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# ESTIMATION OF EXTENDED EXPONENTIATED WEIBULL DISTRIBUTION UNDER VARIOUS LOSS FUNCTIONS

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***ABSTRACT***

*In this paper, extended exponentiated Weibull distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors.*

***Keywords:*** *Bayesian method, extended exponentiated Weibull distribution, quasi and gamma priors, squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions.*

## 1. INTRODUCTION

The extended exponentiated Weibull distribution has been proposed by Mahmoudi *et al.* [1]. They observe that it contains exponentiated Weibull and extended generalized exponential distribution as special cases and it can be used quite effectively for analyzing lifetime data. The probability density function of extended exponentiated Weibull distribution is given by

$$f\left(x;\theta\right)=\lambda\theta\left(1-\lambda cx\right)^{\frac{1}{c}-1}\left[1-\left(1-\lambda cx\right)^{\frac{1}{c}}\right]^{\theta-1};\ x>0. \tag{1}$$

The joint density function or likelihood function of (1) is given by

$$f\left(\underline{x};\theta\right)=\left(\lambda\theta\right)^{n}\left(\prod_{i=1}^{n}\left(1-\lambda cx_i\right)^{\frac{1}{c}-1}\right)exp\left[\left(\theta-1\right)\sum_{i=1}^{n}log\left(1-\left(1-\lambda cx_i\right)^{\frac{1}{c}}\right)\right]. \tag{2}$$

The log likelihood function is given by

$$log\ f\left(\underline{x};\theta\right)=n\,log\left(\lambda\theta\right)+log\left(\prod_{i=1}^{n}\left(1-\lambda cx_i\right)^{\frac{1}{c}-1}\right)+\left(\theta-1\right)\sum_{i=1}^{n}log\left(1-\left(1-\lambda cx_i\right)^{\frac{1}{c}}\right). \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given as:

$$\hat{\theta}=n\left(\sum_{i=1}^{n}log\left[1-\left(1-\lambda cx_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{4}$$

## 2. BAYESIAN METHOD OF ESTIMATION

The Bayesian inference procedures have been developed generally under squared error loss function

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^{2}. \tag{5}$$

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The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, *i.e*,

$$\hat{\theta}_S = E(\theta). \tag{6}$$

Zellner [2], Basu and Ebrahimi [3] have recognized that the inappropriateness of using symmetric loss function. Norstrom [4] introduced precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}. \tag{7}$$

The Bayes estimator under this loss function is denoted by $\hat{\theta}_P$ and is obtained as: $\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2}$. (8)

Calabria and Pulcini [5] points out that a useful asymmetric loss function is the entropy loss

$$L(\delta) \propto \left[\delta^p - p\ log_e(\delta) - 1\right]$$

where $\delta = \dfrac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta} = \theta$. Also, the loss function $L(\delta)$ has been used in Dey *et al.* [6] and Dey and Liu [7], in the original form having $p = 1$. Thus $L(\delta)$ can written be as:

$$L(\delta) = b\left[\delta - log_e(\delta) - 1\right];\ b>0. \tag{9}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation

$$\hat{\theta}_E = \left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{10}$$

Wasan [8] proposed the K-loss function which is given as:

$$L\left(\hat{\theta},\theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as:

$$\hat{\theta}_K = \left[\frac{E(\theta)}{E(1/\theta)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [9] introduced a new loss function which is given as:

$$L\left(\hat{\theta},\theta\right) = \theta^c\left(\hat{\theta}-\theta\right)^2. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as:

$$\hat{\theta}_{Al} = \frac{E\left(\theta^{c+1}\right)}{E\left(\theta^c\right)}. \tag{14}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.
(i) Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d}\ ;\ \theta > 0,\ d \ge 0, \tag{15}$$

where *d = 0* leads to a diffuse prior and *d = 1*, a non-informative prior.

(ii) Gamma prior: Generally, the gamma density is used as prior distribution of the parameter θ given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\ ;\ \theta > 0. \tag{16}$$

 

## 3. POSTERIOR DENSITY UNDER $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{(\lambda\theta)^n \left(\prod_{i=1}^{n}(1-\lambda c x_i)^{\frac{1}{c}-1}\right) exp\left[(\theta-1)\sum_{i=1}^{n} log\left(1-(1-\lambda c x_i)^{\frac{1}{c}}\right)\right]\theta^{-d}}{\int_0^\infty (\lambda\theta)^n \left(\prod_{i=1}^{n}(1-\lambda c x_i)^{\frac{1}{c}-1}\right) exp\left[(\theta-1)\sum_{i=1}^{n} log\left(1-(1-\lambda c x_i)^{\frac{1}{c}}\right)\right]\theta^{-d} d\theta}$$

$$= \frac{\theta^{n-d}\; e^{-\theta\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}}}{\int_0^\infty \theta^{n-d}\; e^{-\theta\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}} d\theta}$$

$$= \frac{\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)}\; \theta^{n-d}\; e^{-\theta\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}} \tag{17}$$

**Theorem 1:** On using (17), we have

$$E(\theta^c) = \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{-c}. \tag{18}$$

**Proof:** By definition,

$$E(\theta^c) = \int \theta^c f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{(n-d+c)}\; e^{-\theta\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}} d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+c+1)}{\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{n-d+c+1}}$$

$$= \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{-c}.$$

From equation (18), for $c=1$, we have

$$E(\theta) = (n-d+1)\left(\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{19}$$

From equation (18), for $c=2$, we have

$$E(\theta^2) = \left[(n-d+2)(n-d+1)\right]\left[\sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1}\right]^{-2}. \tag{20}$$

From equation (18), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n-d)}\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}. \tag{21}$$

From equation (18), for $c=c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma(n-d+c+2)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{-(c+1)}. \tag{22}$$

## 4. BAYES ESTIMATORS UNDER $g_1(\theta)$

From equation (6), on using (19), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=(n-d+1)\left(\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{23}$$

From equation (8), on using (20), the Bayes estimator of θ under precautionary loss function is obtained as:

$$\hat{\theta}_P=\left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{24}$$

From equation (10), on using (21), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E=(n-d)\left(\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{25}$$

From equation (12), on using (19) and (21), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K=\left[(n-d+1)(n-d)\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{26}$$

From equation (14), on using (18) and (22), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al}=(n-d+c+1)\left(\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{27}$$

## 5. POSTERIOR DENSITY UNDER $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as:

$$f(\theta/\underline{x})=\frac{(\lambda\theta)^n\left(\prod_{i=1}^{n}(1-\lambda cx_i)^{\frac{1}{c}-1}\right)exp\left[(\theta-1)\sum_{i=1}^{n} log\left(1-(1-\lambda cx_i)^{\frac{1}{c}}\right)\right]\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}}{\int_0^\infty(\lambda\theta)^n\left(\prod_{i=1}^{n}(1-\lambda cx_i)^{\frac{1}{c}-1}\right)exp\left[(\theta-1)\sum_{i=1}^{n} log\left(1-(1-\lambda cx_i)^{\frac{1}{c}}\right)\right]\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta}$$

$$=\frac{\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)\theta}}{\int_0^\infty\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)\theta}d\theta}$$

$$=\frac{\theta^{n+\alpha-1}\,e^{-\left(\beta+\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)\theta}}{\Gamma(n+\alpha)\Big/\left(\beta+\sum_{i=1}^{n} log\left[1-(1-\lambda cx_i)^{\frac{1}{c}}\right]^{-1}\right)^{n+\alpha}}$$

$$= \frac{\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)} \theta^{n+\alpha-1}\, e^{-\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)\theta} \tag{28}$$

**Theorem 2:** On using (28), we have

$$E\left(\theta^c\right)=\frac{\Gamma\left(n+\alpha+c\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-c}. \tag{29}$$

**Proof:** By definition,

$$E\left(\theta^c\right)=\int \theta^c f\left(\theta/\underline{x}\right)d\theta$$

$$= \frac{\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)} \int_0^\infty \theta^{n+\alpha+c-1}\, e^{-\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)\theta} d\theta$$

$$= \frac{\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)} \frac{\Gamma\left(n+\alpha+c\right)}{\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{n+\alpha+c}}$$

$$= \frac{\Gamma\left(n+\alpha+c\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-c}.$$

From equation (29), for $c=1$, we have

$$E\left(\theta\right)=\left(n+\alpha\right)\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{30}$$

From equation (29), for $c=2$, we have

$$E\left(\theta^2\right)=\left[\left(n+\alpha+1\right)\left(n+\alpha\right)\right]\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-2}. \tag{31}$$

From equation (29), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{\left(n+\alpha-1\right)}\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right). \tag{32}$$

From equation (29), for $c=c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma\left(n+\alpha+c+1\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-(c+1)}. \tag{33}$$

## 6. BAYES ESTIMATORS UNDER $g_2\left(\theta\right)$

From equation (6), on using (30), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=\left(n+\alpha\right)\left(\beta+\sum_{i=1}^{n} log\left[1-\left(1-\lambda c x_i\right)^{\frac{1}{c}}\right]^{-1}\right)^{-1}. \tag{34}$$

From equation (8), on using (31), the Bayes estimator of θ under precautionary loss function is obtained as:

$$\hat{\theta}_P = \left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}} \left( \beta + \sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1} \right)^{-1}. \tag{35}$$

From equation (10), on using (32), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = (n+\alpha+1) \left( \beta + \sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1} \right)^{-1}. \tag{36}$$

From equation (12), on using (30) and (32), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[(n+\alpha)(n+\alpha-1)\right]^{\frac{1}{2}} \left( \beta + \sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1} \right)^{-1}. \tag{37}$$

From equation (14), on using (29) and (33), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al} = (n+\alpha+c) \left( \beta + \sum_{i=1}^{n} log\left[1-(1-\lambda c x_i)^{\frac{1}{c}}\right]^{-1} \right)^{-1}. \tag{38}$$

## CONCLUSION

In this paper, we have obtained a number of estimators of parameter of extended exponentiated Weibull distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (23), (24), (25), (26) and (27) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (34), (35), (36), (37) and (38) we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution. We therefore recommend that the estimator's choice lies according to the value of the prior distribution which in turn depends on the situation at hand.

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