

ON THE STABILITY
OF A RADICAL QUINTIC FUNCTIONAL EQUATION IN QUASI- β -SPACES

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ABSTRACT

In this paper we introduce and solve the radical quintic functional equation.

$$q\left(\sqrt[5]{u^5 + v^5}\right) = q(u) + q(v),$$

in a quasi- β -Banach spaces and stability by using a subadditive function for the quintic functional equation in (β, p) -Banach spaces.

Keywords: Radical function equations, Hyers-Ulam-Rassias stability, quasi- β -normed spaces, quintic functional equation.

1. INTRODUCTION

The problem of stability of functional equations originated from a question of S. M. Ulam [25] concerning the stability of group homomorphisms:

“Let G_1 be a group and G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h: G_1 \rightarrow G_2$ satisfies the inequality $d(h(u * v), h(u) * h(v)) < \delta$, for all $u, v \in G_1$, then there exist a homomorphism $H: G_1 \rightarrow G_2$ with $d(h(u) * H(u)) < \varepsilon$, for all $u \in G_1$?”

If the answer is affirmative, we say that the equation of homomorphism

$$H(u * v) = H(u) * H(v),$$

is stable. The concept of stability for a functional equation comes when we replace the functional equation comes when we change the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability problem of functional equations is how do the solutions of the inequality differ from those of the given functional equation?

The first answer to Ulam's question came within a year when D. H. Hyers [13] has excellently answered this problem of Ulam for Banach Spaces.

“Let E_1 and E_2 be Banach spaces. Consider $f: E_1 \rightarrow E_2$ satisfies

$$\|q(u + v) - q(u) - q(v)\| \leq \varepsilon,$$

for all $u, v \in E_1$ and $\varepsilon > 0$. Then there exists a unique additive mapping $T: E_1 \rightarrow E_2$ such that $\|q(u) - T(u)\| \leq \varepsilon$, for all $u \in E_1$. Then Aoki [1] and Bourgin [3] solved the stability problem with unbounded Cauchy differences.”

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In 1978, an approach was made to weaken the condition for the bound of the norm of the Cauchy difference by Th. M. Rassias [21]. In 1991, Z. Gajda [7] as well as by Rassias and Semrl [22] that one cannot prove a stability theorem of the additive equation for a specific function. P. Gavruta [8] gave a further generalization by replacing the Cauchy differences by a Control mapping in the spirit of Rassias approach. In the last few years several mathematicians have published on various generalizations and applications of generalized Hyers-Ulam stability to a number of functional equations and mappings (see for instance [4, 9-11, 6, 14-16, 19, 23]). In 2005, Lee, Im and Hawng [18] solved the cubic functional equation satisfying the mapping $q(x) = dx^3$ is a solution of that cubic functional equation.

“In 2010, Cho, Kang and Koh [27] used the following functional equation

$$2q(2u + v) + 2q(2u - v) + q(u + 2v) + q(u - 2v) = 20 [q(u + v) + q(u - v)] + 90q(u). \quad (1.1)$$

It is easy to see that the mapping $q(x) = du^5$ is a solution of the functional equation (1.1), which is called the quintic functional equation.”

Recently, the stability problem of the radical functional equations in various spaces was proved in [26, 27, 5, 2].

Before we present our results, we may define some basic definitions.

Definition 1.1: [27] “Let E_1 be a linear space and let E_2 be a real complete linear space. Then a mapping $f: E_1 \rightarrow E_2$ is called quintic if the quintic functional equation.

$$2q(2u + v) + 2q(2u - v) + q(u + 2v) + q(u - 2v) = 20 [q(u + v) + q(u - v)] + 90q(u), \quad (1.2)$$

holds for all $u, v \in E_1$.

Note that the mapping f is called quintic because of the following algebraic identity.

$$2(2u + v)^5 + 2(2u - v)^5 + (u + 2v)^5 + (u - 2v)^5 = 20 [(u + v)^5 + (u - v)^5] + 90u^5, \quad (1.3)$$

holds for every $u, v \in E_1$.”

Let β be a real number with $0 < \beta \leq 1$ and K be either R or C .

Definition 1.2[27]: “Let E_1 be a linear space over K . A quasi- β -norm $\|\cdot\|$ is a real-valued function on E_1 satisfying the following conditions:

1. $\|u\| \geq 0$ for all $u \in E_1$ and $\|u\| = 0$ if and only if $u=0$,
2. $\|\lambda u\| = |\lambda|^\beta \cdot \|u\|$ for all $\lambda \in R$ and $u \in E_1$.
3. There is a constant $K \geq 1$ such that $\|u + v\| \leq K (\|u\| + \|v\|)$ for every $u, v \in E_1$.

Then $(E_1, \|\cdot\|)$ is called a quasi- β -normed space with norm $\|\cdot\|$. A quasi - β -Banach space is a complete quasi- β -normed space.”

A quasi - β -norm $\|\cdot\|$ is called a (β, p) - norm ($0 < p \leq 1$) if $\|u + v\| \leq \|u\|^p + \|v\|^p$ for every $u, v \in E_1$. In this case, a quasi - β - Banach space is called a (β, p) - Banach space. For more information, readers are referred to [12, 17, 24].

In this paper, we introduce the following quintic functional equation.

$$"q\left(\sqrt[5]{u^5 + v^5}\right) = q(u) + q(v)." \quad (1.4)$$

We use a direct method to prove the Hyers-Ulam stability of the functional equation (1.4) in a quasi - β - normed spaces.

2. SOLUTION OF EQUATION (1.4)

During this section, R, Z and Q denote the sets of real, integer and rational numbers respectively and E_1 a linear space.

Theorem 2.1: If a function $q : R \rightarrow E_1$ satisfies the functional equation (1.4), then f is a quintic function.

Proof: Letting $u = v = 0$ in (1.4), we find $q(0) = 0$.

Putting $v = u$ in (1.4), we have $q(-u) = -q(u)$ for every $u \in R$.

Letting $v = u$ in (1.4), we find that $q(\sqrt[5]{2u}) = 2q(u)$ for all $u \in R$. Putting $v = \sqrt[5]{2}u$ in (1.4) and using $q(\sqrt[5]{2u}) = 2q(u)$, we have $q(\sqrt[5]{3u^5}) = 3q(u)$ for all $u \in R$. By induction, we get $q(\sqrt[5]{nu^5}) = nq(u)$ for all $u \in R$ and $n \in Z$. We have

$$q\left(\frac{u}{\sqrt[5]{n}}\right) = \frac{1}{n}q(u),$$

and so,

$$q\left(\sqrt[5]{\frac{m}{n}}\right) = \frac{m}{n}q(u),$$

for all $u \in R$ and $m, n \in Z$. So, we get

$$q(\sqrt[5]{ru}) = r q(u), \tag{2.1}$$

for every $u \in R$ and $r \in Q$.

Replacing u and v by $\sqrt[5]{u}$ and $\sqrt[5]{v}$ in (1.4) respectively, we get

$$q(\sqrt[5]{u+v}) = q(\sqrt[5]{u} + \sqrt[5]{v}), \tag{2.2}$$

for all $u, v \in R$. Putting $u = 2u + v, v = 2u - v$ in (1.4), using (2.1) and (2.2), we have

$$\begin{aligned} q(2u+v) + q(2u-v) &= q(\sqrt[5]{64u^5 + 160u^3v^2 + 20uv^4}) \\ &= 64q(u) + 160q(\sqrt[5]{u^3v^2}) + 20q(\sqrt[5]{uv^4}), \end{aligned} \tag{2.3}$$

for all $u, v \in R$. Now, putting $u = u + 2v$ and $v = u - 2v$ in (1.4) and using (2.1) and (2.2), we have

$$\begin{aligned} q(u+v) + q(u-2v) &= q(\sqrt[5]{2u^5 + 80u^3v^2 + 160uv^4}) \\ &= 2q(u) + 80q(\sqrt[5]{u^3v^2}) + 160q(\sqrt[5]{uv^4}), \end{aligned} \tag{2.4}$$

for all $u, v \in R$. Replacing u by $u + v$ and v by $u - v$ in (1.4), and using (2.1) and (2.2), we get

$$\begin{aligned} q(u+v) + q(u-v) &= q(\sqrt[5]{2u^5 + 20u^3v^2 + 10uv^4}) \\ &= 2q(u) + 20q(\sqrt[5]{u^3v^2}) + 10q(\sqrt[5]{uv^4}), \end{aligned} \tag{2.5}$$

for all $u, v \in R$. Now using (2.3), (2.4) and (2.5), we obtain that q satisfies (1.1), that is

$$\begin{aligned} 2q(2u+v) + 2q(2u-v) + q(u+2v) + q(u-2v) \\ &= 130q(u) + 400q(\sqrt[5]{u^3v^2}) + 200q(\sqrt[5]{uv^4}), \\ &= 20[q(u+v) + q(u-v)] + 90q(u), \end{aligned} \tag{2.6}$$

Hence, q is quintic function and this completes the proof.

3. APPROXIMATION OF THE RADICAL CUBIC FUNCTIONAL EQUATION (1.4)

During this section, we tend to modify the generalized Hyers-Ulam stability of the radical equation (1.4) in quasi – β - normed spaces and (β, p) - Banach spaces respectively.

First, let E_1 be a quasi – β -Banach space and let $\phi: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ be a function. A function $q: \mathbb{R} \rightarrow E_1$ is said to be “ ϕ - approximately radical quintic function” if

$$\left\| q\left(\sqrt[5]{u^5 + v^5}\right) - q(u) - q(v) \right\| \leq \phi(u, v). \tag{3.1}$$

Theorem 3.1: Let $q: \mathbb{R} \rightarrow E_1$ be a “ ϕ - approximately radical quintic function” with $q(0) = 0$. Consider a function $\phi: \mathbb{R} \rightarrow [0, \infty)$ satisfies

$$\Phi(u) = \sum_{j=1}^{\infty} \left(\frac{1}{2^\beta}\right)^j \phi\left(2^{j/5}u, 2^{j/5}v\right) < \infty,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{2^{\beta n}} \phi\left(2^{n/5}u, 2^{n/5}v\right) = 0,$$

for all $u, v \in \mathbb{R}$, then the limit

$$Q^*(u) = \lim_{n \rightarrow \infty} \frac{1}{2^n} q\left(2^{n/5}u\right),$$

exists for all $u \in \mathbb{R}$ and there exists a function $Q^*: \mathbb{R} \rightarrow E_1$ fulfilling the functional equation (1.4) and the inequality

$$\left\| q(u) - Q^*(u) \right\| \leq \frac{K}{2^\beta} \Phi(u), \tag{3.2}$$

for every $u \in \mathbb{R}$.

Proof: Substituting $v = u$ in (3.1), we have

$$\left\| q\left(\sqrt[5]{2}u\right) - 2q(u) \right\| \leq \phi(u, u), \tag{3.3}$$

and so

$$\left\| q(u) - \frac{1}{2}q\left(\sqrt[5]{2}u\right) \right\| \leq \frac{1}{2^\beta} \phi(u, u), \tag{3.4}$$

for every $u \in \mathbb{R}$. For any integers m, k with $m > k \geq 0$,

$$\left\| \frac{1}{2^k} q\left(2^{k/2}u\right) - \frac{1}{2^m} q\left(2^{m/2}u\right) \right\| \leq \frac{K}{2^\beta} \sum_{j=k}^{m-1} \left(\frac{1}{2^\beta}\right)^j \phi\left(2^{j/5}u, 2^{j/5}u\right), \tag{3.5}$$

for every $u \in \mathbb{R}$. Then a sequence $\left\{ \frac{1}{2^n} q\left(2^{n/2}u\right) \right\}$ is a Cauchy sequence in a quasi – β – Banach space E_1 and so it converges. We can define a function $Q^*: \mathbb{R} \rightarrow E_1$ by

$$Q^*(u) = \lim_{n \rightarrow \infty} \frac{1}{2^n} q\left(2^{n/5}u\right),$$

for every $u \in \mathbb{R}$. From (3.1), the following inequality holds:

$$\begin{aligned} \left\| Q^*\left(\sqrt[5]{u^5 + v^5}\right) - Q^*(u) - Q^*(v) \right\| \\ = \lim_{n \rightarrow \infty} \frac{1}{2^{\beta n}} \left\| Q^*\left(\sqrt[5]{2^n u^5 + 2^n v^5}\right) - Q^*\left(2^{n/5}u\right) - Q^*\left(2^{n/5}v\right) \right\| \\ = \lim_{n \rightarrow \infty} \frac{1}{2^{\beta n}} \phi\left(2^{n/5}u, 2^{n/5}v\right) = 0, \end{aligned}$$

for all $u, v \in \mathbb{R}$. Hence

$$Q^*\left(\sqrt[5]{u^5 + v^5}\right) - Q^*(u) - Q^*(v) = 0,$$

and by Theorem 2.1, F is a radical quintic function. Putting the limit m tends to ∞ in (3.5), with k equal to zero, we get a function F satisfies

$$\|q(u) - Q^*(u)\| \leq \frac{K}{2^\beta} \Phi(u).$$

Now, we consider that there exists another quintic function $G: \mathbb{R} \rightarrow E_1$ which assures the functional equation (1.4) and inequality (3.2). Since G satisfies (3.2), it is straight forward to point out that $G\left(2^{\frac{n}{5}}u\right) = 2^n G(u)$, for every $u \in \mathbb{R}$ and $n \geq 1$. Then we get

$$\begin{aligned} \|Q^*(u) - G(u)\| &= \left\| \frac{1}{2^n} Q^*\left(2^{\frac{n}{5}}u\right) - \frac{1}{2^n} G\left(2^{\frac{n}{5}}u\right) \right\| \\ &\leq \frac{K}{2^{\beta n}} \left(\left\| Q^*\left(2^{\frac{n}{5}}u\right) - q\left(2^{\frac{n}{5}}u\right) \right\| + \left\| q\left(2^{\frac{n}{5}}u\right) - G\left(2^{\frac{n}{5}}u\right) \right\| \right) \\ &\leq \frac{K}{2^{\beta n}} \Phi\left(2^{\frac{n}{5}}u\right), \text{ for all } u \in \mathbb{R}. \end{aligned}$$

Making n tends to ∞ , we have $Q^*(u) = G(u)$ for all $u \in \mathbb{R}$.

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