u| < \overset{\frown}{1}$, then $\frac{1}{u}$ is unifying partner ,
- the 0 has not unifying partners, (see (3.47) and Remark 2.16, point 4)
- the discriminators have not unifying partners (see Remark 2.16, point 4 and (3.50) – (3.54)).

***Remark* 3.67:** Definitions 2.10 and 2.17 show the commutativity of addition and multiplication, respectively. Unfortunately, the most important traditional laws are not valid in algebra $\left(\overset{\frown}{\mathbb{R}}\;,+\,,\cdot\right)$.

- Counterexample to the associativity of addition: $\left(\overset{\frown}{2}+2\right)+\overset{\frown}{\left(\frac{3}{2}\right)} \neq \overset{\frown}{2}+\left(2+\overset{\frown}{\left(\frac{3}{2}\right)}\right)$ because $2\;and\;\overset{\frown}{\left(\frac{3}{2}\right)}$ are addition – incompetent partners. (See Definition 2.3 and (2.5).)
- Counterexample to the associativity of multiplication: $\left(\overset{\frown}{2}\cdot\left(-\frac{1}{6}\right)\right)\cdot\overset{\frown}{\left(\frac{6}{5}\right)} \neq \overset{\frown}{2}\cdot\left(\left(-\frac{1}{6}\right)\cdot\overset{\frown}{\left(\frac{6}{5}\right)}\right)$ because $\left(-\frac{1}{6}\right)\;and\;\overset{\frown}{\left(\frac{6}{5}\right)}$ are multiplication – incompetent partners. (See Definition 2.11 and (2.13).)
- Counterexample to the distributivity: $\left(\overset{\frown}{2}+2\right)\cdot\frac{1}{2} \neq \overset{\frown}{2}\cdot\frac{1}{2}+2\cdot\frac{1}{2}$ because $\overset{\frown}{2}\;and\;\frac{1}{2}$ are multiplication – incompetent partners. (See Definition 2.11 and (2.13).)

Finding the necessary and sufficient conditions for the above laws to investigate is a nice task.

4. TRADITIONAL SUBTRACTION IN ALGEBRA $\left(\overset{\frown}{\mathbb{R}}\;,\;\leq,+\;,\;\cdot\right)$

In [1] we have already mentioned the identity

(4.1) $u - v = \left(u\mathcal{S}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{A}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{S}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{A}(v\mathcal{M}|u|)|\right)$

for any pair *u* , *v* of real numbers. (See, [1], Theorem 2.5.)

***Example* 4.2:** For any $v \in \mathbb{R}$, (4.1) yields: $0 - v = (0\mathcal{S}v)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|v|\mathcal{A}|0\mathcal{S}v|\right) = \overbrace{\left(\frac{0-\underset{\smile}{v}}{1-\left|\underset{\smile}{v}\right|+\left|0-\underset{\smile}{v}\right|}\right)} = -v.$

Our purpose to extend the identity (4.1) for exploded numbers as a definition.

Formula (1.22) shows that the denominator of a super – division cannot be 0, so we give

***Definition* 4.3:** The exploded numbers *u* and *v* are called subtraction – incompetent partners if

(4.4) $\quad \overset{\frown}{1}\;\mathcal{S}\;|u|\;\mathcal{S}\;|v|\;\mathcal{A}\;|u\;\mathcal{M}\;v|\;\mathcal{A}\;|u\;\mathcal{S}\;v\;\mathcal{S}\;(u\;\mathcal{M}\;|v|)\;\mathcal{A}\;(v\;\mathcal{M}\;|u|)| = 0.$

Otherwise we say that *u* and *v* are subtraction - competent partners.■

For example, real numbers are subtraction competent partners.

Using the inversion formulas with (1.21), (1.7), (1.8), (1.35) and (1.33) we can write

$$\overset{\frown}{1}\ \mathcal{S}\ |u|\ \mathcal{S}\ |v|\ \mathcal{A}\ |u\ \mathcal{M}\ v|\ \mathcal{A}\ |u\ \mathcal{S}\ v\ \mathcal{S}\ (u\ \mathcal{M}\ |v|)\ \mathcal{A}\ (v\ \mathcal{M}\ |u|)| = \\ = \overbrace{1 - \left|\underset{\smile}{u}\right| - \left|\underset{\smile}{v}\right| + \left|\underset{\smile}{u}\cdot\underset{\smile}{v}\right| + \left|\underset{\smile}{u} - \underset{\smile}{v} - \underset{\smile}{u}\cdot\left|\underset{\smile}{v}\right| + \underset{\smile}{v}\cdot\left|\underset{\smile}{u}\right|\right|}$$

The set

(4.5) $\quad \mathbb{C}_{\mathcal{S}} = \left\{(u,v) \in \overset{\frown}{\mathbb{R}^2} \middle| 1 - \left|\underset{\smile}{u}\right| - \left|\underset{\smile}{v}\right| + \left|\underset{\smile}{u}\cdot\underset{\smile}{v}\right| + \left|\underset{\smile}{u} - \underset{\smile}{v} - \underset{\smile}{u}\cdot\left|\underset{\smile}{v}\right| + \underset{\smile}{v}\cdot\left|\underset{\smile}{u}\right|\right| = 0\right\}$

is called the super – curve of the subtraction– incompetence.

The following should be for the simplicity $\underset{\smile}{u} = x$ *and* $\underset{\smile}{v} = y$. Of course, we have that $-\infty < x < \infty$ and $-\infty < y < \infty$. Clearly,

(4.6) $\quad \underbrace{\mathbb{C}_{\mathcal{S}}} = \{(x,y) \in \mathbb{R}^2 | 1 - |x| - |y| + |x\cdot y| + \big|x - y - x\cdot|y| + y\cdot|x|\big| = 0\}.$

In (4.6) replacing *y* with (- *y*) yields $\underbrace{\mathbb{C}_{\mathcal{A}}}$ (see (2.6)), so $\underbrace{\mathbb{C}_{\mathcal{A}}}$ (see Fig. 2.7) is a reflection of $\underbrace{\mathbb{C}_{\mathcal{S}}}$ on the „x" axis.

***Definition* 4.7:**

Assuming that the exploded numbers *u* and *v* are subtraction – competent partners

$u - v =^{def} \left(u\mathcal{S}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{A}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overset{\frown}{1}\ \mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{S}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{A}(v\mathcal{M}|u|)|\right)$. ■

Definition 4.7 yields

***Property* 4.8:** If $|u| \neq 1$ then for any $u \in \overset{\frown}{\mathbb{R}}$ the equality $u - u = 0$ is valid.

Because " $-$ " **is used both the subtraction and additive inverse of number *u*** (see Theorem 3.28) it is important to see the following identities:

(4.9) $\quad 0 - u = -u \quad , \ u \in \overset{\frown}{\mathbb{R}}.$

(4.10) $\quad u - v = u + (-v) \quad , (u,v) \notin \mathbb{C}_{\mathcal{S}}$. (See, (4.5).)

and

(4.11) $\quad (-u) - v = -(u + v) \quad , (u,v) \notin \mathbb{C}_{\mathcal{A}}$. (See, (2.5).)

***Remark* 4.12:**

1. $\underbrace{\mathbb{R}^2} \cap \underbrace{\mathbb{C}_{\mathcal{S}}} = \{\quad\}$, so for any real number each each real number is subtraction– competent partner.

2. 0 has not subtraction – incompetent partner.

3. The positive discriminator $\overset{\frown}{1}$ is an subtraction – incompetent partner to himself.

4. The negative discriminator $-\overset{\frown}{1}$ is an subtraction – incompetent partner to himself.

5. The discriminators are subtraction – icompetent partners.

Considering Definition 4.7 the esteemed reader can easily prove

***Property* 4.13:** (The behavior of discriminators for subtraction.) Assuming that $|v| \neq \overset{\frown}{1}$, for any $v \in \overset{\frown}{\mathbb{R}}$

(4.14) $\quad \overset{\frown}{1} - v = \begin{cases} \overset{\frown}{1} & if\ |v| < \overset{\frown}{1} \\ \overset{\frown}{-1} & if\ |v| > \overset{\frown}{1} \end{cases}$, (see [1], (3.8)*),

and

(4.15) $\quad \overset{\frown}{-1} - v = \begin{cases} \overset{\frown}{-1} & if\ |v| < \overset{\frown}{1} \\ \overset{\frown}{1} & if\ |v| > \overset{\frown}{1} \end{cases}$

hold.■

***Theorem* 4.16:** (It is important for the concept of „difference".)

Let u and $v \neq 0$ be arbitrarily given exploded numbers such that $|u| \neq \overset{\frown}{1}$, $|v| \neq \overset{\frown}{1}$ *and* $u \neq v$. Considering the equation

(4.17) $\quad u + w = v$

Cases and solutions are

Case I./a) $-\infty < \underset{\smile}{u} < -1$ *and* $-\infty < \underset{\smile}{v} < \underset{\smile}{u}$ or $1 < \underset{\smile}{u} < \infty$ *and* $\underset{\smile}{u} < \underset{\smile}{v} < \infty$

$$w_1 = v - u = \overbrace{\left(\frac{\underset{\smile}{v} - \underset{\smile}{u}}{1 - 2\left|\underset{\smile}{u}\right| + \underset{\smile}{u}\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v} - \underset{\smile}{u} - 2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}{1 - 2\left|\underset{\smile}{u}\right| + 2\underset{\smile}{v}\left(sgn\,\underset{\smile}{u}\right) - 3\underset{\smile}{u}\underset{\smile}{v}}\right)}.$$

Case I/b) $-\infty < \underset{\smile}{u} < -1$ *and* $\underset{\smile}{u} < \underset{\smile}{v} < 0$ or $1 < \underset{\smile}{u} < \infty$ *and* $0 < \underset{\smile}{v} < \underset{\smile}{u}$

$$w_1 = v - u = \overbrace{\left(\frac{\underset{\smile}{v} - \underset{\smile}{u}}{1 - 2\underset{\smile}{v}\left(sgn\ \underset{\smile}{u}\right) + \underset{\smile}{u}\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v} - \underset{\smile}{u} - 2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}{1 - 2\left|\underset{\smile}{u}\right| + 2\underset{\smile}{v}\left(sgn\,\underset{\smile}{u}\right) - 3\underset{\smile}{u}\underset{\smile}{v}}\right)}.$$

Case I/c.) $-\infty<\underbrace{u}<-1$ and $0<\underbrace{v}<-\frac{1}{\underbrace{u}}$ or $1<\underbrace{u}<\infty$ and $-\frac{1}{\underbrace{u}}<\underbrace{v}<0$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}.$$

Case I/d.) $-\infty<\underbrace{u}<-1$ and $-\frac{1}{\underbrace{u}}\le\underbrace{v}\le\frac{2\underbrace{u}+1}{3\underbrace{u}+2}$ or $1<\underbrace{u}<\infty$ and $\frac{2\underbrace{u}-1}{3\underbrace{u}-2}\le\underbrace{v}\le-\frac{1}{\underbrace{u}}$

there exists only one solution $w=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}$.

Case I/e.) $-\infty<\underbrace{u}<-1$ and $\frac{2\underbrace{u}+1}{3\underbrace{u}+2}<\underbrace{v}<\infty$ or $1<\underbrace{u}<\infty$ and $-\infty<\underbrace{v}<-\frac{2\underbrace{u}-1}{3\underbrace{u}-2}$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1-2|\underbrace{u}|+2\underbrace{v}(sgn\,\underbrace{u})-3\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}.$$

Case II/a.) $-1<\underbrace{u}\le-\frac{2}{3}$ and $-\infty<\underbrace{v}<\underbrace{u}$ or $\frac{2}{3}\le\underbrace{u}<1$ and $\underbrace{u}<\underbrace{v}<\infty$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1-2|\underbrace{u}|+2\underbrace{v}(sgn\,\underbrace{u})-3\underbrace{u}\underbrace{v}}\right)}.$$

Case II/b.) $-1<\underbrace{u}\le-\frac{2}{3}$ and $\underbrace{u}<\underbrace{v}<0$ or $\frac{2}{3}\le\underbrace{u}<1$ and $0<\underbrace{v}<\underbrace{u}$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2\underbrace{v}(sgn\,\underbrace{u})+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1-2|\underbrace{u}|+2\underbrace{v}(sgn\,\underbrace{u})-3\underbrace{u}\underbrace{v}}\right)}.$$

Case II/c.) $-1<\underbrace{u}\le-\frac{2}{3}$ and $0<\underbrace{v}\le-\frac{1}{\underbrace{u}}$ or $\frac{2}{3}\le\underbrace{u}<1$ and $-\frac{1}{\underbrace{u}}\le\underbrace{v}<0$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}$$

Case II/d.) $-1<\underbrace{u}\le-\frac{2}{3}$ and $-\frac{1}{\underbrace{u}}\le\underbrace{v}\le\frac{2\underbrace{u}+1}{3\underbrace{u}+2}$ or $\frac{2}{3}\le\underbrace{u}<1$ and $-\frac{2\underbrace{u}-1}{3\underbrace{u}-2}\le\underbrace{v}\le-\frac{1}{\underbrace{u}}$

there exists only one solution $w=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}$.

Case II/e.) $-1<\underbrace{u}<-\frac{2}{3}$ and $\frac{2\underbrace{u}+1}{3\underbrace{u}+2}<\underbrace{v}<\infty$ or $\frac{2}{3}<\underbrace{u}<1$ and $-\infty<\underbrace{v}<-\frac{2\underbrace{u}-1}{3\underbrace{u}-2}$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1-2|\underbrace{u}|+2\underbrace{v}(sgn\,\underbrace{u})-3\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}.$$

Case III/a.) $-\frac{2}{3}<\underbrace{u}\le-\frac{1}{\sqrt{3}}$ and $-\infty<\underbrace{v}\le\frac{2\underbrace{u}+1}{3\underbrace{u}+2}$ or $\frac{1}{\sqrt{3}}\le\underbrace{u}<\frac{2}{3}$ and $-\frac{2\underbrace{u}-1}{3\underbrace{u}-2}\le\underbrace{v}<\infty$

there exists only one solution $w=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}$.

Case III/b.) $-\frac{2}{3}<\underbrace{u}\le-\frac{1}{\sqrt{3}}$ and $\frac{2\underbrace{u}+1}{3\underbrace{u}+2}<\underbrace{v}<\underbrace{u}$ or $\frac{1}{\sqrt{3}}\le\underbrace{u}<\frac{2}{3}$ and $\underbrace{u}<\underbrace{v}<-\frac{2\underbrace{u}-1}{3\underbrace{u}-2}$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1-2|\underbrace{u}|+2\underbrace{v}(sgn\,\underbrace{u})-3\underbrace{u}\underbrace{v}}\right)}.$$

Case III/c.) $-\frac{2}{3}<\underbrace{u}\le-\frac{1}{\sqrt{3}}$ and $\underbrace{u}<\underbrace{v}<0$ or $\frac{1}{\sqrt{3}}\le\underbrace{u}<\frac{2}{3}$ and $0<\underbrace{v}<\underbrace{u}$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2\underbrace{v}(sgn\,\underbrace{u})+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1-2|\underbrace{u}|+2\underbrace{v}(sgn\,\underbrace{u})-3\underbrace{u}\underbrace{v}}\right)}.$$

Case III/d.) $-\frac{2}{3}<\underbrace{u}\le-\frac{1}{\sqrt{3}}$ and $0<\underbrace{v}<-\frac{1}{\underbrace{u}}$ or $\frac{1}{\sqrt{3}}\le\underbrace{u}<\frac{2}{3}$ and $-\frac{1}{\underbrace{u}}<\underbrace{v}<0$

$$w_1=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}-2|\underbrace{u}|\underbrace{v}}{1+\underbrace{u}\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}.$$

Case III/e.) $-\frac{2}{3}<\underbrace{u}\le-\frac{1}{\sqrt{3}}$ and $-\frac{1}{\underbrace{u}}\le\underbrace{v}<\infty$ or $\frac{1}{\sqrt{3}}\le\underbrace{u}<\frac{2}{3}$ and $-\infty<\underbrace{v}\le-\frac{1}{\underbrace{u}}$ $<$

there exists only one solution $w=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}$.

Case IV/a.) $-\frac{1}{\sqrt{3}}\le\underbrace{u}\le-\frac{1}{2}$ and $-\infty<\underbrace{v}<\underbrace{u}$ or $\frac{1}{2}\le\underbrace{u}\le\frac{1}{\sqrt{3}}$ and $\underbrace{u}<\underbrace{v}<\infty$

there exists only one solution $w=v-u=\overbrace{\left(\frac{\underbrace{v}-\underbrace{u}}{1-2|\underbrace{u}|+\underbrace{u}\underbrace{v}}\right)}$.

Case IV/b.) $-\frac{1}{\sqrt{3}}\le\underbrace{u}\le-\frac{1}{2}$ and $\underbrace{u}<\underbrace{v}\le\frac{2\underbrace{u}+1}{3\underbrace{u}+2}$ or $\frac{1}{2}\le\underbrace{u}\le\frac{1}{\sqrt{3}}$ and $-\frac{2\underbrace{u}-1}{3\underbrace{u}-2}\le\underbrace{v}<\underbrace{u}$

there exists only one solution $w = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\underset{\smile}{v}\left(sgn\,\underset{\smile}{u}\right)+\underset{\smile}{u}\underset{\smile}{v}}\right)}$.

Case IV/c.) $-\frac{1}{\sqrt{3}} \le \underset{\smile}{u} < -\frac{1}{2}$ *and* $\frac{2\underset{\smile}{u}+1}{3\underset{\smile}{u}+2} < \underset{\smile}{v} < 0$ or $\frac{1}{2} < \underset{\smile}{u} \le \frac{1}{\sqrt{3}}$ *and* $0 < \underset{\smile}{v} < -\frac{2\underset{\smile}{u}-1}{3\underset{\smile}{u}-2}$

$$w_1 = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\underset{\smile}{v}\left(sgn\,\underset{\smile}{u}\right)+\underset{\smile}{u}\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}{1-2\left|\underset{\smile}{u}\right|+2\underset{\smile}{v}\left(sgn\,\underset{\smile}{u}\right)-3\underset{\smile}{u}\underset{\smile}{v}}\right)}.$$

Case IV/d.) $-\frac{1}{\sqrt{3}} \le \underset{\smile}{u} \le -\frac{1}{2}$ *and* $0 < \underset{\smile}{v} < -\frac{1}{\underset{\smile}{u}}$ or $\frac{1}{2} \le \underset{\smile}{u} \le \frac{1}{\sqrt{3}}$ *and* $-\frac{1}{\underset{\smile}{u}} < \underset{\smile}{v} < 0$

$$w_1 = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}{1+\underset{\smile}{u}\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\left|\underset{\smile}{u}\right|+\underset{\smile}{u}\underset{\smile}{v}}\right)}.$$

Case IV/e) $-\frac{1}{\sqrt{3}} \le \underset{\smile}{u} < -\frac{1}{2}$ *and* $-\frac{1}{\underset{\smile}{u}} \le \underset{\smile}{v} < \infty$ or $\frac{1}{2} < \underset{\smile}{u} \le \frac{1}{\sqrt{3}}$ *and* $-\infty < \underset{\smile}{v} \le -\frac{1}{\underset{\smile}{u}}$

there exists only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\left|\underset{\smile}{u}\right|+\underset{\smile}{u}\underset{\smile}{v}}\right)}$.

Case V/a.) $-\frac{1}{2} \le \underset{\smile}{u} < 0$ *and* $-\infty < \underset{\smile}{v} < \underset{\smile}{u}$ or $0 < \underset{\smile}{u} \le \frac{1}{2}$ *and* $\underset{\smile}{u} < \underset{\smile}{v} < \infty$

there exists only one solution $w = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\left|\underset{\smile}{u}\right|+\underset{\smile}{u}\underset{\smile}{v}}\right)}$.

Case V/b.) $-\frac{1}{2} \le \underset{\smile}{u} < 0$ *and* $\underset{\smile}{u} < \underset{\smile}{v} < 0$ or $0 < \underset{\smile}{u} \le \frac{1}{2}$ *and* $0 < \underset{\smile}{v} < \underset{\smile}{u}$

there exists only one solution $w = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\underset{\smile}{v}\left(sgn\,\underset{\smile}{u}\right)+\underset{\smile}{u}\underset{\smile}{v}}\right)}$.

Case V/c.) $-\frac{1}{2} \le \underset{\smile}{u} < 0$ *and* $0 \le \underset{\smile}{v} \le -\frac{1}{\underset{\smile}{u}} - 2$ or $0 < \underset{\smile}{u} \le \frac{1}{2}$ *and* $-\frac{1}{\underset{\smile}{u}} + 2 \le \underset{\smile}{v} \le 0$

there exists only one solution $w = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}{1+\underset{\smile}{u}\underset{\smile}{v}}\right)}$.

Case V/d.) $-\frac{1}{2} \le \underset{\smile}{u} < 0$ *and* $-\frac{1}{\underset{\smile}{u}} - 2 \le \underset{\smile}{v} < -\frac{1}{\underset{\smile}{u}}$ or $0 < \underset{\smile}{u} \le \frac{1}{2}$ *and* $-\frac{1}{\underset{\smile}{u}} < \underset{\smile}{v} \le -\frac{1}{\underset{\smile}{u}} + 2$

$$w_1 = v - u = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}{1+\underset{\smile}{u}\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\left|\underset{\smile}{u}\right|+\underset{\smile}{u}\underset{\smile}{v}}\right)}.$$

Case V/e.) $-\frac{1}{2} \le \underset{\smile}{u} < 0$ *and* $-\frac{1}{\underset{\smile}{u}} \le \underset{\smile}{v} < \infty$ or $0 < \underset{\smile}{u} \le \frac{1}{2}$ *and* $-\infty < \underset{\smile}{v} \le -\frac{1}{\underset{\smile}{u}}$

there exists only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}-\underset{\smile}{u}}{1-2\left|\underset{\smile}{u}\right|+\underset{\smile}{u}\underset{\smile}{v}}\right)}$

Proof:
By Definition 2.10 we have to solve the equation

$$\left(u\,\mathcal{A}\,w\,\mathcal{S}(u\mathcal{M}|w|)\,\mathcal{S}\,(w\mathcal{M}|u|)\right)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|w|\,\mathcal{A}|u\mathcal{M}w|\,\mathcal{A}|u\,\mathcal{A}\,w\,\mathcal{S}(u\mathcal{M}|w|)\,\mathcal{S}\,(w\mathcal{M}|u|)|\right) = v.$$

Denoting $\underset{\smile}{u} = x$, $\underset{\smile}{v} = y$ *and* $\underset{\smile}{w} = z$ and using the inversion formulas (1.5) , (1.6), operations (1.7) , (1.8) , (1.21) and (1.22) , with (1.33) , equation (4.17) has the form

(4.18) $$\frac{x+z-x\cdot|z|-z\cdot|x|}{1-|x|-|z|+|xz|+|x+z-x\cdot|z|-z\cdot|x||} = y.$$

Of course, by (2.5) we have that $1-|x|-|z|+|xz|+\left|x+z-x\cdot|z|-z\cdot|x|\right| \neq 0$.

By the assuptions $v \neq 0$ and $u \neq v$ we have that $y \neq 0$ and $x \neq y$. If $x = 0$, then (4.18) has solution $z = y$, only. So, we may assume, that $x \neq 0$. If $z = 0$ then (4.18) is reduced to $x = y$, so $z = 0$ is excluded. Given the absolute values we will consider eight cases.

A) $-\infty < x(\neq -1) < 0$.

The transmuted (4.18) is

(4.19) $$\frac{x+z-x\cdot|z|+zx}{1+x-|z|-x\cdot|z|+|x+z-x\cdot|z|+zx|} = y.$$

a) $-\infty < z < 0$.

The transmuted (4.19) is

(4.20) $$\frac{x+z+2xz}{1+x+z+xz+|x+z+2xz|} = y.$$

i) $x + z + 2xz \le 0$.

The transmuted (4.20) is

(4.21) $$\frac{x+z+2xz}{1-xz}=y \Leftrightarrow z=\frac{y-x}{1+2x+xy}=\frac{y-x}{1-2|x|+xy}.$$

This gives the solutions
w_1 of Case I/a , w_2 of Case of I/c, w of Case I/d and w_2 of Case I/e,
w_1 of Case II/a, w_2 of Case II/c, w of Case II/d, w_2 of Case II/e,
w of Case III/a, w_1 of Case III/b, w_2 of Case III/d, w of Case III/e,
w of Case IV/a, w_2 of Case IV/d, w of Case IV/e,
w of Case V/a, w_2 of Case V/d, w of Case V/e.
ii) $x+y+2xz \geq 0$.

The transmuted (4.20) is

(4.22) $$\frac{x+y+2xz}{1+2x+2y+3xz}=y \Leftrightarrow z=\frac{y-x+2xy}{1+2x-2y-3xy}=\frac{y-x-2|x|y}{1-2|x|+2y(sgnx)-3xy}$$

This gives the solutions
w_2 of Cases I/a and I/b, w_1 of Case I/e,
w_2 of Cases II/a and II/b, w_1 of Case II/e ,
w_2 of Cases III/b and Case III/c,
w_2 of Case IV/c.

b) $0<z<\infty$.
The transmuted (4.19) is

(4.23) $$\frac{x+z}{1+x-z-xz+|x+z|}=y.$$

i) $x+z \leq 0$.
The transmuted (4.23) is

(4.24) $$\frac{x+z}{1-2z-xz}=y \Leftrightarrow z=\frac{y-x}{1+2y+xy}=\frac{y-x}{1-2y(sgnx)+xy}.$$

This gives the solutions
w_1 of Case I/b,
w_1 of Case II/b,
w_1 of Case III/c,
w of Case IV/b, w_1 of Case IV/c,
w of Case V/b.
ii) $x+y \geq 0$.
The transmuted (4.23) is

(4.25) $$\frac{x+z}{1+2x-xz}=y \Leftrightarrow z=\frac{y-x+2xy}{1+xy}=\frac{y-x-2|x|y}{1+xy}$$

This gives the solutions
w_1 of Case I/c,
w_1 of Case II/c,
w_1 of Case III/d,
w_1 of Case IV/d,
w of Case V/c, w_1 of Case V/d.
B) $0<x(\neq 1)<\infty$.
The transmuted (4.18) is

(4.26) $$\frac{x+z-x\cdot|z|-zx}{1-x-|z|+x\cdot|z|+|x+z-x\cdot|z|-zx|}=y.$$

a) $-\infty<z<0$.
The transmuted (4.26) is

(4.27) $$\frac{x+z}{1-x+z-xz+|x+z|}=y.$$

i) $x+z \leq 0$.
The transmuted (4.27) is

(4.28) $$\frac{x+z}{1-2x-xz}=y \Leftrightarrow z=\frac{y-x-2xy}{1+xy}=\frac{y-x-2|x|y}{1+xy}$$

This gives the solutions
w_1 of Case I/c,
w_1 of Case II/c,
w_1 of Case III/d,
w_1 of Case IV/d,
w of Case V/c, w_1 of Case V/d.
ii) $x+z \geq 0$.

The transmuted (4.27) is

(4.29) $$\frac{x+z}{1+2z-xz}=y \Leftrightarrow z=\frac{y-x}{1-2y+xy}=\frac{y-x}{1-2y(sgnx)+xy}.$$

This gives the solutions
w_1 of Case I/b,
w_1 of Case II/b,
w_1 of Case III/c,
w of Case IV/b, w_1 of Case IV/c,
w of Case V/b.

b) $0<z<\infty$.
The transmuted (4.26) is

(4.30) $$\frac{x+z-2xz}{1-x-z+xz+|x+z-2xz|}=y.$$

i) $x+z-2xz\leq 0$.
The transmuted (4.30) is

(4.31) $$\frac{x+z-2xz}{1-2x-2z+3xz}=y \Leftrightarrow z=\frac{y-x-2xy}{1-2x+2y-3xy}=\frac{y-x-2|x|y}{1-2|x|+2y(sgnx)-3xy}.$$

This gives the solutions
w_2 of Cases I/a and I/b, w_1 of Case I/e,
w_2 of Cases II/a and II/b, w_1 of Case II/e ,
w_2 of Cases III/b and Case III/c,
w_2 of Case IV/c.
ii) $x+z-2xz\geq 0$.
The transmuted (4.30) is

(4.32) $$\frac{x+z-2xz}{1-xz}=y \Leftrightarrow z=\frac{y-x}{1-2x+xy}=\frac{y-x}{1-2|x|+xy}$$

This gives the solutions
w_1 of Case I/a , w_2 of Case of I/c, w of Case I/d and w_2 of Case I/e,
w_1 of Case II/a, w_2 of Case II/c, w of Case II/d,w_2 of Case II/e,
w of Case III/a, w_1 of Case III/b, w_2 of Case III/d, w of Case III/e,
w of Case IV/a, w_2 of Case IV/d, w of Case IV/e,
w of Case V/a, w_2 of Case V/d, w of Case V/e.
Finally, in the Cases I/a , I/b , I/c , I/e , II/a , II/b , II/c . II/e , III/a, III/b , III/c , III/d , IV/a , IV/b, IV/c, IV/d, V/a , V/b , V/c abd V/d we may apply Definition 4.7. So, the proof of Theorem 4.16 is complete. ■
Istennek Hála! 2020. február 27. (csütörtök) 15.10, Szalay István

***Remark* 4.33:** For any given exploded numbers u and v, Theorems 4.16, 3.28 (case $v=0$) and 3.2 (case $v=u$) give the full solution of the linear equation (4.17).

***Remark* 4.34:** Considering the linear equation (4.17) we can see that in Cases I/d, II/d, III/e, IV/e and V/e the solution w is not $v-u$ but in the cases only one of u and v is a real number the other is invisible exploded number.

***Remark* 4.35:** If u and v are real numbers then the traditional interpretation of „difference" remains: „$v-u$" is the real number that we add to u give v.

Definition* 4.36:** If $w\neq v-u$ but adding u to give v, then it is called the **pseudo – difference of the ordered pair *v* , *u. In the case of $v=u$ for $\boldsymbol{w}\neq\mathbf{0}$ we can use the terms **pseudo – zero for *u*** (see Remark 3.21) and in the case $v=0$ for $\boldsymbol{w}\neq-\boldsymbol{u}$ the **pseudo – additive inverse of *u*** (see Remark. 3.39).■

For example, the real number $u=2$ has two addition - value partners (see Definition 3.23). One of them is $w_1=0$ and one $w_2=\overbrace{\left(\frac{8}{3}\right)}$ is the pseudo – zero for 2 (see Theorem 3.2.B). So, the pseudo -difference of the pair 2 , 2 is the invisible exploded number $\overbrace{\left(\frac{8}{3}\right)}\neq 0$ because by the extended addition (see Definition 2.10) $2+\overbrace{\left(\frac{8}{3}\right)}=2$. On the other hand, the real number $u=2$ has two nullifying partners (see Definition 3.41). One of them is $w_1=-2$ and one $w_2=\overbrace{2}$ is the pseudo – additive inverse of 2 (see Theorem 3.28, second statement). So, the pseudo - difference of the pair 0 , 2 is the invisible exploded number $\overbrace{2}\neq -2$ because by the extended addition (see Definition 2.10) $2+\overbrace{2}=0$.

5. TRADITIONAL DIVISION IN ALGEBRA $\left(\overbrace{\mathbb{R}}\,,\,\leq,+\,,\,\cdot\right)$

In [1] we have already mentioned the identity

$$\frac{u}{v}=\left(u\mathcal{M}\left(\overbrace{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\overbrace{sgn}\left(v\mathcal{M}\left(\overbrace{1}\,\mathcal{S}|u|\right)\right)\right)\mathcal{D}\left(\left|u\mathcal{M}\left(\overbrace{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\mathcal{M}\left(\overbrace{1}\,\mathcal{S}|u|\right)\right|\right)$$

for any pair $u\,,v\neq 0$ of real numbers. (See, [1], Theorem 2.7.)

***Definition* 5.1:** Assuming that the exploded number $v \neq 0$, for any $u \in \overset{\frown}{\mathbb{R}}$ and $|u| \neq \overset{\frown}{1}$ (among others $\frac{\overset{\frown}{1}}{\overset{\frown}{1}}, \frac{\overset{\frown}{-1}}{\overset{\frown}{1}}, \frac{\overset{\frown}{-1}}{\overset{\frown}{-1}}, \frac{\overset{\frown}{1}}{\overset{\frown}{-1}}$ are forbidden symbols) we define

$$\frac{u}{v} = \left(u\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\overset{\frown}{sgn}\left(v\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\right)\right)\mathcal{D}\left(\left|u\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\right|\right). \blacksquare$$

***Remark* 5.2:** If $u \in \mathbb{R}$, *then* $\overset{\frown}{1}\,\mathcal{S}|u| > 0$, we may use

(5.2) $\quad \frac{u}{v} = \left(u\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\overset{\frown}{sgn}\,v\right)\mathcal{D}\left(\left|u\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\right|\right).$

***Exercise* 5.3:** If $u = 1$ then for any $v(\neq 0) \in \mathbb{R}$, by the definition of super – function $\overset{\frown}{sign}$, using Lemma 1.15 and applying the formulas (5.2),(1.1),(1.33) and (1.4) the esteemed reader can prove that $\left(1\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\overset{\frown}{sgn}\ v\right)\mathcal{D}\left(\left|1\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\,\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}\,1\right)\right|\right) = \frac{1}{v}$.

If $u = 1$ then for any $|v| > \overset{\frown}{1}$ Definition 5.1 yields

$$\frac{1}{v} = \left(1\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\overset{\frown}{sgn}\left(v\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|1|\right)\right)\right)\mathcal{D}\left(\left|1\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|1|\right)\right|\right) =$$

$$= \overbrace{\left(\frac{\left(1 - \left|\underset{\smile}{v}\right|\right) sgn\ \underset{\smile}{v}}{2\left|\underset{\smile}{v}\right| - 1}\right)} = \frac{1}{\underset{\smile}{v}} - sgn\ \underset{\smile}{v}. \blacksquare$$

Casting a glance at (2.13) we get **if $|v| > \overset{\frown}{1}$** then the pair $\left(v, \frac{1}{v}\right) \in \mathbb{C}_{\mathcal{M}}$, that is v and $\frac{1}{v}$ are multiplication – incompetent partners, so, their **extended product $v \cdot \frac{1}{v}$ is undefined**. (See Definition 2.10.) This means that if $|u| > \overset{\frown}{1}$ then $\frac{1}{u}$ is not an unifying partner for u (see Definition 3.66) but it has the unifying partner $\left(sgn\ \underset{\smile}{u}\right)\frac{1-\left|\underset{\smile}{u}\right|}{3\left|\underset{\smile}{u}\right|-1}$. (See Theorem 3.55, Case C.)

For the forbidden symbol $\frac{\overset{\frown}{1}}{\overset{\frown}{1}}$ see [1], (3.12). Of course, the symbol $\frac{0}{0}$ is forbidden, too.

The most important novelties given by Definition 5.1 are

(5.4) $\quad \frac{u}{\overset{\frown}{1}} = 0 \quad , \ u \in \overset{\frown}{\mathbb{R}}, \quad |u| \neq \overset{\frown}{1}, \qquad$ (see, [1], (3.14))

and

(5.5) $\quad \frac{u}{\overset{\frown}{-1}} = 0 \quad , \ u \in \overset{\frown}{\mathbb{R}}, \quad |u| \neq \overset{\frown}{1}.$

[By (5.2) we mention the following (apocryphal) extensions which is out of Definition 5.1.

(5.6) $\quad \frac{\overset{\frown}{1}}{u} =^{def} \begin{cases} \overset{\frown}{1} \ if \ \ u < \overset{\frown}{-1} \\ \overset{\frown}{-1} \ \ if \ \ \overset{\frown}{-1} < u < 0 \\ \overset{\frown}{1} \ \ if \ \ 0 < u < \overset{\frown}{1} \\ \overset{\frown}{-1} \ if \ \ \overset{\frown}{1} < u \end{cases}$, (see, [1], (3.13))

and

(5.7) $\quad \frac{\overset{\frown}{-1}}{u} =^{def} \begin{cases} \overset{\frown}{-1} \ if \ \ u < \overset{\frown}{-1} \\ \overset{\frown}{1} \ \ \ if \ \ \overset{\frown}{-1} < u < 0 \\ \overset{\frown}{-1} \ \ if \ \ 0 < u < \overset{\frown}{1} \\ \overset{\frown}{1} \ if \ \ \overset{\frown}{1} < u \end{cases}$.]

Having Definition 5.1 the esteemed reader can prove that

(5.8) $\quad \frac{u}{u} = 1, \ u \in \overset{\frown}{\mathbb{R}} \ \ but \ \ u \neq 0 \ \ and \ |u| \neq \overset{\frown}{1},$

is obtained. On the other hand having that

(5.9) $\quad \frac{u}{1} = \begin{cases} u & if \ \ u \in \mathbb{R} \\ \frac{-\underset{\smile}{u}}{\left|\underset{\smile}{u}\right|-1} = & if \ \ |u| > \overset{\frown}{1} \end{cases}$

holds. (If $|u| > \overset{\frown}{1}$ then $\overbrace{\left(\frac{-\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)} = \left(\left(sgn\ \frac{-\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)\frac{\left\{\left|\frac{-\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right|\right\}}{1-\left\{\left|\frac{-\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right|\right\}}, 0\right) = \left(-sgn\ \underset{\smile}{u}\right)\frac{\frac{\left|\underset{\smile}{u}\right|}{2\left|\underset{\smile}{u}\right|-1}}{1-\frac{\left|\underset{\smile}{u}\right|}{2\left|\underset{\smile}{u}\right|-1}} = \frac{-\left|\underset{\smile}{u}\right|}{\left|\underset{\smile}{u}\right|-1}$ is used.)

Moreover, by Definition 4.1 for the invisible exploded numbers we give

Corollary 5.10. If $|u| > \overset{\frown}{1}$then

(5.11) $\quad u - \frac{u}{1} = 0.$

So, the real number $\frac{-u}{1}$ $(\neq u)$is one of nullifying partners, that is pseudo-additive inverse of u. (See Definitions 3.41 and 4.36.) We may refer to Theorem 3.28 where the second solution of the equation $u + v = 0$ is

$v_2 = \overbrace{\left(\frac{\underset{\smile}{u}}{2\left|\underset{\smile}{u}\right|-1}\right)} = \frac{\underset{\smile}{u}}{\left|\underset{\smile}{u}\right|-1} = \frac{-u}{1}$ (See (5.9.)

***Remark* 5.12:** In Excersise 5.3 we can see that if $|u| > \overbrace{1}$, then

$$\frac{1}{u} = \overbrace{\left(\frac{\left(1-\left|\underbrace{u}\right|\right) sgn\ \underbrace{u}}{2\left|\underbrace{u}\right|-1}\right)}.$$

By (5.9) we have

$$\frac{u}{1} = \overbrace{\left(\frac{-\underbrace{u}}{2\left|\underbrace{u}\right|-1}\right)}, \qquad |u| > \overbrace{1}\,.$$

Using Definition 2.17 the estemeed reader is able to prove the interesing identity

(5.13) $$\frac{1}{u}\cdot\frac{u}{1} = 1 \quad , \quad |u| > \overbrace{1}\,.$$

Clearly, if $u(\neq 0)$is a real number, that is $|u| < \overbrace{1}$, then u is an unifying partner for its reciprocial value $\frac{1}{u}$. Now, (5.13) shows that if $|u| > \overbrace{1}$ then $\frac{u}{1}\,(\neq u)$ is an unifying partner for $\frac{1}{u}$. (See Definition 3.66.) At the same time, if $|u| > \overbrace{1}$ then $\frac{1}{u}$ and u are multiplication – incompetent partners.

Istennek Hála, 2020-03-07.: 11.44, and 2020-04-21: 9.12. Sz.I.

***Remark* 5.14:**

An invisible exploded number may have a lot of new partners. For example the invisible exploded number $\overbrace{2}$ has seven traditional or funny partners:

- 0 is the traditional addition - value partner (see Property 3.1.)
- $\frac{8}{3}$ is another addition – value partner because $\overbrace{2} + \frac{8}{3} = \overbrace{2}$ (see Definition 3.23.)
- $2\ and\ \overbrace{-2}$ are nullifying partners of $\overbrace{2}$ because $\overbrace{2} + 2 = 0 = \overbrace{2} + \overbrace{-2}$ (see Definition 3.41). We seem to have found the contradiction $2 = \overbrace{-2}$. (Having that $2 = \overbrace{\left(\frac{2}{3}\right)}$, see [1], Theorem 1.2). It's just an illusion because in the third line of the necessary derivation

$$\begin{gathered}\overbrace{2} + 2 = \overbrace{2} + \overbrace{-2}\\ \overbrace{-2} + \left(\overbrace{2} + 2\right) = \overbrace{-2} + \left(\overbrace{2} + \overbrace{-2}\right)\\ \left(\overbrace{-2} + 2\right) + 2 = \left(\overbrace{-2} + \overbrace{2}\right) + \overbrace{-2}\\ 0 + 2 = 0 + \overbrace{-2}\\ 2 = \overbrace{-2}\end{gathered}$$

  unproven associativity is used. From here we say that **disregarding pairs for addition** the **algebra** $\left(\overbrace{\mathbb{R}}, +\right)$ **is a commutative (Abelian) grupoid vith unit element 0.** (See (2.5) and Fig. 2.7.)
- 1 is the traditional multiplication – value partner (see Property 3.43)
- $-\frac{1}{3}$ is another multiplication – value partner because $\overbrace{2}\cdot\left(-\frac{1}{3}\right) = \overbrace{2}$ (see Definition 3.48)
- $-\boldsymbol{\frac{1}{6}}$ is (unique) unifying partner because $\overbrace{2}\cdot\left(-\frac{1}{6}\right) = 1$ (see Definition 3.66)

Turning towards the real numbers: In the case of $\frac{1}{4}$, the real number 4 and the invisible exploded number $\overbrace{-4}$ are unifying partners of $\frac{1}{4}$, (see Theorem 3.55, Case A) because $\frac{1}{4}\cdot 4 = 1 = \frac{1}{4}\cdot\overbrace{-4}$. We seem to have found the contradiction $4 = \overbrace{-4}$. (Having that $4 = \overbrace{\left(\frac{4}{5}\right)}$, see [1], Theorem 1.2). It's just an illusion because in the third line of the necessary derivation

$$\begin{gathered}\frac{1}{4}\cdot 4 = \frac{1}{4}\cdot\overbrace{-4}\\ 4\cdot\left(\frac{1}{4}\cdot 4\right) = 4\cdot\left(\frac{1}{4}\cdot\overbrace{-4}\right)\\ \left(4\cdot\frac{1}{4}\right)\cdot 4 = \left(4\cdot\frac{1}{4}\right)\cdot\overbrace{-4}\\ 1\cdot 4 = 1\cdot\overbrace{-4}\\ 4 = \overbrace{-4}\end{gathered}$$

unproven associativity is used. From here we say that **disregarding pairs for multiplication** the **algebra** $\left(\overbrace{\mathbb{R}}\,,\cdot\right)$ **is a commutative grupoid vith unit element 1.** (See (2.13) and Fig. 2.15.) ■

***Example* 5.15:** (Continuation of Exercise 5.3) Having that for any $u(\neq 0) \in \overset{\frown}{\mathbb{R}}$ we get that $\left|\frac{\left(1-\left|\underset{\smile}{u}\right|\right)\cdot sgn\ \underset{\smile}{u}}{\left|1-\left|\underset{\smile}{u}\right|\right|+\left|\underset{\smile}{u}\right|}\right| = \frac{\left|1-\left|\underset{\smile}{u}\right|\right|}{\left|1-\left|\underset{\smile}{u}\right|\right|+\left|\underset{\smile}{u}\right|} < 1$ is obtained and assumed that $|u| \neq \overset{\frown}{1}$, by the explosion formula (1.1) we compute

$$\left|\overbrace{\frac{\left(1-\left|\underset{\smile}{u}\right|\right)\cdot sgn\ \underset{\smile}{u}}{\left|1-\left|\underset{\smile}{u}\right|\right|+\left|\underset{\smile}{u}\right|}}\right| = \left(sgn\ \frac{\left(1-\left|\underset{\smile}{u}\right|\right)\cdot sgn\ \underset{\smile}{u}}{\left|1-\left|\underset{\smile}{u}\right|\right|+\left|\underset{\smile}{u}\right|}\right)\frac{\left|1-\left|\underset{\smile}{u}\right|\right|}{\left|\underset{\smile}{u}\right|} = \left(sgn\ \underset{\smile}{u}\right)\cdot\left(sgn\ \left|1-\left|\underset{\smile}{u}\right|\right|\right)\cdot\frac{\left|1-\left|\underset{\smile}{u}\right|\right|}{\left|\underset{\smile}{u}\right|} = \frac{1-\left|\underset{\smile}{u}\right|}{\underset{\smile}{u}} =$$

$$= \begin{cases} \dfrac{1-\underset{\smile}{u}}{\underset{\smile}{u}} & if\ u > 0 \\ \dfrac{1+\underset{\smile}{u}}{\underset{\smile}{u}} & if\ u < 0 \end{cases} = \begin{cases} \dfrac{1}{\underset{\smile}{u}} + 1 & if\ u < -\overset{\frown}{1} \\ \dfrac{1}{\underset{\smile}{u}} & if\ -\overset{\frown}{1} < u(\neq 0) < \overset{\frown}{1} \\ \dfrac{1}{\underset{\smile}{u}} - 1 & if\ u > \overset{\frown}{1} \end{cases}. \blacksquare$$

***Remark* 5.16:** If $|u| > \overset{\frown}{1}$ then u and $\frac{1}{u}$ are multiplication – incompetent partners (see Definition 2.11). Really, by (5.2) having $\frac{1}{u} = \left(1\ \mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\mathcal{M}\ \widetilde{sgn}\ v\right)\mathcal{D}\left(\left|1\ \mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\ \mathcal{M}\left(\overset{\frown}{1}\,\mathcal{S}\ 1\right)\right|\right)$ and denoting $x = \underset{\smile}{u}\ and$ $y = \underset{\smile}{\left(\frac{1}{u}\right)} = \frac{\left(1-\left|\underset{\smile}{u}\right|\right)\cdot sgn\ \underset{\smile}{u}}{\left|1-\left|\underset{\smile}{u}\right|\right|+\left|\underset{\smile}{u}\right|}$, moreover, considering (2.14) we have $|x| > 1$, so

$$1 - |x| - |y| + 2\cdot|x|\cdot|y| = 1 - |x| - \frac{\left|1-|x|\right|}{\left|1-|x|\right|+|x|} + 2\cdot|x|\cdot\frac{\left|1-|x|\right|}{\left|1-|x|\right|+|x|} = 0.$$

Moreover, If $|u| > \overset{\frown}{1}$ then Theorem 3.55 Case C shows that u has the unifying partner (see Definition 3.66) $\left(\widetilde{sgn}\,u\right)\mathcal{M}\left(\left(\overset{\frown}{1}\,\mathcal{S}|u|\right)\mathcal{D}\left(\overset{\frown}{4}\,\mathcal{M}|u|\ \mathcal{S}\,\overset{\frown}{1}\right)\right) = \overbrace{\left(\frac{\left(1-\left|\underset{\smile}{u}\right|\right)\cdot sgn\ \underset{\smile}{u}}{4\left|\underset{\smile}{u}\right|-1}\right)}$. For example $-\frac{1}{6}$ is unifying partner for $\overset{\frown}{2}$, while $\underset{\smile}{\frac{1}{2}} = -\frac{1}{2}$.
■

***Theorem* 5.17:** (It is important for the concept of „quotient".)
Let $u \neq 0$ and $|v| \neq 1$ be arbitrarily given exploded numbers such that $|u| \neq \overset{\frown}{1}, |v| \neq \overset{\frown}{1}\ and\ u \neq v$. Considering the equation

(5.18) $$u \cdot w = v$$

Cases and solutions are

Case I/a) $-\infty < \underset{\smile}{u} < -1\ \ and\ -\infty < \underset{\smile}{v} < -1$ or $1 < \underset{\smile}{u} < \infty\ \ and\ 1 < \underset{\smile}{v} < \infty$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case I/b) $-\infty < \underset{\smile}{u} < -1\ \ and\ -1 < \underset{\smile}{v} < \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}}$ or $1 < \underset{\smile}{u} < \infty\ \ and\ \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}} < \underset{\smile}{v} < 1$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case I/c) $-\infty < \underset{\smile}{u} < -1\ \ and\ \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}} \leq \underset{\smile}{v} < 0$ or $1 < \underset{\smile}{u} < \infty\ \ and\ 0 < \underset{\smile}{v} \leq \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}}$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$

Case I/d) $-\infty < \underset{\smile}{u} < -1\ \ and\ 0 < \underset{\smile}{v} \leq \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}}$ or $1 < \underset{\smile}{u} < \infty\ \ and\ \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}} \leq \underset{\smile}{v} < 0$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$

Case I/e) $-\infty < \underset{\smile}{u} < -1\ \ and\ \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}} < \underset{\smile}{v} < 1$ or $1 < \underset{\smile}{u} < \infty\ \ and\ -1 < \underset{\smile}{v} < \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\ \underset{\smile}{u}}$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case I/f) $-\infty < \underset{\smile}{u} < -1$ and $1 < \underset{\smile}{v} < \infty$ or $1 < \underset{\smile}{u} < \infty$ and $-\infty < \underset{\smile}{v} < -1$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case II/a) $-1 < \underset{\smile}{u} < -\frac{1}{2}$ and $-\infty < \underset{\smile}{v} \le \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$ or $\frac{1}{2} < \underset{\smile}{u} < 1$ and $\frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} \le \underset{\smile}{v} < \infty$

there is no solution of equation (5.18)

Case II/b) $-1 < \underset{\smile}{u} < -\frac{1}{2}$ and $\frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} < \underset{\smile}{v} < -1$ or $\frac{1}{2} < \underset{\smile}{u} < 1$ and $1 < \underset{\smile}{v} < \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$

Case II/c) $-1 < \underset{\smile}{u} < -\frac{1}{2}$ and $-1 < \underset{\smile}{v} < 0$ or $\frac{1}{2} < \underset{\smile}{u} < 1$ and $0 < \underset{\smile}{v} < 1$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} = \frac{v}{u}$

Case II/d) $-1 < \underset{\smile}{u} < -\frac{1}{2}$ and $0 < \underset{\smile}{v} < 1$ or $\frac{1}{2} < \underset{\smile}{u} < 1$ and $-1 < \underset{\smile}{v} < 0$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} = \frac{v}{u}$

Case II/e) $-1 < \underset{\smile}{u} < -\frac{1}{2}$ and $1 < \underset{\smile}{v} < \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$ or $\frac{1}{2} < \underset{\smile}{u} < 1$ and $\frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} < \underset{\smile}{v} < -1$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$

Case II/f) $-1 < \underset{\smile}{u} < -\frac{1}{2}$ and $\frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} \le \underset{\smile}{v} < \infty$ or $\frac{1}{2} < \underset{\smile}{u} < 1$ and $-\infty < \underset{\smile}{v} \le \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$

there is no solution of equation (5.18)

Case III/a) $-\frac{1}{2} < \underset{\smile}{u} \le -\frac{1}{3}$ and $-\infty < \underset{\smile}{v} < \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$ or $\frac{1}{3} \le \underset{\smile}{u} < \frac{1}{2}$ and $\frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} < \underset{\smile}{v} < \infty$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case III/b) $-\frac{1}{2} < \underset{\smile}{u} < -\frac{1}{3}$ and $\frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} \le \underset{\smile}{v} < -1$ or $\frac{1}{3} < \underset{\smile}{u} < \frac{1}{2}$ and $1 < \underset{\smile}{v} \le \frac{-\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$

Case III/c) $-\frac{1}{2} < \underset{\smile}{u} \le -\frac{1}{3}$ and $-1 < \underset{\smile}{v} < 0$ or $\frac{1}{3} \le \underset{\smile}{u} < \frac{1}{2}$ and $0 < \underset{\smile}{v} < 1$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} = \frac{v}{u}$

Case III/d) $-\frac{1}{2} < \underset{\smile}{u} \le -\frac{1}{3}$ and $0 < \underset{\smile}{v} < 1$ or $\frac{1}{3} \le \underset{\smile}{u} < \frac{1}{2}$ and $-1 < \underset{\smile}{v} < 0$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} = \frac{v}{u}$

Case III/e) $-\frac{1}{2} < \underset{\smile}{u} < -\frac{1}{3}$ and $1 < \underset{\smile}{v} \le \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$ or $\frac{1}{3} < \underset{\smile}{u} < \frac{1}{2}$ and $\frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} \le \underset{\smile}{v} < -1$

there exist only one solution $w = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$

Case III/f) $-\frac{1}{2} < \underset{\smile}{u} \le -\frac{1}{3}$ and $\frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}} < \underset{\smile}{v} < \infty$ or $\frac{1}{3} \le \underset{\smile}{u} < \frac{1}{2}$ and $-\infty < \underset{\smile}{v} < \frac{\left|\underset{\smile}{u}\right|}{2\underset{\smile}{u}-sgn\,\underset{\smile}{u}}$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case IV/a) $-\frac{1}{3} \le \underset{\smile}{u} < 0$ and $-\infty < \underset{\smile}{v} < -1$ or $0 < \underset{\smile}{u} \le \frac{1}{3}$ and $1 < \underset{\smile}{v} < \infty$

$$w_1 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)} \text{ and } w_2 = \overbrace{\left(\frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}\right)}$$

Case IV/b) $-\frac{1}{3}<\underbrace{u}<0\ and\ -1<\underbrace{v}<\frac{-|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}$ or $0<\underbrace{u}<\frac{1}{3}\ and\ \frac{-|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}<\underbrace{v}<1$

$$w_1=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}+\underbrace{v}-2|\underbrace{u}|\underbrace{v}}\right)}=\frac{v}{u} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}-\underbrace{v}+2|\underbrace{u}|\underbrace{v}}\right)}$$

Case IV/c) $-\frac{1}{3}<\underbrace{u}<0\ and\ \frac{-|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}\le\underbrace{v}<0$ or $0<\underbrace{u}<\frac{1}{3}\ and\ 0<\underbrace{v}\le\frac{-|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}$

there exist only one solution $w=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}+\underbrace{v}-2|\underbrace{u}|\underbrace{v}}\right)}=\frac{v}{u}$

Case IV/d) $-\frac{1}{3}<\underbrace{u}<0\ and\ 0<\underbrace{v}\le\frac{|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}$ or $0<\underbrace{u}<\frac{1}{3}\ and\ \frac{|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}\le\underbrace{v}<0$

there exist only one solution $w=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}-\underbrace{v}+2|\underbrace{u}|\underbrace{v}}\right)}=\frac{v}{u}$

Case IV/e) $-\frac{1}{3}<\underbrace{u}<0\ and\ \frac{|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}<\underbrace{v}<1$ or $0<\underbrace{u}<\frac{1}{3}\ and -1<\underbrace{v}<\frac{|\underbrace{u}|}{2\underbrace{u}-sgn\,\underbrace{u}}$

$$w_1=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}-\underbrace{v}+2|\underbrace{u}|\underbrace{v}}\right)}=\frac{v}{u} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}+\underbrace{v}-2|\underbrace{u}|\underbrace{v}}\right)}$$

Case IV/f) $-\frac{1}{3}\le\underbrace{u}<0\ and\ 1<\underbrace{v}<\infty$ or $0<\underbrace{u}\le\frac{1}{3}\ and -\infty<\underbrace{v}<-1$

$$w_1=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}-\underbrace{v}+2|\underbrace{u}|\underbrace{v}}\right)} \text{ and } w_2=\overbrace{\left(\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}+\underbrace{v}-2|\underbrace{u}|\underbrace{v}}\right)}$$

Proof:

Considering Definition 2.17 we transcribe the linear equation (5.18) by super- operations

$$(u\mathcal{M}w)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|w|\mathcal{A}\,\overset{\frown}{2}\,\mathcal{M}|u\mathcal{M}w|\right)=v$$

where we assume that the exploded numbers *u* and *w* are multiplication – competent partners. Using (1.7), (1.8) , (1.21) and (1.22) with (1.5) , (1.6), (1.32), (1.33) and (1.35) we have to solve equation

$$\frac{\underbrace{u}\cdot\underbrace{w}}{1-|\underbrace{u}|-|\underbrace{w}|+2\cdot|\underbrace{u}|\cdot|\underbrace{w}|}=\underbrace{v}\quad ,where\ \underbrace{u}\ne 0\,,|\underbrace{v}|\ne\frac{1}{2}\,,|\underbrace{u}|\ne 1\ and\ |\underbrace{v}|\ne 1$$

Denoting $\underbrace{u}=x\,,\underbrace{v}=y\ and\ \underbrace{w}=z$ and using the inversion formulas we solve equation

(5.19) $\frac{xz}{1-|x|-|z|+2\cdot|x|\cdot|z|}=y$, where $x\ne 0\,,|y|\ne\frac{1}{2}\,,|x|\ne 1\ and\ |y|\ne 1$

such that

(5.20) $1-|x|-|z|+2\cdot|x|\cdot|z|\ne 0.$

We analyse (5.19) in the following four cases

Case A) $x<0\ \ and\ \ z\ge 0$

The equation (5.19) has the form $\frac{xz}{1+x-z-2xz}=y$ and

(5.21) $z_A^+=\frac{y(1+x)}{x+y+2xy}=\frac{\underbrace{v}\left(1-|\underbrace{u}|\right)}{\underbrace{u}+\underbrace{v}-2|\underbrace{u}|\underbrace{v}}$

is obtained. Moreover we can compute that $1-|x|-|z_A^+|+2\cdot|x|\cdot|z_A^+|=\frac{x(1+x)}{x+y+2xy}$. Given the $x+y+2xy\ne 0$ requirements (5.21) gives the following solutions to the left of the following cases

w_1 for Case I/a , Case I/b , Case I/e , Case I/f and w for Case I/d
w for Case II/b , II/c
w_1 for Case III/a , III/f and w for Case III/b , III/c
w_1 for Case IV/a , IV/b and w for IV/c and w_2 for IV/e , IV/f.

Case B) $x<0\ \ and\ \ z\le 0$

The equation (5.19) has the form $\frac{xz}{1+x+z+2xz}=y$ and

(5.22) $$z_B^- = \frac{y(1+x)}{x-y-2xy} = \frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}$$

is obtained. Moreover we can compute that $1-|x|-|z_B^-|+2\cdot|x|\cdot|z_B^-| = \frac{x(1+x)}{x-y-2xy}$. Given the $x-y-2xy \neq 0$ requirements (5.22) gives the following solutions to the left of the following cases

w_2 for Case I/a , Case I/b , Case I/e , Case I/f and w for Case I/c
w for Case II/d , II/e
w_2 for Case III/a , III/f and w for Case III/d , III/e
w_2 for Case IV/a , IV/b and w for IV/d and w_1 for IV/e , IV/f.

Case C) $x > 0 \ \ and \ \ z \geq 0$

The equation (5.19) has the form $\frac{xz}{1-x-z+2xz} = y$ and

(5.23) $$z_C^+ = \frac{y(1-x)}{x+y-2xy} = \frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}+\underset{\smile}{v}-2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}$$

is obtained. Moreover we can compute that $1-|x|-|z_C^+|+2\cdot|x|\cdot|z_C^+| = \frac{x(1-x)}{x+y-2xy}$. Given the $x+y-2xy \neq 0$ requirements (5.23) gives the following solutions to the left of the following cases

w_1 for Case I/a , Case I/b , Case I/e , Case I/f and w for Case I/d
w for Case II/b , II/c
w_1 for Case III/a , III/f and w for Case III/b , III/c
w_1 for Case IV/a , IV/b and w for IV/c and w_2 for IV/e , IV/f.

Case D) $x > 0 \ \ and \ \ z \leq 0$
The equation (5.19) has the form $\frac{xz}{1-x+z-2xz} = y$ and

(5.24) $$z_D^- = \frac{y(1-x)}{x-y+2xy} = \frac{\underset{\smile}{v}\left(1-\left|\underset{\smile}{u}\right|\right)}{\underset{\smile}{u}-\underset{\smile}{v}+2\left|\underset{\smile}{u}\right|\underset{\smile}{v}}$$

is obtained. Moreover we can compute that $1-|x|-|z_D^-|+2\cdot|x|\cdot|z_D^-| = \frac{x(1-x)}{x-y+2xy}$. Given the $x-y+2xy \neq 0$ requirements (5.24) gives the following solutions to the left of the following cases

w_2 for Case I/a , Case I/b , Case I/e , Case I/f and w for Case I/c
w for Case II/d , II/e
w_2 for Case III/a , III/f and w for Case III/d , III/e
w_2 for Case IV/a , IV/b and w for IV/d and w_1 for IV/e , IV/f.
Considering Case II/c, II/d, III/c, III/d, IV/c and IV/d we apply Definition 5.1, too.

Finally, we investigate Case II/a, II/f.
To the left-side of II/a and II/f.
Strarting from (5.19) we state that $z = 0$ is not solution of the equation (5.18).
For $z < 0$ equation $\frac{xz}{1+x+z+2xz} = y$ is obtained. Hence, $z = \frac{y(1+x))}{x-y-2xy} > 0$ is a contradiction.
For $z > 0$ equation $\frac{xz}{1+x-z-2xz} = y$ is obtained. Hence, $z = \frac{y(1+x))}{x+y+2xy} < 0$ is a contradiction.

So, in these cases the equation (5.18) has not solution.
To the right-side of II/a and II/f.
Strarting from (5.19) we state that $z = 0$ is not solution of the equation.
For $z < 0$ equation $\frac{xz}{1-x+z-2xz} = y$ is obtained. Hence, $z = \frac{y(1-x))}{x-y+2xy} > 0$ is a contradiction.
For $z > 0$ equation $\frac{xz}{1-x-z+2xz} = y$ is obtained. Hence, $z = \frac{y(1-x))}{x+y-2xy} < 0$ is a contradiction.

So, in these cases the equation (5.18) has not solution. ■

***Remark* 5.25:** By Definition 2.17 we can see immedialely that, if $u = 0 \ \ and \ \ v = 0$ then every exploded number w exception of $|w| = \overset{\frown}{1}$ is a solution of equation (5.18). (See Remark 2.16, point **4**.) If $u = 0$ but $v \neq 0$ then equation (5.18) is insoluble. For $|u| = \overset{\frown}{1}$ or $|v| = \overset{\frown}{1}$ we may use Property 3.49. If $u = v$ or $v = 1$ then Theorems 3.44 and 3.55 are valid, respectively. For $v = -1$ we give the following theorem without any proof.

***Theorem* 5.26:** Let u be an arbitrarily given exploded number such that $u \neq 0$ and $|u| \neq \overbrace{1}$. The solution of the equation

$$u \cdot v = -1$$

Case A) If $0 < |u| < \frac{1}{3}$ then solutions are

$$v_1 == -\frac{1}{u} \qquad \text{and} \qquad v_2 = \overbrace{\left(\frac{|\underbrace{u}|-1}{4\underbrace{u}-sgn\,\underbrace{u}}\right)} = \overbrace{\left(\frac{(sgn\ \underbrace{u})(|\underbrace{u}|-1)}{4|\underbrace{u}|-1}\right)} = \overbrace{\left(\frac{\underbrace{u}-sgn\ \underbrace{u}}{4|\underbrace{u}|-1}\right)}$$

Case B) If $\frac{1}{3} \leq |u| < \overbrace{1}$ then there exists only one solution $v = -\frac{1}{u}$.

Case C) If $|u| > \overbrace{1}$ then there exists only one solution $v = \overbrace{\left(\frac{|\underbrace{u}|-1}{4\underbrace{u}-sgn\,\underbrace{u}}\right)} = \left(sgn\ \underbrace{u}\right)\frac{|\underbrace{u}|-1}{3|\underbrace{u}|-2}$.

By Remark 5.25 and Theorem 5.26 we have the full solution of the linear equation (5.18).■

***Remark* 5.27:** Observing Cases II/c , II/d , III/c , III/d , IV/c and IV/d we can see that u and v are real numbers and the unique solution of equation (5.18) the traditional $w = \frac{v}{u}$ is obtained. In Cases IV/b and IV/e despite of that u and v are real numbers we get two solutions. One them is the traditional $w_1 = \frac{v}{u}$ while the second is the invisible number w_2. For example if $u = \frac{1}{4}\ and\ v = 3$ (then $\underbrace{u} = \frac{1}{5}$ vith $\frac{-|\underbrace{u}|}{2\underbrace{u}-sgn\ \underbrace{u}} = \frac{1}{3}$ and $\underbrace{v} = \frac{3}{4}$, so we use Case IV/b) then the solutions of equation $\frac{1}{4} \cdot w = 3$ are the traditional $w_1 = 12$ and the invisible exploded number $w_2 = \overbrace{\left(-\frac{12}{5}\right)} = \left(-\frac{2}{3}, -2\right)$. (See the exploder formula (1.1).)

***Remark* 5.28:** If $v \neq 0$ and v are real numbers then the traditional interpretation of „quotient" remains: „$\frac{v}{u}$" is the real number if multiplied by u gives v.

***Definition* 5.29:** If $w \neq \frac{v}{u}$, $\left(u \neq 0, |u| \neq \overbrace{1}\right)$ but multiplied by u to gives v, then it is called the **pseudo –quotient of the pair v , u**. The special case of ordered pair $v = 1$, $u \neq 1$ was already mentioned in Definition 3.66 as pseudo reciprocial value for *u.* (In the special case of pair 1 , 1 identity (5.8) and Theorem 3.44, A) show that their pseudo reciprocical value does not exist.)

For example, in the case of pair $v = 3\ and\ u = \frac{1}{4}$ the traditional quotient $\frac{v}{u} = 12$, while the pseudo-quotient is $\overbrace{\left(-\frac{12}{5}\right)}$, because $\overbrace{\left(-\frac{12}{5}\right)} \neq \frac{v}{u}$ but $\overbrace{\left(-\frac{12}{5}\right)} \cdot u = \overbrace{\left(-\frac{12}{5}\right)} \cdot \frac{1}{4} = \overbrace{\left(\frac{\frac{1}{5}\cdot\frac{-12}{5}}{1-\frac{1}{5}-\frac{12}{5}+2\cdot\frac{1}{5}\cdot\frac{12}{5}}\right)} = \overbrace{\left(\frac{3}{4}\right)} = 3 = v.$

In the case of $v = 1\ and\ u = \frac{1}{4}$ the traditional reciprocial value $\frac{1}{u} = 4$, while the pseudo reciprocial value is $\overbrace{(-4)}$, because $\overbrace{(-4)} \neq \frac{1}{u}$ but $\overbrace{(-4)} \cdot u = \overbrace{(-4)} \cdot \frac{1}{4} = \overbrace{\left(\frac{\frac{1}{5}\cdot(-4)}{1-\frac{1}{5}-4+2\cdot\frac{1}{5}\cdot 4}\right)} = \overbrace{\left(\frac{1}{2}\right)} = 1.$■

***Remark* 5.30:** You have to make friends with the facts that algebra $\left(\overbrace{\mathbb{R}}, +, \cdot, \leq\right)$ is much harder than normal algebra $(\mathbb{R}, +, \cdot, \leq)$. For example the rule „dividing by a real number ($\neq 0$) multipliying its reciprocal" invalidates. The rule

$$\frac{u}{\underbrace{u}} = u \cdot \frac{1}{\underbrace{u}} \quad , \quad u(\neq 0) \in \mathbb{R} \tag{5.30}$$

is obvious, but for exploded numbers it is just an equation. Let's solve it! If $u(\neq 0) \in \overbrace{\mathbb{R}}$ we use the Definitions 5.1 and 2.17, respectively. Denoting $\underbrace{u} = x$ and using the second inversion formula (1.6) we have that equation (5.30) is equivalent with the equation

$$\frac{\left(x\left(1-|\underbrace{x}|\right)\right)\cdot sgn\left(\underbrace{x}(1-|x|)\right)}{\left|x\left(1-|\underbrace{x}|\right)\right|+\left|\underbrace{x}(1-|x|)\right|} = \frac{x\cdot\underbrace{\left(\frac{1}{x}\right)}}{1-|x|-\left|\underbrace{\left(\frac{1}{x}\right)}\right|+2\left|x\cdot\underbrace{\left(\frac{1}{x}\right)}\right|}. \tag{5.31}$$

We consider three cases. First we assume that $|x| < 1$. Using that $sgn\left(\underbrace{x}(1-|x|)\right) = sgn\ \underbrace{x} = sgn\,x$, and (1.33) by Lemma 1.15 ($\left|\underbrace{x}\right| < |x|$) the equation (5.31) has the form

$$\frac{\left(x\left(1-|\underbrace{x}|\right)\right)\cdot sgn\ \underbrace{x}}{|x|\left(1-|\underbrace{x}|\right)+|\underbrace{x}|\,(1-|x|)} = \frac{x\cdot\underbrace{\left(\frac{1}{x}\right)}}{1-|x|-\left|\underbrace{\frac{1}{x}}\right|+2|x|\left|\underbrace{\frac{1}{x}}\right|}.$$

Applying the geometric compressor formula (1.4), $\underbrace{x} = \frac{|x|}{1+|x|}$ and $\underbrace{\left(\frac{1}{x}\right)} = \frac{1}{x}\cdot\frac{|x|}{1+|x|}$ are obtained and we get

$$\frac{sgn\ \underbrace{x}}{|x|(2-|x|)} = \frac{sgn\,x\,,}{|x|(2-|x|)}.$$

Because $|x| < 1$ we can see, that every x is a solution for (5.31), that is every reall number is a solution of the equation (5.30).
Second we assume $|x| = 1$. Now the left hand side of (5.30) is equal 0, but by its right hand side $x = 0$ which is excluded.
Finally, we assume $|x| > 1$, that is $|u| > \overbrace{1}$. Similarly to the first step we have

$$\frac{-sgn\ \underbrace{x}}{|x|(2-|x|)} = \frac{sgn\ x\ ,}{|x|(2-|x|)}.$$

Here, $|x|(2-|x|) \neq 0$ because $x \neq 0$ and $2-|x| \neq 0$ because if $|x| = 2$ then $\left|\underbrace{\left(\frac{1}{x}\right)}\right| = \frac{1}{1+|x|} = \frac{1}{3}$

$$1-|x|-\left|\underbrace{\left(\frac{1}{x}\right)}\right| + 2|x|\left|\underbrace{\left(\frac{1}{x}\right)}\right| = 1-2-\frac{1}{3}+2\cdot 2\cdot\frac{1}{3} = 0$$

so both $\left(2,\frac{1}{3}\right)$ and $\left(-2,-\frac{1}{3}\right) \in \underbrace{\mathbb{C}_M}$ (see (2.14) or Fig. 2.15) .Observing Definition 2.10 we can see that $\overbrace{2}$ and $\overbrace{\left(\frac{1}{3}\right)} = \frac{1}{2} = \frac{1}{\underbrace{(2)}}$ are multiplication – incompetent partners. Similarly $\overbrace{-2}$ with $\overbrace{\left(-\frac{1}{3}\right)} = -\frac{1}{2} = \frac{1}{\underbrace{(-2)}}$ are
are multiplication – incompetent partners. So, $\overbrace{2}\cdot\frac{1}{\underbrace{(2)}}$ and $\overbrace{-2}\cdot\frac{1}{\underbrace{(-2)}}$ are undetermined. On the other hand by
Definition 5.1 the surpisinng $\frac{\overbrace{2}}{\underbrace{(2)}} = 1$ and $\frac{\overbrace{-2}}{\underbrace{(-2)}} = 1$ are obtained. (They are funny results, but true.) Moreover, if $x \neq 0$ the equgality $-sgn\ \underbrace{x} = sgn\ x$ is impossible. Consequently, if $|u| > \overbrace{1}$ then equation (5.30) has no solution.

6. SOLUTION OF THE GENERAL LINEAR EQUATION

Without using the associativity of extended addition (see Definition 2.10) and multiplication (see Definition 2.10) we solve the linear equation

(6.1) $\quad u\xi + b = 0 \quad$ where $\quad u \neq 0\,, |u| \neq \overbrace{1}\;$ and $\;|b| \neq \overbrace{1}\,,$

on the set of exploded numbers. The *solution model* has two steps
Step 1.

Using Theorem 3.28 we solve the equation

(6.2) $\quad b + v = 0\ \ ,\ where\ \ |b| \neq \overbrace{1}\,.$

Step 2. Using one of Theorems 3. 55 or 5. 17 or 5.26 we solve the equation

(6.3) $\quad u\xi = v\quad$ where $\quad u \neq 0\,, |u| \neq \overbrace{1}\;$ and $\;|v| \neq \overbrace{1}\,.$

According to the solution method by the commutativity of extended addition ($b + u\xi = u\xi + b$) the **linear equation (6.1) has at least one and at most four solutions** on the set of exploded numbers.
There are two possible solutions both the equations (6.2) and (6.3) can be up to four solution of equation (6.1). Then consider the following exercises.
Exercise 6.4. Solve the linear equation $3\xi + \frac{1}{4} = 0$.
Step 1. For (6.2) we have a solution $v = -\frac{1}{4}$, only.
Step 2. For (6.3) by Theoem 5.17 (Case II/d, right hand side) gives the traditional $\xi = -\frac{1}{12}$.
So, the equation (6.1) has only one solution.
Exercise 6.5. Solve the linear equation $3\xi - 5 = 0$.
Step 1. For (6.2) we have two solutions $v_1 = 5$ and $v_2 = \overbrace{\left(-\frac{5}{4}\right)}$.
Step 2. For (6.3) we get the equation $3\xi = 5$ with the unique solution $\frac{5}{3}$ (see Theorem 5.17, Case II/c right) and the equation $3\xi = \overbrace{\left(-\frac{5}{4}\right)}$ with the unique solution $\xi_2 = \overbrace{\left(-\frac{5}{2}\right)}$ (see Theoem 5.17 , II/e right,)
So, the equation (6.1) has two solutions.
Exercise 6.6. Solve the linear equation $\overbrace{2}\,\xi + \overbrace{2} = 0$.
Step 1. For (6.2) we have two solutions $v_1 = -\overbrace{2} = \overbrace{-2}$ and $v_2 = 2$.
Step 2. For (6.3) we get the equation $\overbrace{2}\,\xi = \overbrace{-2}$ with two solutions $\xi_{11} = \frac{1}{3}$ and $\xi_{12} = -1$ (see Theorem 5.17, Case I/c right) and the equation $\overbrace{2}\,\xi = 2$ with the unique solution $\xi_2 = -\frac{1}{5}$ (see Theoem 5.17 , I/c right,)
So, the equation (6.1) has three solutions.
Exercise 6.7. Solve the linear equation $\overbrace{2}\,\xi + 3 = 0$.
Step 1. For (6.2) we have two solutions $v_1 = -3$ and $v_2 = \overbrace{\left(\frac{3}{2}\right)}$.

Step 2. For (6.3) we get the equation $\overbrace{2}\,\xi = -3$ with two solutions $\xi_{11} = \frac{3}{14}$ and $\xi_{12} = -3$ (see Theorem 5.17, Case I/e right)
and
the equation $\overbrace{2}\,\xi = \overbrace{\left(\frac{3}{2}\right)}$ with two solutions $\xi_{21} = \frac{3}{2}$ and $\xi_{22} = -\frac{3}{10}$ (see Theoem 5.1 , I/a right,)
So, the equation (6.1) has four solutions. All of them are real numbers.

Finally, we have yet to prove that the equation* (6.1*) always has a solution.
In Step 1. the equation (6.2) always has solution (one or two).
Case α) Be

(6.8) $$|b| \le 1 \quad \Leftrightarrow \quad \left|\underbrace{b}\right| \le \frac{1}{2}.$$

Now, (6.2) has the solution $v = -b$, only. Let us consider step 2.
If $|b| = 1$ then by Theorems 3.55 and 5.26 the equation $u\xi = -b$ (see (6.3)) can always be solved, so, the equation (6.1) has a solution. If

(6.9) $$|b| < 1 \quad \Leftrightarrow \quad \left|\underbrace{b}\right| < \frac{1}{2}$$

is fulfilled, then there is no solution to equation (6.3) if we get Theorem 5.17 for II/a and II/f. So, there is no solution to equation $u\xi = -b$ in the following cases:

II/a left $\quad -1 < \underbrace{u} < -\frac{1}{2}$ and $-\infty < -\underbrace{b} \le \frac{-\underbrace{u}}{2\underbrace{u}+1} < -1$,

II/a right $\quad \frac{1}{2} < \underbrace{u} < 1$ and $1 < \frac{\underbrace{u}}{2\underbrace{u}-1} \le -\underbrace{b} < \infty$,

II/f left $\quad -1 < \underbrace{u} < -\frac{1}{2}$ and $1 < \frac{\underbrace{u}}{2\underbrace{u}+1} \le -\underbrace{b} < \infty$

and

II/f right $\quad \frac{1}{2} < \underbrace{u} < 1$ and $-\infty < -\underbrace{b} \le \frac{-\underbrace{u}}{2\underbrace{u}-1} < -1$.

In all four cases we get a contradiction with the condition (6.9), so the equation (6.3) can be solved.
Summary for Case α: Under the condition (6.8) the equation (6.1) has a solution.
Case β)
Be

(6.10) $$|b| > 1 \quad \Leftrightarrow \quad \left|\underbrace{b}\right| > \frac{1}{2} \qquad \text{such that} \quad |b| \ne \overbrace{1}.$$

In Step 1. the equation (6.2) has two solutions $v_1 = -b$ and $v_2 = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$.
Considering Step 2. there is no solution to equation (6.3) if we get Theorem 5.17 for II/a and II/f.
We distindwish four cases.
We may use Theorem 5.17.

Case II/a left
First investigation. Considering the condition (6.10) we assume that

$$b < -1 \Leftrightarrow < \underbrace{b} < -\frac{1}{2}$$

and we turn towards the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$. It has not a solution if

$$-1 < \underbrace{u} < -\frac{1}{2} \quad \text{and} \quad -\infty < \frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1} \le \frac{-\underbrace{u}}{2\underbrace{u}+1} (< -1) \left(\Leftrightarrow \underbrace{b} > \underbrace{u}\right).$$

However, in this domain u and $(-b)$ are real numbers and the equation $u\xi = -b$ has a solution, so the eqation (6.1) has a solution.
Second investigation. Considering the condition (6.10) again, we assume that

$$b > 1 \Leftrightarrow \underbrace{b} > \frac{1}{2}$$

and we turn towards the equation $u\xi = -b$. It has not a solution if

$$-1 < \underbrace{u} < -\frac{1}{2} \quad \text{and} \quad -\infty < -b \le \frac{-\underbrace{u}}{2\underbrace{u}+1} (< -1) \left(\Leftrightarrow (1 <) \frac{\underbrace{u}}{2\underbrace{u}+1} < \underbrace{b} < \infty\right).$$

However, in this domain u and $\overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ are real numbers and the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ has a solution, so the eqation (6.1) has a solution.
Case II/a right
First investigation. Considering the condition (6.10) we assume that

$$b < -1 \Leftrightarrow \underbrace{b} < -\frac{1}{2}$$

and we turn towards the equation $u\xi = -b$. It has not a solution if

$$\frac{1}{2} < \underbrace{u} < 1 \quad \text{and} \quad \frac{\underbrace{u}}{2\underbrace{u}-1} \le -\underbrace{b} < \infty \left(\Leftrightarrow -\infty < \underbrace{b} \le \frac{-\underbrace{u}}{2\underbrace{u}-1} \right).$$

However, in this domain u and $\overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ are real numbers and the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ has a solution, so the eqation (6.1) has a solution.

Second investigation. Considering the condition (6.10) again, we assume that

$$b > 1 \Leftrightarrow \underbrace{b} > \frac{1}{2}$$

and we turn towards the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$. It has not a solution if

$$\frac{1}{2} < \underbrace{u} < 1 \quad \text{and} \quad (1 <)\frac{\underbrace{u}}{2\underbrace{u}-1} \le \frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1} < \infty \left(\Leftrightarrow \underbrace{b} < \underbrace{u} \right).$$

However, in this domain u and $(-b)$ are real numbers and the equation $u\xi = -b$ has a solution, so the eqation (6.1) has a solution.

Case II/f left

First investigation. Considering the condition (6.10) we assume that

$$b < -1 \Leftrightarrow \underbrace{b} < -\frac{1}{2}$$

and we turn towards the equation $u\xi = -\underbrace{b}$ It has not a solution if

$$-1 < \underbrace{u} < -\frac{1}{2} \quad \text{and} \; (1 <)\frac{\underbrace{u}}{2\underbrace{u}+1} \le -\underbrace{b} < \infty \left(\Leftrightarrow -\infty < \underbrace{b} \le \frac{-\underbrace{u}}{2\underbrace{u}+1} (< -1) \right).$$

However, in this domain u and $\overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ are real numbers and the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ has a solution, so the eqation (6.1) has a solution.

Second investigation. Considering the condition (6.10) again, we assume that

$$b > 1 \Leftrightarrow \underbrace{b} > \frac{1}{2}$$

and we turn $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ towards the equation. It has not a solution if

$$-1 < \underbrace{u} < -\frac{1}{2} \quad \text{and} \quad (1 <)\frac{\underbrace{u}}{2\underbrace{u}+1} \le \frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1} < \infty \left(\Leftrightarrow \underbrace{b} \le -\underbrace{u} \right).$$

However, in this domain u and $-\,b$ are real numbers and the equation $u\xi = -b$ has a solution, so the eqation (6.1) has a solution.

Case II/f right

First investigation. Considering the condition (6.10) we assume that

$$b < -1 \Leftrightarrow \underbrace{b} < -\frac{1}{2}$$

and we turn towards the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$. It has not a solution if

$$\frac{1}{2} < \underbrace{u} < 1 \quad \text{and} \quad -\infty < \frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1} \le \frac{-\underbrace{u}}{2\underbrace{u}-1} (< -1) \left(\Leftrightarrow \underbrace{b} \ge -\underbrace{u} \right).$$

However, in this domain u and $(-b)$ are real numbers and the equation $u\xi = -b$ has a solution, so the eqation (6.1) has a solution.

Second investigation. Considering the condition (6.10) again, we assume that

$$b > 1 \Leftrightarrow \underbrace{b} > \frac{1}{2}$$

and we turn $u\xi = -b$ towards the equation. It has not a solution if

$$\frac{1}{2} < \underbrace{u} < 1 \quad \text{and} \quad -\infty < -\underbrace{b} \le \frac{-\underbrace{u}}{2\underbrace{u}-1} (< -1) \left(\Leftrightarrow (1 <) \frac{\underbrace{u}}{2\underbrace{u}-1} \le \underbrace{b} < \infty \right)$$

However, in this domain u and $\overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ are real numbers and the equation $u\xi = \overbrace{\left(\frac{\underbrace{b}}{2\left|\underbrace{b}\right|-1}\right)}$ has a solution, so the eqation (6.1) has a solution

Summary for Case β: Under the condition (6.10) the equation (6.1) has a solution.

Summary for together Case α and Case β: ***The equation* (6.1) *has a solution.***

Istennek Hála, 2020-04-08, 18:35 Szalay István

HISTORICAL DISCUSSION

In the seventh century the great Indian mathematician and astronomer Brahmagupta entering the set of negative numbers solved the general linear equation. (See [2].) We, in the set of exploded numbers have done this in a very difficult way. (See (6.1) and part **6**.) Moreover, Brahmagupta extended the rules of arithmetic manipulations that apply to zero, but his description division by zero differs from our understanding: with respect to $\frac{a}{0}$,$(a \neq 0)$ he did not commit himself but if $a = 0$ then $\frac{0}{0} = 0$. As regards the set of exploded number we maintain the prohibition of division by 0 and $\frac{0}{0}$ is not allowed either. Novelties that $\frac{u}{\overbrace{1}} = 0$ and $\frac{u}{\overbrace{-1}} = 0$ such that $|u| \neq \overbrace{1}$ and $0 \cdot \overbrace{1}$ and $0 \cdot \overbrace{-1}$ are not allowed. (See (5.4) - (5.7) and Remark. 2.16 point 4.)

In terms of astronomy, we try to get a glimpse into the mysteries of the Multiverse by traditional operations, considering our Universe as a set of real numbers and the set of exploded numbers as the Multiverse.

REFERENCES

[1] I. Szalay, Geometric explosion of the real numbers and the cardinal number $_0\aleph$. International Journal of Mathematical Archieve-11(1), 2020, 7-19.

[2] wikipedia.org/wiki/Brahmagupta (assorted 08. April, 2020).

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