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EXPANSIVE MAPPING IN METRIC SPACES AND FIXED POINT THEOREM

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ABSTRACT

***I**n this paper, we obtain unique fixed point theorems for expansive mapping by using rational expression in metric spaces. These results extend the main results of many comparable results from the current literature.*

Keyword: *Fixed point, Complete Metric space, Rational expressions, Expansive mapping.*

1. INTRODUCTION

The study of expansive mappings is a very interesting research area in the fixed point theory. Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $T\,x = x$, is the most widely used fixed point theorem in all of analysis. Rhoades (1977) summarized contractive mappings of some types and discussed on fixed points. Wang, *et al.* (1984) proved some fixed point theorems on expansive mappings, which corresponds to some contractive mappings in metric spaces. Further, Khan, *et al.* (1986) generalized the result of Wang *et al.* (1984) by using functions. Park and Rhoades (1988) proved some fixed point theorems for expansion mappings. Also, Rhoades (1985) and Taniguchi (1989) generalized the results of (1984) for pair's mappings. Furthermore, Kang and Rhoades (1992) and Kang (1993) extend and generalized the results of Khan, *et al.* (1986) Rhoades (1985) and Taniguchi(1989). In (1992) the author defined on expanding condition for a pair of mappings and proved some common fixed point theorems for two mapping in complete metric spaces. In 1996, Pathak et al. (1996) proved some fixed point theorem for expansion mappings in metric space, which improve the results of Kang (1993), Khan *et al.* (1986), (1985) and Taniguchi (1989).

Recently, Samet *et al.* (2011), introduced a new concept of α-φ contractive type mappings and established fixed point theorems for such mappings in complete metric spaces. In 2012, Priya, *et al.* (2012) proved fixed point theorems for (ξ, α) $-$expansive mapping in complete metric spaces, which is extended, generalized and improved the results of (2011). He is also derived coupled fixed point theorems in metric space for α-φ contractive type mappings. In 2014, Shrivasatava, et al. (2014), proved some fixed point theorems for expansion mappings in complete metric space. In the same year, he is also proved fixed point and common fixed point results for same mappings in 2-metric spaces. In 2017, Gornicki, J. (2017) introduced a new concept of F-contraction and obtained some new fixed point results, especially on a complete G- metric space. Very recently, Yesikaya, S.S. and Aydin, (2020), proved fixed point result for expansive mappings in metric space.

The aim of this paper is to obtain a unique fixed point theorem for expansive mapping in metric spaces satisfying rational expressions. Our results extend, generalize and improve several results from existing literature, especially the work of Shrivastava *et al.* (2014.)

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2. PRELIMINARIES

The definition and properties of metric spaces are as follows:

Definition 2.1[Shrivastava et al. (2014.)]: Let X be a non-empty set and $d: X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions:

(i) $d(x, y) \geq 0$, for all $x, y \in X$;
(ii) $d(x, y) = 0 \Leftrightarrow x = y$, for all $x, y \in X$;
(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iv) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$.

If d is distance function on X. Then the pair (X, d) is called metric space.

Definition 2.2[Shrivastava et al. (2014.)]: A sequence $\{x_n\}$ in metric space(X, d) is called Cauchy **s**equence, if for given $\varepsilon > 0$ there exists $n_0 \in N$ such that, for all $m, n > n_0$

$$d(x_n, x_m) < \varepsilon. i.e. \lim_{n\to\infty} d(x_n, x_m) < \varepsilon.$$

Definition 2.3 [Shrivastava et al. (2014.)]: A sequence $\{x_n\}$ in metric space(X, d) is convergent to $x \in X$, if

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad \text{or} \quad x_n \rightarrow x.$$

Definition 2.4[Shrivastava et al.(2014)]: A metric spaces (X, d) is called complete, if every Cauchy sequence is convergent.

Definition 2.5: Let (X, d) be a metric space and let $S: X \rightarrow X$ be a mapping on X. Then $S$is called expansive mapping, if for every $x, y \in X$, there exists a number $r > 1$ such that

$$d(Sx, Sy) \geq rd(x, y)$$

3. MAIN RESULTS

In this section, we shall prove, generalize and extend the result of [13] and obtain unique fixed point for expansive mappings satisfying rational expressions.

Theorem 3.1: Let (X, d) be a complete metric space and let $S: X \rightarrow X$ be a surjective mapping satisfying the following condition

$$d(S\alpha, S\beta) \geq \mathrm{A}\left[\frac{d(\beta, S\alpha) + d(\alpha, S\beta)}{1 + d(\beta, S\beta).d(\alpha, S\beta)}\right] + B\left[\frac{d(\beta, S\alpha) + d(\alpha, S\beta)}{d(\alpha, \beta) + d(\beta, S\beta)}\right].d(\beta, S\beta) + Cd(\alpha, \beta) \tag{3.1.1}$$

For all $\alpha, \beta \in X$, where $A, B, C \geq 0$ are real constants and $B + C > 1 + 2A, C > 1 + 2A$. Then S has a unique fixed point on X.

Proof: Choose $\alpha_0 \in X$ be an arbitrary point. We define the iterative sequence $\{\alpha_i\}$, $n \in \mathrm{N}$ as follows

$\alpha_0 \in X, \alpha_0 = S\alpha_1,\ \alpha_1 = Sx_2, \alpha_2 = S\alpha_3 \ldots\ldots\ldots \alpha_i = S\alpha_{i+1}$

$$d(S\alpha_{i+1}, S\alpha_{i+2}) \geq A\left[\frac{d(\alpha_{i+2}, S\alpha_{i+1}) + d(\alpha_{i+1}, S\alpha_{i+2})}{1 + d(\alpha_{i+2}, S\alpha_{i+2}).d(\alpha_{i+1}, S\alpha_{i+2})}\right] + B\left[\frac{d(\alpha_{i+2}, S\alpha_{i+1}) + d(\alpha_{i+1}, S\alpha_{i+2})}{d(\alpha_{i+1}, \alpha_{i+2}) + d(\alpha_{i+2}, S\alpha_{i+2})}\right] d(\alpha_{i+2}, S\alpha_{i+2}) + Cd(\alpha_{i+1}, \alpha_{i+2})$$

$$d(\alpha_{i+}, \alpha_{i+1}) \geq A\left[\frac{d(\alpha_{i+2}, \alpha_i) + d(\alpha_{i+1}, \alpha_{i+1})}{1 + d(\alpha_{i+2}, \alpha_{i+1}).d(\alpha_{i+1}, \alpha_{i+1})}\right] + B\left[\frac{d(\alpha_{i+2}, \alpha_i) + d(\alpha_{i+1}, \alpha_{i+1})}{d(\alpha_{i+1}, \alpha_{i+2}) + d(\alpha_{i+2}, \alpha_{i+1})}\right] d(\alpha_{i+2}, \alpha_{i+1}) + Cd(\alpha_{i+1}, \alpha_{i+2})$$

$$\begin{aligned} d(\alpha_{i+}, \alpha_{i+1}) &\geq Ad(\alpha_{i+2}, \alpha_i) + \frac{B}{2} d(\alpha_{i+2}, \alpha_i) + C\, d(\alpha_{i+1}, \alpha_{i+2}) \\ &\geq A[d(\alpha_i, \alpha_{i+1}) + d(\alpha_{i+1}, \alpha_{i+2}] + \frac{B}{2}[d(\alpha_i, \alpha_{i+1}) + d(\alpha_{i+1}, \alpha_{i+2})] + C\, d(\alpha_{i+1}, \alpha_{i+2}) \end{aligned}$$

Therefore,

$$\left(1 - A - \frac{B}{2}\right) d(\alpha_{i+}, \alpha_{i+1}) \geq \left(A + \frac{B}{2} + C\right) d(\alpha_{i+1}, \alpha_{i+2})$$

$$\Rightarrow d(\alpha_{i+1}, \alpha_{i+2}) \leq \left[\frac{1 - A - \frac{B}{2}}{A + \frac{B}{2} + C}\right] d(\alpha_{i+}, \alpha_{i+1})$$

$$\Rightarrow d(\alpha_{i+1}, \alpha_{i+2}) \leq kd(\alpha_{i+}, \alpha_{i+1}), \text{ where, } k = \left[\frac{1 - A - \frac{B}{2}}{A + \frac{B}{2} + C}\right].$$

In general, we can write

$$d(\alpha_{i+1}, \alpha_{i+2}) \leq k^n d(x_1, x_0).$$

Since $0 \leq k < 1$ as $i \to \infty, k^n \to 0$. So, we have $d(\alpha_{i+1}, \alpha_{i+2}) \to 0$.

Hence $\{\alpha_i\}$ is Cauchy sequence in the complete metric space X. So, there exists $u \in X$ such that $\alpha_i \to u$. Since S is continuous. So, we have $Su = \lim_{i \to 1} \alpha_{i+1} = u$. Thus u is a fixed point of S.

Since S is a surjective map and hence, there exists a point $v \in X$ such that $u = Sv$.

Consider

$$\begin{aligned} d(\alpha_i, u) &= d(S\alpha_{i+1}, Sv) \\ &\geq A\left[\frac{d(v, S\alpha_{i+1}) + d(\alpha_{i+1}, Sv)}{1 + d(v, Sv).d(\alpha_{i+1}, Sv)}\right] + B\left[\frac{d(v, S\alpha_{i+1}) + d(\alpha_{i+1}, Sv)}{d(\alpha_{i+1}, v) + d(v, Sv)}\right].d(v, Sv) + Cd(\alpha_{i+1}, v) \\ &= \left[\frac{d(v, \alpha_i) + d(\alpha_{i+1}, Sv)}{1 + d(v, Sv).d(\alpha_{i+1}, Sv)}\right] + B\left[\frac{d(v, \alpha_i) + d(\alpha_{i+1}, Sv)}{d(\alpha_{i+1}, v) + d(v, Sv)}\right].d(v, Sv) + Cd(\alpha_{i+1}, v). \end{aligned}$$

Since $\{\alpha_{i+1}\}$ is a sub sequence of $\{\alpha_i\}$, so, $\{\alpha_i\} \to v \Rightarrow \{\alpha_{i+1}\} \to v$, when $i \to \infty$.

Therefore,

$$0 \geq A\left[\frac{d(v,v) + d(v,u)}{1 + d(v,v).d(v,u)}\right] + B\left[\frac{d(v,v) + d(v,u)}{d(v,v) + d(u,v)}\right].d(v,u) + Cd(v,v)$$

$$\Rightarrow 0 \geq (A + B)d(u, v)$$

$$\Rightarrow d(u, v) = 0 \Rightarrow u = v.$$

Thus, u has a unique fixed point of S. This completes the proof of the theorem.

Theorem 3.2: Let (X, d) be a complete metric space and $S: X \to X$ be a surjective self-map satisfying the following condition

$$d(Sp, Sq) \geq \alpha d(p, q) + \beta \frac{d(p, Sp).d(q, Sq)}{d(p, q)} + \gamma \frac{d(q, Sq).d(p, Sq)}{d(p, q) + d(q, Sq)} \tag{3.2.1}$$

For all $p, q \in X$, where $\alpha, \beta, \gamma > 0$ are all real constants $\beta + \gamma > 1 + 2\alpha, \gamma > 1, \gamma > 1 + 2\alpha$. Then S has a unique fixed point.

Proof: Let p_0 be an arbitrary point in X. There is p_1 in X such that $S(P_1) = p_0$. In this way, we define a sequence $\{p_k\}$ as follows:

$$p_k = Sp_{k+1}, \text{ for } k = 0,1,2 \ldots \ldots$$

If $p_k = p_{k+1}$, for some k,then we see p_k is a fixed point of S. Therefore, we suppose that no two-consecutive term of sequence $\{p_k\}$ are equal. Now we consider,

$$\begin{aligned} d(p_k, p_{k+1}) &= d(Sp_{k+1}, Sp_{k+2}) \\ &\geq \alpha d(p_{k+1}, p_{k+2}) + \beta \frac{d(p_{k+1}, Sp_{k+1}).d(p_{k+2}, Sp_{k+2})}{d(p_{k+1}, p_{k+2})} + \gamma \frac{d(p_{k+2}, Sp_{k+2}).d(p_{k+1}, Sp_{k+2})}{d(p_{k+1}, p_{k+2}) + d(p_{k+2}, Sp_{k+2})} \\ &= \alpha d(p_{k+1}, p_{k+2}) + \beta \frac{d(p_{k+1}, p_k).d(p_{k+2}, p_{k+1})}{d(p_{k+1}, p_{k+2})} + \gamma \frac{d(p_{k+2}, p_{k+1}).d(p_{k+1}, p_{k+1})}{d(p_{k+1}, p_{k+2}) + d(p_{k+2}, p_{k+1})} \\ &\geq \alpha d(p_{k+1}, p_{k+2}) + \beta d(p_{k+1}, p_k) \end{aligned}$$

Therefore,

$$\begin{aligned} &(1 - \beta)d(p_k, p_{k+1}) \geq \alpha d(p_{k+1}, p_{k+2}) \\ &\Rightarrow d(p_{k+1}, p_{k+2}) \leq \frac{1 - \beta}{\alpha} d(p_k, p_{k+1}) \\ &\Rightarrow d(p_{k+1}, p_{k+2}) \leq h\, d(p_k, p_{k+1}), \text{ where } \frac{1-\beta}{\alpha} = h. \end{aligned}$$

In general, we can write

$$\Rightarrow d(p_{k+1}, p_{k+2}) \leq h^n\, d(p_0, p_1).$$

Since $0 \leq h < 1$ as $k \to \infty, h^n \to 0$, we have

$$d(p_{k+1}, p_{k+2}) \to 0.$$

Hence $\{p_k\}$ is a Cauchy sequence in X. As X is a complete metric space, so, there exists a point $u \in X$ such that$\{p_k\} \to u$,since$S$ is a continuous, we have

$$Su = \lim_{k \to \infty} p_{k+1} = u.$$

Thus, u is a fixed point of S.

Since S is a surjective self-map and hence there exists $v \in X$such that $u = Sv$.

Consider,

$$\begin{aligned} d(p_k,u) &= d(Sp_{k+1},Sv) \\ &\geq \alpha d(p_{k+1},v) + \beta \frac{d(p_{k+1},Sp_{k+1}).d(v,Sv)}{d(p_{k+1},v)} + \gamma \frac{d(v,Sv).d(p_{k+1},Sv)}{d(p_{k+1},v)+d(v,Sv)} \\ &= \alpha d(p_{k+1},v) + \beta \frac{d(p_{k+1},p_k).d(v,Sv)}{d(p_{k+1},v)} + \gamma \frac{d(v,Sv).d(p_{k+1},Sv)}{d(p_{k+1},v)+d(v,Sv)}. \end{aligned}$$

Since $\{p_{k+1}\}$ is a subsequence of $\{p_k\}$. So, $\{p_k\} \to u, \{p_{k+1}\} \to u$, when $k \to \infty$.

So,

$$0 \geq \alpha d(u,v) + \beta \frac{d(u,u).d(v,u)}{d(u,v)} + \gamma \frac{d(v,u).d(u,u)}{d(u,v)+d(v,u)}$$

$$0 \geq \alpha d(u,v)$$

$$\Rightarrow d(u,v) \leq 0.$$

Hence $d(u,v) = 0 \Rightarrow u = v$. Thus,$u$ has a unique fixed point of S.

This completes the prof of theorem.

Theorem 3.3: Let (X,d) be a complete metric space and $S: X \to X$ be a surjective self-map satisfying the following condition

$$d(Sp,Sq) \geq r_1 d(p,q) + r_2 d(p,Sp) + r_3 d(q,Sq) + r_4 d(p,Sq) + r_5 \frac{d(p,Sp).d(q,Sq)}{d(p,q)} + r_6 \frac{[1+d(p,Sp)].d(p,Sq)}{d(p,q)} + r_7 \frac{d(q,Sq).d(p,Sq)}{d(p,q)}.$$

For all $p,q \in X$, where $r_i \geq 0, r_1 + r_2 + r_3 + r_5 > 1$and $r_1 + r_2 > 0$. Then S has a unique fixed point on X.

Proof: Let p_0 be an arbitrary point in X. There is p_1 in X such that $S(P_1) = p_0$. In this way, we define a sequence $\{p_n\}$ as follows:

$$p_n = Sp_{n+1}, \text{ for } k = 0,1,2 \ldots \ldots$$

If $p_n = p_{n+1}$, for some n,then we see p_n is a fixed point of S. Therefore, we suppose that no two-consecutive term of sequence $\{p_k\}$ are equal. Now we consider,

$$\begin{aligned} d(p_n,p_{n+1}) &= d(Sp_{n+1},Sp_{n+2}) \\ &\geq r_1 d(p_{n+1},p_{n+2}) + r_2 d(p_{n+1},Sp_{n+1}) + r_3 d(p_{n+2},Sp_{n+2}) \\ &\quad + r_4 d(p_{n+1},Sp_{n+2}) + r_5 \frac{d(p_{n+1},Sp_{n+1}).d(p_{n+2},Sp_{n+2})}{d(p_{n+1},p_{n+2})} + r_6 \frac{[1+d(p_{n+1},Sp_{n+1})]d(p_{n+1},sp_{n+2})}{d(p_{n+1},p_{n+2})} \\ &\quad + r_7 \frac{d(p_{n+2},Sp_{n+2})\, d(p_{n+1},Sp_{n+2})}{d(p_{n+1},p_{n+2})} \\ &= r_1 d(p_{n+1},p_{n+2}) + r_2\, d(p_{n+1},p_n) + r_3\, d(p_{n+2},p_{n+1}) \\ &\quad + r_4 d(p_{n+1},p_{n+1}) + r_5 \frac{d(p_{n+1},p_n).d(p_{n+2},p_{n+1})}{d(p_{n+1},p_{n+2})} + r_6 \frac{[1+d(p_{n+1},p_n)]d(p_{n+1},p_{n+1})}{d(p_{n+1},p_{n+2})} \\ &\quad + r_7 \frac{d(p_{n+2},p_{n+1})\, d(p_{n+1},p_{n+1})}{d(p_{n+1},p_{n+2})} \\ &\geq (r_1 + r_3) d(p_{n+1},p_{n+2}) + (r_2 + r_5) d(p_{n+1},p_n). \end{aligned}$$

Implies that

$$1 - (r_2 + r_5) d(p_n,p_{n+1}) \geq (r_1 + r_3) d(p_{n+1},p_{n+2})$$

$$\Rightarrow d(p_{n+1},p_{n+2}) \leq \frac{1-(r_2+r_5)}{(r_1+r_3)}\, d(p_n,p_{n+1})$$

$$\Rightarrow d(p_{n+1},p_{n+2}) \leq \delta d(p_n,p_{n+1}), \text{ where } \delta = \frac{1-(r_2+r_5)}{(r_1+r_3)} \text{ as } r_1 + r_2 + r_3 + r_5 > 1.$$

So, in general we can write

$$\Rightarrow d(p_{n+1},p_{n+2}) \leq \delta^n\, d(p_0,p_1).$$

Since $0 \leq \delta < 1$ as $n \to \infty, \delta^n \to 0$, we have

$$d(p_{n+1},p_{n+2}) \to 0.$$

Hence $\{p_n\}$ is a Cauchy sequence in X. As X is a complete metric space, so, there exists a point $x \in X$ such that $\{p_k\} \to x$.

Since S is a continuous, we have

$$Sx = \lim_{n\to\infty} p_{n+1} = x.$$

Thus, x is a fixed point of S.

Since S is a surjective self-map and hence there exists $y \in X$such that $x = Sy$.

Consider,

$$\begin{aligned} d(p_n,x) &= d(Sp_{n+1},Sy) \\ &\geq r_1 d(p_{n+1},y) + r_2\, d(p_{n+1},Sp_{n+1}) + r_3\, d(y,Sy) + r_4 d(p_{n+1},Sy) \\ &\quad + r_5 \frac{d(p_{n+1},Sp_{n+1}).d(y,Sy)}{d(p_{n+1},y)} + r_6 \frac{[1+d(p_{n+1},Sp_{n+1})]d(p_{n+1},Sy)}{d(p_{n+1},y)} + r_7 \frac{d(y,Sy)\, d(p_{n+1},Sy)}{d(p_{n+1},y)} \\ &= r_1 d(p_{n+1},y) + r_2\, d(p_{n+1},p_n) + r_3\, d(y,Sy) + r_4 d(p_{n+1},Sy) \\ &\quad + r_5 \frac{d(p_{n+1},p_n).d(y,Sy)}{d(p_{n+1},y)} + r_6 \frac{[1+d(p_{n+1},p_n)]d(p_{n+1},Sy)}{d(p_{n+1},y)} + r_7 \frac{d(y,Sy)\, d(p_{n+1},Sy)}{d(p_{n+1},y)}. \end{aligned}$$

Since $\{p_{n+1}\}$ is a subsequence of $\{p_n\}$. So, $\{p_n\} \to x, \{p_{n+1}\} \to x$, when $n \to \infty$.

So,

$$0 \geq r_1 d(x,y) + r_2\, d(x,x) + r_3\, d(y,x) + r_4 d(x,x) + r_5 \frac{d(x,x).d(y,x)}{d(x,y)} + r_6 \frac{[1+d(x,x)]d(x,x)}{d(x,y)} + r_7 \frac{d(y,x)\, d(x,x)}{d(x,y)}$$

$$\Rightarrow 0 \geq (r_1 + r_3)\, d(x,y)$$

$\Rightarrow d(x,y) = 0 \Rightarrow x = y$. Thus x has a unique fixed point of S.
This completes the proof of the theorem.

Theorem 3.4: Let (X,d) be a complete metric space and $S: X \to X$ be a surjective self-map satisfying the following condition

$$d(Sx,Sy) \geq q max\left\{d(x,y), \frac{d(x,Sx).d(y,Sy)}{d(x,y)}, \frac{[1+d(x,Sx)].d(x,Sy)}{d(x,y)}, \frac{d(y,Sy).d(x,Sy)}{d(x,y)}\right\} \tag{3.4.1}$$

For all $x, y \in X$ and $q > 1$. Then$S$has a unique fixed point.

Proof: Construct a sequence $\{\alpha_k\}$ as in proof of theorem 3.1, we claim that inequality 3.4.1, for put $x = \alpha_{k+1}$ and $y = \alpha_{k+2}$. Then we have

$$d(S\alpha_{k+1}, S\alpha_{k+2}) \geq q max \left\{ \begin{array}{c} d(\alpha_{k+1},\alpha_{k+2}), \dfrac{d\big(\alpha_{k+1}, S\alpha_{K+1}\big).\, d(\alpha_{k+2}, S\alpha_{k+2})}{d(\alpha_{k+1},\alpha_{k+2})}, \\ \dfrac{[1+d(\alpha_{k+1},S\alpha_{k+1})]d(\alpha_{k+1},S\alpha_{k+2})}{d(\alpha_{k+1},\alpha_{k+2})}, \dfrac{d(\alpha_{k+2},S\alpha_{k+2})d(\alpha_{k+1},S\alpha_{k+2})}{d(\alpha_{k+1},\alpha_{k+2})} \end{array} \right\}$$

$$\begin{aligned} d(\alpha_k,\alpha_{k+1}) &\geq q max \left\{ \begin{array}{c} d(\alpha_{k+1},\alpha_{k+2}), \frac{d(\alpha_{k+1},\alpha_k).d(\alpha_{k+2},\alpha_{k+1})}{d(\alpha_{k+1},\alpha_{k+2})}, \\ \frac{[1+d(\alpha_{k+1},\alpha_k)]d(\alpha_{k+1},\alpha_{k+1})}{d(\alpha_{k+1},\alpha_{k+2})}, \frac{d(\alpha_{k+2},\alpha_{k+1})d(\alpha_{k+1},\alpha_{k+1})}{d(\alpha_{k+1},\alpha_{k+2})} \end{array} \right\} \\ &= q max\{d(\alpha_{k+1},\alpha_{k+2}), d(\alpha_{k+1},\alpha_k)\} \end{aligned}$$

Case-I: $d(\alpha_k,\alpha_{k+1}) \geq q max\{d(\alpha_k,\alpha_{k+1})\} \Rightarrow 1 \geq q$, This is contradiction.

Case-II:

$$d(\alpha_k,\alpha_{k+1}) \geq q d(\alpha_{k+1},\alpha_{k+2})$$

$$\Rightarrow d(\alpha_{k+1},\alpha_{k+2}) \leq \frac{1}{q} d(\alpha_k,\alpha_{k+1})$$

$\Rightarrow d(\alpha_{k+1},\alpha_{k+2}) \leq \lambda d(\alpha_k,\alpha_{k+1})$, where$\frac{1}{q} = \lambda < 1$ as $q > 1$.

So, in general we have

$$d(\alpha_k,\alpha_{k+1}) \leq \lambda d(\alpha_{k-1},\alpha_k), \text{ for } k = 0,1,2 \ldots.$$

Therefore, $d(\alpha_k,\alpha_{k+1}) \leq \lambda^n d(\alpha_0,\alpha_1)$ (3.4.2)

We find that $\{\alpha_k\}$ is Cauchy sequence using (3.4.2) as proved theorem 3.1. As X is complete metric space, so, there exists a point $x^* \in X$ such that $\{\alpha_k\} \to x^{*.}$

Since S is a surjective self-map and hence there exists point $y^* \in X$ such that

$$x^* = Sy^*.$$

Now, Consider,

$$\begin{aligned} d(\alpha_k,x^*) &= d(S\alpha_{k+1},Sy^*) \\ &\geq q max \left\{ \begin{array}{c} d(\alpha_{k+1},y^*), \dfrac{d(\alpha_{k+1},S\alpha_{k+1})d(y^*,Sy^*)}{d(\alpha_{k+1},y^*)}, \dfrac{[1+d(\alpha_{k+1},S\alpha_{k+1})]d(\alpha_{k+1},Sy^*)}{d(\alpha_{k+1},y^*)} \\ \dfrac{d(y^*,Sy^*)d(\alpha_{k+1},Sy^*)}{d(\alpha_{k+1},y^*)} \end{array} \right\} \end{aligned}$$

$$\geq qmax\left\{\begin{matrix} d(\alpha_{k+1},y^*), \dfrac{d(\alpha_{k+1},\alpha_k)d(y^*,x^*)}{d(\alpha_{k+1},y^*)}, \dfrac{[1+d(\alpha_{k+1},\alpha_k)]d(\alpha_{k+1},x^*)}{d(\alpha_{k+1},y^*)} \\ \dfrac{d(y^*,x^*)d(\alpha_{k+1},x^*)}{d(\alpha_{k+1},y^*)} \end{matrix}\right\}$$

Since $\{\alpha_{k+1}\}$ is a subsequence of$\{\alpha_k\}$. So, $\{\alpha_k\} \to x^*, \{\alpha_{k+1}\} \to x^*$, when $k \to \infty$.
Therefore

$$d(x^*,x^*) \geq qmax\left\{d(x^*,y^*), \frac{d(x^*,x^*)d(y^*,x^*)}{d(x^*,y^*)}, \frac{[1+d(x^*,x^*)]d(x^*,x^*)}{d(x^*,y^*)}, \frac{d(y^*,x^*).d(x^*,x^*)}{d(x^*,y^*)}\right\}$$

$$0 \geq qd(x^*,y^*)$$

$\Rightarrow qd(x^*,y^*) = 0 \Rightarrow x^* = y^*$. Thus, x^* has a unique fixed point of S.
This completes the proof of theorem.

Theorem 3.5: Let (X,d) be a complete metric space and $S: X \to X$ be a surjective self-map satisfying the following conditions

$$d^2(Su,Sv) \geq \alpha d(u,u)d(u,Su) + \beta d(v,Sv)d(u,v) + \gamma d(u,Su)d(v,Sv) \tag{3.5.1}$$

For all $u,v \in X$, where $\alpha,\gamma \geq 0, \beta > 0$ and $\alpha+\beta+\gamma > 1$. Then S has a unique fixed point of X.

Proof: Construct a sequence $\{u_k\}$ as in proof of theorem 3.1, we claim that inequality 3.5.1, for put $u = u_{k+1}$ and $v = u_{k+2}$. Then we have

$$\begin{aligned} d^2(Su_{k+1},Su_{k+2}) &\geq \alpha d(u_{k+1},u_{k+2})d(u_{k+1},Su_{k+1}) + \beta d(u_{k+1},Su_{k+2})d(u_{k+1},u_{k+2}) \\ &\quad + \gamma d(u_{k+1},Su_{k+1})d(u_{k+2},Su_{k+2}) \\ &= \alpha d(u_{k+1},u_{k+2})d(u_{k+1},u_k) + \beta d(u_{k+1},u_{k+1})d(u_{k+1},u_{k+2}) \\ &\quad + \gamma d(u_{k+1},u_k)d(u_{k+2},u_{k+1}) \end{aligned}$$

Therefore,

$$d^2(u_k,u_{k+1}) \geq d(u_{k+1},u_{k+2})(\alpha+\beta+\gamma).\min\{d(u_k,u_{k+1})d(u_{k+1},u_{k+2})\}$$

Case-I: $d^2(u_k,u_{k+1}) \geq (\alpha+\beta+\gamma)d(u_{k+1},u_{k+2})d(u_k,u_{k+1})$

$$\Rightarrow d(u_{k+1},u_{k+2}) \leq \frac{1}{(\alpha+\beta+\gamma)}d(u_k,u_{k+1})$$

$\Rightarrow d(u_{k+1},u_{k+2}) \leq r_1 d(u_k,u_{k+1})$, where $r_1 = \frac{1}{(\alpha+\beta+\gamma)}$ as $\alpha+\beta+\gamma > 1$.

Case-II:

$$d^2(u_k,u_{k+1}) \geq (\alpha+\beta+\gamma)d(u_{k+1},u_{k+2})d(u_{k+1},u_{k+2})$$

$$\Rightarrow d^2(u_{k+1},u_{k+2}) \leq \frac{1}{(\alpha+\beta+\gamma)}d^2(u_k,u_{k+1})$$

$$\Rightarrow d(u_{k+1},u_{k+2}) \leq \left[\frac{1}{\alpha+\beta+\gamma}\right]^{1/2} d(u_k,u_{k+1})$$

$\Rightarrow d(u_{k+1},u_{k+2}) \leq r_2\, d(u_k,u_{k+1})$, where $r_2 = \left[\frac{1}{\alpha+\beta+\gamma}\right]^{1/2} < 1$, as $\alpha+\beta+\gamma > 1$.

Let $r = max\{r_1,r_2\}$. Hence $r < 1$. So, in general, we can write

$$d(u_k,u_{k+1}) \leq rd(u_{k-1,}u_k), \text{ for } k = 0,1,2\ldots$$

So,

$$d(u_k,u_{k+1}) \leq r^n d(u_{0,}u_1).$$

Since $0 \leq r < 1$ as $n \to \infty, r^n \to 0$, we have

$$d(u_k,u_{k+1}) \to 0.$$

Hence $\{u_k\}$ is a Cauchy sequence in X. As X is a complete metric space, so, there exists a point $u^* \in X$ such that$\{u_k\} \to u^*$.

Since S be a surjective self map and hence there exists a point $v^* \in X$ such that

$$u^* = Sv^*.$$

Now consider,

$$d(u_k,u^*) = d(Su_{k+1},Sv^*)$$

$$d^2(Su_{k+1},Sv^*) \geq \alpha d(u_{k+1},v^*).d(u_{k+1},Su_{k+1}) + \beta d(v^*,Sv^*)d(u_{k+1},v^*) + \gamma d(u_{k+1,}Su_{k+1})d(v,Sv^*)$$

$$d(u_k,u^*) \geq \alpha d(u_{k+1},v^*)d(u_{k+1},u_k) + \beta d(v^*,u^*)d(u_{k+1},v^*) + \gamma d(u_{k+1},u_k)d(v^*,u^*).$$

Since $\{u_{k+1}\}$ is a subsequence of$\{u_k\}$. So, $\{u_k\} \to u^*, \{u_{k+1}\} \to u^*$, when $k \to \infty$.
So,

$$d(u^*, v^*) \geq \alpha d(u^*, u^*) d(u^*, v^*) + \beta d(v^*, u^*) d(u^*, v^*) + \gamma d(u^*, u^*) d(v^*, u^*)$$

$$0 \geq \beta d^2(u^*, v^*)$$

$\Rightarrow d(u^*, v^*)= 0 \Rightarrow u^* = v^*$.Thus,$u^*$ has a unique fixed point of S.
This completes the proof of theorem.

CONCLUSION

In the present work, we obtained some unique fixed point results for expansive type contractive mappings with rational expression in metric spaces. Our results 3.1, 3.2, 3.3, 3.4 and 3.5 extend and improve some recent results of Shrivastava et al.(2014).

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