

APPLICATIONS OF FOURIER SERIES IN DOUBLE SINE AND COSINE SERIES

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ABSTRACT

In this paper we have introduce the new concept of double sine and cosine series in Fourier series of two variable functions and implementing these result by the examples of two variable Fourier series.

Keywords: Fourier series, double sine series, double cosine series.

Subject Classification 2010: 42A16, 42B05.

1. INTRODUCTION

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions.

“Any functions $f(x)$ can be expressed as a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

In the interval $(0, 2\pi)$ or $(-\pi, \pi)$, where a_0, a_n, b_n are constant.

$f(x)$ is periodic, single valued and finite

$f(x)$ has a finite number of finite discontinuities in any one period

$f(x)$ has a finite number of maxima and minima

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

Even function $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$

Odd function $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$

2. MAIN RESULT

Fourier series of two variable functions

Any functions $f(x, y)$ can be expressed as a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cos ny + \sum_{n=1}^{\infty} b_n \sin nx \sin ny$$

In the interval $(0, 2\pi)$ or $(-\pi, \pi)$, where a_0, a_n, b_n are constant.

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \int_0^{\pi} f(x, y) dx dy$$

Even function $a_n = \frac{2}{\pi} \int_0^{\pi} \int_0^{\pi} f(x, y) \cos(nx) \cos(ny) dx dy$

Odd function $b_n = \frac{2}{\pi} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin(nx) \sin(ny) dx dy$

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2.1 Examples: Find the Fourier series of the function $f(x, y) = x^2 + y^2$ in $-\pi < x < \pi$

Solution: Let the Fourier series of the function $f(x, y) = x^2 + y^2$ be

$$f(x, y) = x^2 + y^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx \cos ny + b_n \sin nx \sin ny)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) dx dy$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (x^2 + y^2) dx dy \\ &= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} + \frac{1}{\pi} \left[\frac{y^3}{3} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] = \frac{1}{\pi} \frac{4\pi^3}{3} = \frac{4\pi^2}{3} \\ a_0 &= \frac{4\pi^2}{3} \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \cos(nx) \cos(ny) dx dy$$

$$a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (x^2 + y^2) \cos(nx) \cos(ny) dx dy$$

$$\frac{2}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx + \int_{-\pi}^{\pi} y^2 \cos ny dy$$

$$\begin{aligned} \int_{-a}^a \int_{-a}^a f(x, y) dx dy &= \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is an even function of } x \\ 0, & \text{if } f(x) \text{ is an odd function of } x \end{cases} \\ &= \frac{2}{\pi} \left[x^2 \frac{\sin nx}{n} - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi} + \\ &\quad \frac{2}{\pi} \left[y^2 \frac{\sin ny}{n} - 2y \left(-\frac{\cos ny}{n^2} \right) + 2 \left(-\frac{\sin ny}{n^3} \right) \right]_0^{\pi} \\ &= \frac{4(-1)^n}{n^2} + \frac{4(-1)^n}{n^2} = \frac{8(-1)^n}{n^2} \quad [\sin n\pi = 0 \text{ and } \cos n\pi = (-1)^n] \\ &\quad \begin{cases} \frac{-8}{n^2}, & \text{when } n \text{ is odd} \\ \frac{8}{n^2}, & \text{when } n \text{ is even} \end{cases} \end{aligned}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \sin nx \sin ny dx dy$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (x^2 + y^2) \sin nx \sin ny dx dy = 0$$

[x^2 is an even function and $\sin nx \sin ny$ is an odd function]

so that $x^2 \sin nx$ and $y^2 \sin ny$ is an odd function]

Substituting the above values in (1), we obtain the required Fourier series of the function $f(x, y) = (x^2 + y^2)$ as given by

$$x^2 + y^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \cos nx + \frac{4(-1)^n}{n^2} \cos ny \right) + \sum_{n=1}^{\infty} 0 \cdot \sin nx \sin ny$$

$$x^2 + y^2 = \frac{\pi^2}{3} + 4 \left[-\cos x + \frac{1}{4} \cos 2x - \frac{1}{9} \cos 3x + \dots \right] + 4 \left[-\cos y + \frac{1}{4} \cos 2y - \frac{1}{9} \cos 3y + \dots \right]$$

Putting $x = 0, y = 0,$

$$\begin{aligned} 0 &= \frac{\pi^2}{3} + 4 \left[-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \dots \right] + 4 \left[-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \dots \right] \\ &= \frac{\pi^2}{3} + 8 \left[-2 + \frac{1}{2} - \frac{2}{9} + \frac{1}{8} - \dots \right] \end{aligned}$$

$$\frac{\pi^2}{24} = 2 - \frac{1}{2} + \frac{2}{9} - \frac{1}{8} + \dots$$

2.2 Examples: Obtain the Fourier series representing the function $f(x, y) = x + y$ in the interval $0 < x < 2\pi$ and $0 < y < 2\pi$.

Solution: Let the Fourier series representing the functions

$$f(x, y) = x + y$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} f(x, y) dx dy = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} (x + y) dx dy$$

$$\frac{1}{\pi} \left[\frac{x^2}{2} + \frac{y^2}{2} \right]_0^{2\pi} = 2\pi + 2\pi = 4\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \cos nx \cos ny dx dy$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} (x + y) \cos nx \cos ny dx dy$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cos nx dx + \frac{1}{\pi} \int_0^{2\pi} y \cos ny dy$$

$$= \frac{1}{\pi} \left[x \cdot \frac{\sin nx}{n} - 1 \cdot \left(\frac{-\cos nx}{n^2} \right) \right]_0^{2\pi} + \frac{1}{\pi} \left[y \cdot \frac{\sin ny}{n} - 1 \cdot \left(\frac{-\cos ny}{n^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2} \right] + \frac{1}{\pi} \left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{1}{n^2\pi} (1 - 1) + \frac{1}{n^2\pi} (1 - 1) = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \sin nx \sin ny dx dy$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx + \frac{1}{\pi} \int_0^{2\pi} y \sin ny dy$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - 1 \cdot \left(-\frac{\sin nx}{n^2} \right) \right]_0^{2\pi} + \frac{1}{\pi} \left[y \left(-\frac{\cos ny}{n} \right) - 1 \cdot \left(-\frac{\sin ny}{n^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-\frac{2\pi \cos 2n\pi}{n} \right] + \frac{1}{\pi} \left[-\frac{2\pi \cos 2n\pi}{n} \right] = -\frac{4}{n}$$

Putting the value's of a 's and b 's in (1), we get

$$x = \pi - 2 \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right]$$

$$y = \pi - 2 \left[\sin y + \frac{1}{2} \sin 2y + \frac{1}{3} \sin 3y + \dots \right]$$

3. RESULT

To solve the examples of two variable Fourier series we see that the result are obtained as similar in two variable Fourier series.

4. CONCLUSIONS

The necessity of dealing with Fourier series by double sine and cosine series in different applications for Engineering together with the difficulty of computing them in double valued function. In this paper to be self-contained, where the basic theoretical concepts on double sine and cosine series are solved by Fourier series. This theoretical concepts have been used to compute Fourier series of double valued functions.

REFERENCES

1. R. P. BOAS, Integrability of trigonometric series, Duke Math. J. 18 (1951), 787-793.
2. B.Paris, R.DeGraeve, Giac/Xcasversion1.2.2,2016.<http://www-fourier.ujf-grenoble.fr/~parisse/giac.html> (accessed August 29, 2016)
3. G. H. HARDY, On the convergence of certain multiple series, Proc. Cambridge Philosoph. Soc. 19 (1916-1919), 8695.
4. F. Mitycz, On the convergence in a restricted sense of multiple series, Anal. Math. 5 (1979), 135-147.
5. F. M-RIKZ, Convergence and integrability of double trigonometric series with coefficients of bounded variation, Proc. Amer. Math. Soc. 102 (1988), 633-640.
6. F. M-RIKZ, On the integrability of double cosine and sine series. I, J. Math. Anal. Appl. 154 (1991), 452-465.

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