# APPLICATION OF UNDIRECTED GRAPH IN METRIC SPACE <br> MANISHA BHADORIYA <br> Research Scholar 

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#### Abstract

We give some graph theory in the setting of metric spaces endowed with a Undirected graphs. The presented results extend and improve several well-known results in the literature. In particular, we discuss a some example\& theorems of metric space by the using of undirected graph theory.


Keywords: Undirected graph, Metric space, Self-loop, adjacency, incidence.

## INTRODUCTION

In many problems dealing with discrete objects and binary relations, a graphical representation of the object and the binary relations on them is very convenient form of representation. This leads to naturally to a study of graph, graph theory has a very wide range of applications in engineering in physical, social and mathematical science. In this paper, we shall study with basic terminology of graphs and metric space.

Definition: An undirected graph G is defined abstractly as an ordered pair $(\mathrm{V}, \mathrm{E})$ where V is an non empty set and E is an multiples of two elements from V .

An undirected graph can be represented geometrically as a set of marked points V and set of lines E between the points.


Fig.-1.1
Example: $\mathrm{G}=(\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\},\{\mathrm{a}, \mathrm{b}\},\{\mathrm{a}, \mathrm{d}\},\{\mathrm{b}, \mathrm{c}\},\{\mathrm{b}, \mathrm{d}\},\{\mathrm{c}, \mathrm{c}\})$ is an undirected graph.
Definition: A graph $G=(V, E)$ that has neither self loop nor parallel edges is called a simple graph.
Let $\left\{v=v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and $E=\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ where $e_{1}=\left(v_{1}, v_{2}\right), e_{2}=\left(v_{2}, v_{3}\right), e_{3}=\left(v_{1}, v_{3}\right)$, and $e_{4}=\left(v_{3}, v_{4}\right)$.
Then $G=(V, E)$ is simple graph as shown in fig 1.2


Fig.-1.2
Definition: A graph with a finite number of vertices as well as finite number of edges is called a finite graph otherwise it is an infinite.

Definition: If $G=(V, E)$ is a finite graph, then the number of vertices is denoted by $|V|$ and is called the order of the graph $G$. The number of edges is denoted by $|E|$

Definition: Let $e_{k}$ be an edge joining two vertices $v_{i} a n d v_{j}$ of graph $G=(V, E)$. Then the edge $e_{k}$ is said to be incident on each of its vertices $v_{i}$ and $v_{j}$.

Example: In the graph of fig 1.2 edge $e_{2}$ is incident on vertices $v_{2}$ and $v_{3}$.
Definition: Two vertices in a graph $G=(V, E)$ are said to be adjacent if there exists an edge joining the vertices.
Example: In graph of fig 1.2 vertices $v_{1}$ and $v_{3}$ are adjacent while vertices $v_{1}$ and $v_{4}$ are not adjacent .
Definition: The degree of a vertex $v$ in a graph $G$ written as $d(v)$ is equal to the number of edges which are incident on $v$ with self loop counted twice

Example: In graph 1.2 we have

$$
d\left(v_{1}\right)=2, d\left(v_{2}\right)=2, d\left(v_{3}\right)=3, d\left(v_{4}\right)=1
$$

Definition: Let $X$ be a non empty set. A metric(or distance function ) on $X$ is a mapping $d: X \times X \rightarrow R$ which satisfies the following axioms for all $x, y, z \in X$.
$\left(M_{1}\right): d(x, x)=0$
$\left(M_{2}\right): d(x, y)=0 \Rightarrow x=y$
$\left(M_{3}\right): d(x, y)=d(y, x)$.
$\left(M_{4}\right): d(x, z) \leq d(x, y)+d(y, z)$
If $d$ is a metric on $X$, then the ordered pair $(X, d)$ is called metric space.The number $d(x, y)$ is called the distance between the elements $x$ and $y$.The elements of metric space $X$ are sometimes also called point.

The axiom $\left(M_{1}\right)$ says that the distance of a point from itself is zero.The axiom $\left(M_{2}\right)$ means that if the distance is zero, the two points are same.The axiom $\left(M_{3}\right)$ states that the distance does not depend on the order of the points $x$ and $y$.The axiom $\left(M_{4}\right)$ is commonly called the triangular inequality states " the sum of the length of two sides of a triangle is greater than or equal to the length of the third side.

Example: Consider the set R of all real numbers and define a mapping $d$ : $R \times R \rightarrow R$ such that

$$
d(x, y)=|x-y| \quad \forall x, y \in R
$$

$d$ is metric on $R$ follows from the properties of modulus of real numbers
(i) $|x|=0 \Leftrightarrow x=0$
(ii) $|-x|=|x|$
(iii) $|x+y| \leq|x|+|y| \quad \forall x, y \in R$.

Using these properties $d$ satisfies all the postulates as required for a metric

$$
\begin{aligned}
\left(M_{1}\right) d(x, y) & =0 \Leftrightarrow|x-y|=0 \Leftrightarrow x-y \Leftrightarrow x=y \\
\left(M_{3}\right) d(x, y) & =|x-y|=|-(x-y)|=|y-x|=d(y, x) \\
\left(M_{4}\right) d(x, y) & =|x-y|=|x-z+z-y| \\
& \leq|x-z|+|z-y|=d(x, z)+d(z, y) \quad \forall x, y, z \in R
\end{aligned}
$$

Hence $d$ is a metric on $R$ and is called usual metric on $R$. Thus $(R, d)$ is a metric space called usual metric space.

## MAIN RESULT

We use the graph theory solve the problems of metric space and determine the degree of vertices of metric space.
Example: Let $(X, d)$ be a metric space. A mapping $d_{1}$ is defined such that $d_{1}(x, y)=\frac{d(x, y)}{1+d(x, y)} \forall x, y \in X$.Then show that $d_{1}$ is a metric on $X$.

## Solution:



Let $G=(V, E)$ be a finite graph $x, y, z$ be a vertices and $e_{1}, e_{2}, e_{3}$ be edges of metric space $(X, d)$.

$$
\begin{gathered}
\left(M_{1}\right) d_{1}(x, y)=0 \Leftrightarrow \frac{d(x, y)}{1+d(x, y)}=0 \Leftrightarrow d(x, y)=0 \Leftrightarrow x=y \\
{[\therefore d(x, y)=0 \Leftrightarrow x=y]} \\
\left(M_{3}\right) d_{1}(x, y)=\frac{d(x, y)}{1+d(x, y)}=\frac{d(y, x)}{1+d(y, x)}=d_{1}(y, x) \\
{[\therefore d(x, y)=d(y, x)]}
\end{gathered}
$$

$\left(M_{4}\right) \quad$ Let $x, y, z \in X$ be arbitrary. Then

$$
\begin{aligned}
& \frac{d(x, z)}{1+d(x, z)+d(z, y)} \leq \frac{d(x, z)}{1+d(x, z)}=d_{1}(x, z) \\
& \text { and } \frac{d(z, y)}{1+d(x, z)+d(z, y)} \leq \frac{d(z, y)}{1+d(z, y)}=d_{1}(z, y)
\end{aligned}
$$

Since $d$ is a metric

$$
\begin{aligned}
d(x, y) & \leq d(x, z)+d(z, y) \\
d_{1}(x, y) & =\frac{d(x, y)}{1+d(x, y)} \leq \frac{d(x, z)+d(z, y)}{1+d(x, z)+d(z, y)} \\
& =\frac{d(x, z)}{1+d(x, z)+d(z, y)}+\frac{d(z, y)}{1+d(x, z)+d(z, y)} \\
& \leq d_{1}(x, z)+d_{1}(z, y)
\end{aligned}
$$

Hence $d_{1}$ is a metric
Degree of vertex in a metric $(X, d)$

$$
d(x)=2, d(y)=2, \quad d(z)=2
$$

Incidence of Metric space by the graph $\mathrm{G}=(\mathrm{V}, \mathrm{E})$
$x, y, z$ (vertices) are row matrices $e_{1}, e_{2}, e_{3}$ are column(edges)

$$
\mathrm{I}=\begin{array}{r}
e_{1} \\
x \\
y \\
z
\end{array} e_{2} \quad e_{3}
$$

Hence it is an incidence of metric space of $d(x, y)$.

Theorem1: Let $X$ be a nonempty set. Then a mapping $d: X \times X \rightarrow R$ is a metric if and only if the following condition are satisfied
$\left(M_{1}^{*}\right) d(x, y)=0 \quad$ if and only if $x=y \forall x, y \in X$
$\left(M_{2}^{*}\right) d(x, y) \leq d(x, z)+d(y, z) \quad \forall x, y, z \in X$
Proof: Let the postulates of metric space $M_{1}$ to $M_{4}$ hold. Let $G=(V, E) b e$ a graph, then the number of vertices of graph ( $V=x, y, z$ ) and number of edges be $\left(E=e_{1}, e_{2}, e_{3}, e_{4}, e_{5}\right)$ and $e_{5}$ be a self loop.


Fig.-1.4
$\left(M_{1}\right) d(x, x)=0$ if and only if $x=x \quad \forall x \in X$
$\left(M_{2}\right) d(x, y)=0 \Rightarrow x=y \quad \forall x, y \in X$
$\left(M_{3}\right) d(x, y)=d(y, x) \quad \forall x, y \in X$
$\left(M_{4}\right) d(x, y \leq d(x, z)+d(z, y)$
Also by $\left(M_{3}\right)$,

$$
\begin{equation*}
d(z, y)=d(y, z) \tag{2}
\end{equation*}
$$

It follows from (1) \& (2) that

$$
d(x, y) \leq d(x, z)+d(y, z) \forall x, y, z \in X
$$

which is $\left(M_{2}^{*}\right)$ and $\left(M_{1}^{*}\right)$ is the consequence of $\left(M_{1}\right)$ and $\left(M_{2}\right)$
Conversely suppose that the condition $\left(M_{1}^{*}\right)$ and $\left(M_{2}^{*}\right)$ hold. Obviously $\left(M_{1}\right)$ and $\left(M_{2}\right)$ are direct consequences of $\left(M_{1}^{*}\right)$.
Now, let $x, y$ be any two arbitrary points of $X$. Then applying
( $M_{2}^{*}$ ) for $x, y, x$ we get

$$
\begin{align*}
& d(x, y) \leq d(x, x)+d(y, x) \\
& 0+d(y, x) \quad \text { by }\left(M_{1}^{*}\right) \\
& d(x, y) \leq d(y, x) \tag{3}
\end{align*}
$$

Similarly applying $\left(M_{2}^{*}\right)$ for $y, x, y$, we get

$$
\begin{align*}
& d(y, x) \leq d(y, y)+d(x, y) \\
& 0+d(x, y) \quad \text { by }\left(M_{1}^{*}\right) \\
& d(y, x) \leq d(x, y) \tag{4}
\end{align*}
$$

From (3) \& (4) we have

$$
\begin{equation*}
d(x, y)=d(y, x) \tag{5}
\end{equation*}
$$

and so $\left(M_{3}\right)$ is satisfied.
Finally for any $x, y, z \in X, b y\left(M_{2}^{*}\right)$

$$
\begin{aligned}
d(x, y) & \leq d(x, z)+d(y, z) \\
& =d(x, z)+d(z, y) \quad \text { by }(5)
\end{aligned}
$$

Which is $\left(M_{4}\right)$.
By use the graph theory determine the degree of vertex of graph 1.4

$$
d(x)=3, d(y)=4, d(z)=3
$$

Incidence of Matrix of graph 1.4

$$
I=\begin{gathered}
e_{1} \\
x
\end{gathered} e_{2} \quad e_{3} e_{4} \begin{gathered}
e_{5} \\
y \\
z
\end{gathered}\left[\begin{array}{lllll}
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0
\end{array}\right]
$$

## Adjacency matrix of graph 1.4

$$
\left.A=\begin{array}{c}
x \\
y \\
z
\end{array} \begin{array}{ccc}
x & y & z \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right]
$$

## RESULTS

Hence prove that in a metric space has also a vertices and edges of metric $d(x, y)$ and satisfy the condition of undirected graph theory.

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