International Journal of Mathematical Archive-2(10), 2011, Page: 1905-1908
IJMA Available online through www.ijma.info ISSN 2229 – 5046

COMMUTATOR OF AUTOMORPHISMS

[1]H. Khosravi*, [2]H. Golmakani, and [3]H. M. Mohammadinezhad

[1,2]Department Of Mathematics, Faculty Of Science, Mashhad Branch, Islamic Azad University, Mashhad, 91735-413, Iran

[3]Department Of Mathematics, University Of Birjand, Birjand, 91523-318, Iran

E-mail: *Khosravi@mshdiau.ac.ir ,H.golmakani@mshdiau.ac.ir, Hmohmmadin@math.birjand.edu*

(Received on: 12-09-11; Accepted on: 02-10-11)

ABSTRACT

In this paper we define the commutator of automorphisms in groups.Let G be any group and α_1, α_2, ... *be automorphisms of G , then the commutator of automorphisms of wight 2, define as follows;*

$$[\alpha_1,\alpha_2]=\alpha_1^{-1}\,\alpha_2^{-1}\alpha_1\alpha_2.$$

In general, the element

$$[\alpha_1,\alpha_2...,\alpha_{n-1},\alpha_n]=[[\alpha_1,\alpha_2...,\alpha_{n-1}],\alpha_n]$$

is the left norm commutator of automorphisms of wight n≥2,which is called a simple commutator of automorphisms of wight n.

And in particular we prove some properties of them.

Keywords: *Automorphism, Commutator, Commutator of Automorphisms.*

---

1. INTRODUCTION:

Let G be a group and cosider the set Aut(G) is automorphisms of the group.In group theory [2]define the concept of commutator elements of G ,as follows;

Let x_1, x_2... elements of G, then the element $[x_1,x_2]=x_1^{-1}x_2^{-1}x_1x_2$ is the commutator of x_1 and x_2 of wight 2.In general ,the element

$$[x_1,...,x_n]=[[x_1,..., x_{n-1}], x_n]$$

is the left norm commutator of wight n≥2,which is called a simple commutator of wight n. In this paper we are interested in the structure of commutator of automorphisms of groups. We extend the above concept for automorphisms of G in the case of 2 .

2. COMMUTATOR OF AUTOMORPHISMS:

Theorem (2.1): Let α, β, δ ∈ Aut (G), then

(1) $\alpha^{\beta} = \alpha[\alpha, \beta]$

(2) $[\alpha, \beta] = [\alpha, \beta]^{-1}$

(3) $[\alpha,\beta^{-1}] = [\beta, \alpha]^{\beta^{-1}}$ and $[\alpha^{-1}, \beta]= [\beta, \alpha]^{\alpha^{-1}}$

(4) $[\alpha\beta, \delta] = [\alpha, \delta]^{\beta}[\beta,\delta]$

(5) $[\alpha, \beta\delta] = [\alpha,\delta]\,[\alpha, \beta]^{\delta}$

(6) $\alpha\beta = \beta\alpha[\alpha, \beta]$

(7) $[\alpha, \beta]^{\delta}= [\alpha^{\delta},\beta^{\delta}]$

(8) $[\alpha, \beta^{-1}, \delta]^{\beta}[\beta, \delta^{-1}, \alpha]^{\delta}[\delta, \alpha^{-1}, \beta]^{\alpha} = 1$

(9) $[\alpha, \beta,\delta^{\alpha}]\,[\beta,\delta,\alpha^{\beta}]\,[\delta,\alpha,\beta^{\delta}] = 1$

(10) $[\alpha, \beta\delta][\delta, \alpha\beta][\beta, \delta\alpha] = 1$

***Corresponding author:** [1]H. Khosravi*, ***E-mail:** Khosravi@mshdiau.ac.ir*

Proof: The result of **1** and **2** follows by definition.

3. We have, $[\alpha,\beta^{-1}]=\alpha^{-1}\beta\,\alpha\,\beta^{-1}$ (1) and $[\beta,\alpha]^{\beta^{-1}}=\beta(\beta^{-1}\alpha^{-1}\beta\alpha)\beta^{-1}=\alpha^{-1}\beta\alpha\beta^{-1}$ (2) From(1) and (2) the equality holds.

4. By definition we have, $[\alpha,\delta]^{\beta}[\beta,\delta]=(\alpha^{-1}\delta^{-1}\alpha\delta)^{\beta}(\beta^{-1}\delta^{-1}\beta\delta)$

$$=\beta^{-1}\alpha^{-1}\delta^{-1}\alpha\delta\beta\beta^{-1}\delta^{-1}\beta\delta=\beta^{-1}\alpha^{-1}\delta^{-1}\alpha\beta\delta \quad (1)$$

and $$[\alpha\beta,\delta]=(\alpha\beta)^{-1}\delta^{-1}(\alpha\beta)\delta=\beta^{-1}\alpha^{-1}\delta^{-1}\alpha\beta\delta \quad (2)$$

From (1) and (2) the equality holds.

5. The result follows by previous part.

6. The equality holds by definition.

7. We have, $$[\alpha,\beta]^{\delta}=(\alpha^{-1}\beta^{-1}\alpha\beta)^{\delta}=\delta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\delta \quad (1)$$

and we have $$[\alpha^{\delta},\beta^{\delta}]=(\alpha^{\delta})^{-1}(\beta^{\delta})^{-1}\alpha^{\delta}\beta^{\delta}$$

$$=(\delta^{-1}\alpha\delta)^{-1}(\delta^{-1}\beta\delta)^{-1}(\delta^{-1}\alpha\delta)(\delta^{-1}\beta\delta)$$

$$=\delta^{-1}\alpha^{-1}\delta\delta^{-1}\beta^{-1}\delta\delta^{-1}\alpha\,\delta\delta^{-1}\beta\delta=\delta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\delta \quad (2)$$

So from (1) and (2) the equality holds.

9. We have , $$[\alpha,\beta,\delta^{\alpha}]=[\,[\alpha,\beta],\delta^{\alpha}]$$

$$=[\alpha,\beta]^{-1}(\delta^{\alpha})^{-1}[\alpha,\beta]\,\delta^{\alpha} \quad \text{by part (2)}$$

$$=[\beta,\alpha]\,(\delta^{\alpha})^{-1}[\alpha,\beta]\,\delta^{\alpha}=\beta^{-1}\alpha^{-1}\beta\alpha(\alpha^{-1}\delta\alpha)^{-1}(\alpha^{-1}\beta^{-1}\alpha\beta)(\alpha^{-1}\delta\alpha)$$

$$=\beta^{-1}\alpha^{-1}\beta\alpha\alpha^{-1}\delta^{-1}\alpha\alpha^{-1}\beta^{-1}\alpha\beta\alpha^{-1}\delta\alpha=\beta^{-1}\alpha^{-1}\beta\delta^{-1}\beta^{-1}\alpha\beta\alpha^{-1}\delta\alpha \quad (1)$$

$$[\beta,\delta,\alpha^{\beta}]=[\,[\beta,\delta],\alpha^{\beta}]=[\beta,\delta]^{-1}(\alpha^{\beta})^{-1}[\beta,\delta]\,\alpha^{\beta}$$

$$=[\delta,\beta]\,(\alpha^{\beta})^{-1}[\beta,\delta]\,\alpha^{\beta} \quad \text{by part(2)}$$

$$=\delta^{-1}\beta^{-1}\delta\beta(\beta^{-1}\alpha\beta)^{-1}(\beta^{-1}\delta^{-1}\beta\delta)(\beta^{-1}\alpha\beta)$$

$$=\delta^{-1}\beta^{-1}\delta\beta\,\beta^{-1}\alpha^{-1}\beta\beta^{-1}\delta^{-1}\beta\delta\beta^{-1}\alpha\beta$$

$$=\delta^{-1}\beta^{-1}\delta\alpha^{-1}\delta^{-1}\beta\delta\beta^{-1}\alpha\beta \quad (2)$$

$$[\delta,\alpha,\beta^{\delta}]=[\,[\delta,\alpha],\beta^{\delta}]=[\delta,\alpha]^{-1}(\beta^{\delta})^{-1}[\delta,\alpha]\,\beta^{\delta}$$

$$=[\alpha,\delta]\,(\beta^{\delta})^{-1}[\delta,\alpha]\,\beta^{\delta} \quad \text{by part (2)}$$

$$=(\alpha^{-1}\delta^{-1}\alpha\delta)(\delta^{-1}\beta\delta)^{-1}(\delta^{-1}\alpha^{-1}\delta\alpha)(\delta^{-1}\beta\delta)$$

$$=\alpha^{-1}\delta^{-1}\alpha\delta\delta^{-1}\beta^{-1}\delta\,\delta^{-1}\alpha^{-1}\delta\alpha\,\delta^{-1}\beta\delta$$

$$=\alpha^{-1}\delta^{-1}\alpha\beta^{-1}\alpha^{-1}\delta\alpha\,\delta^{-1}\beta\delta \quad (3)$$

Therefore, by (1) and (2) and (3) we have the result.

8. The result follows by previous part.

10. We have, $$[\alpha,\beta\delta]=\alpha^{-1}(\beta\delta)^{-1}\alpha\beta\delta=\alpha^{-1}\delta^{-1}\beta^{-1}\alpha\beta\delta \quad (1)$$

$$[\delta, \alpha\beta]= \delta^{-1}(\alpha\beta)^{-1}\delta(\alpha\beta)= \delta^{-1}\beta^{-1}\alpha^{-1}\delta\alpha\beta \quad (2)$$

$$[\beta, \delta\alpha]= \beta^{-1}(\delta\alpha)^{-1}\beta(\delta\alpha)= \beta^{-1}\alpha^{-1}\delta^{-1}\beta\delta\alpha \quad (3)$$

Therefore, by (1) and (2) and (3) the equality holds.

Remark: The equality of part (10) holds for n element of automorphism of G.

Theorem (2.2): Let $\alpha, \beta, \delta \in$ Aut (G) such that $[\alpha, \beta]$ commutes with α and β .Then

(i) $[\alpha^i, \beta^j] = [\alpha, \beta]^{ij}$, for all integer i, j.

(ii) $(\alpha\beta)^n=\alpha^n\beta^n[\beta, \alpha]^{\frac{n(n-1)}{2}}$,$n\geq 0$.

Proof: **(i)** we proceed by induction on j.For j= 2, by theorem (2.1), we see in [3], we have

$$[\alpha, \beta^2] = [\alpha, \beta\beta] = [\alpha, \beta][\alpha, \beta]^{\beta} = [\alpha, \beta][\alpha, \beta][[\alpha, \beta],\beta] = [\alpha, \beta]^2$$

Now assume the result holds for j, i.e.

$$[\alpha, \beta^j]= [\alpha, \beta]^j.$$

We proved the result for j+1,

$$[\alpha, \beta^{j+1}]=[\alpha, \beta^j\beta]=[\alpha, \beta][\alpha, \beta^j]^{\beta} = [\alpha, \beta]\ [\alpha, \beta^j]\ [[\alpha, \beta^j],\beta]=[\alpha, \beta][\alpha, \beta^j]$$

$$= [\alpha, \beta]\ [\alpha, \beta]^j = [\alpha, \beta]^{j+1} .$$

Now we have

$$[\alpha^j, \beta^j]= [\alpha^j, \beta]^j =([\beta, \alpha^i]^{-1})^j =([\beta, \alpha]^{ij})^{-1} = ([\beta, \alpha]^{-1})^{ij} = [\alpha, \beta]^{ij} .$$

(ii) Use induction on n, the case n=1 being obvious. For n=2, we have

$$(\alpha\beta)^2 = \alpha\beta\alpha\beta = \alpha\alpha\beta[\beta, \alpha]\beta = \alpha^2\beta[\beta, \alpha]\beta = \alpha^2\beta\beta[\beta, \alpha][[\beta, \alpha],\beta] = \alpha^2\beta^2[\beta, \alpha] .$$

Assume the result holds for n-1, i.e,

$$(\alpha\beta)^{n-1} =\alpha^{n-1}\beta^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

then we have

$$(\alpha\beta)^n = (\alpha\beta)\ (\alpha\beta)^{n-1} = \alpha\beta\alpha^{n-1}\beta^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

$$= \alpha\alpha^{n-1}\ \beta[\beta, \alpha^{n-1}]\ \beta^{n-1}[\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

$$= \alpha^n\ \beta\beta^{n-1}[\beta, \alpha^{n-1}]\ [\ [\beta, \alpha^{n-1}], \beta^{n-1}][\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

$$= \alpha^n\beta^n[\beta, \alpha^{n-1}]\ [\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

$$=\alpha^n\beta^n[\beta, \alpha]^{n-1}\ [\beta, \alpha]^{\frac{(n-1)(n-2)}{2}}$$

$$=\alpha^n\beta^n[\beta, \alpha]^{\frac{n(n-1)}{2}} .$$

Theorem (3.2): Let α , β , δ be some elements of a group Aut(G) such that $\alpha\beta = \beta\alpha$ and $[\alpha, \text{Aut(G)}] = <[\alpha,\beta]; \beta \in \text{Aut(G)} >$ be an abelian subgroup of G, then $[\alpha, \beta, \delta]=[\alpha, \delta, \beta]$.

Proof: We have $[\alpha, \beta, \delta]= [[\alpha,\beta],\delta] = [\alpha, \beta]^{-1}\delta^{-1} [\alpha,\beta]\delta$

$$= \alpha^{-1}\ (\alpha\beta^{-1}\alpha^{-1}\beta)(\alpha\delta^{-1}\alpha^{-1}\delta)\ \delta^{-1}\beta^{-1}\alpha\beta\delta$$

$$= \alpha^{-1}[\alpha^{-1}, \beta][\beta^{-1}, \delta]\ \delta^{-1}\beta^{-1}\alpha\beta\delta.$$

Clearly $[\alpha^{-1}, \beta] = ([\alpha, \beta]^{\alpha^{-1}})^{-1} = [\alpha, \beta^{\alpha^{-1}}]^{-1} \in [\alpha, \mathrm{Aut}(G)]$.

And $[\alpha^{-1}, \delta] = [\alpha, \delta^{\alpha^{-1}}]^{-1} \in [\alpha, \mathrm{Aut}(G)]$.

Hence $[\alpha^{-1}, \beta]$ commutes with $[\alpha^{-1}, \delta]$ and therefore;

$$[\alpha, \beta, \delta] = \delta^{-1}\alpha^{-1}\delta\alpha\beta^{-1}\alpha^{-1}\beta\delta^{-1}\beta^{-1}\alpha\beta\delta$$

But $\delta\beta=\beta\delta$,then

$$[\alpha,\beta,\delta] = [\alpha, \delta]^{-1}\beta^{-1}[\alpha, \delta]\,\beta = [\,[\alpha, \delta], \beta] = [\alpha,\delta,\beta] .$$

Teorem (4.1): Let $\alpha, \beta \in \mathrm{Aut}(G)$ such that $[\alpha, \beta]$ commutes with α and β, then

$$[\alpha^{-1}, \beta] = [\alpha, \beta]^{-1} = [\alpha, \beta^{-1}].$$

Proof: Since $[\alpha,\beta]$ commutes with α and β ,it implies that

$$[\,[\alpha, \beta], \alpha] = 1 = [\,[\alpha, \beta], \beta]$$

similarly, $[\,[\alpha, \beta]^{-1}, \alpha^{-1}] = 1 = [\,[\alpha, \beta]^{-1}, \beta^{-1}]$.

Thus by theorem (2.1), see in [1], we have

$$[\alpha^{-1}, \beta] = ([\alpha, \beta]^{-1})^{\alpha^{-1}} = [\alpha, \beta]^{-1}[\,[\alpha, \beta]^{-1}, \alpha^{-1}] = [\alpha, \beta]^{-1}$$

similarly, $[\alpha, \beta^{-1}] = ([\alpha, \beta]^{-1})^{\beta^{-1}} = [\alpha, \beta]^{-1}[\,[\alpha, \beta]^{-1}, \beta^{-1}] = [\alpha, \beta]^{-1}$

Therefore, $[\alpha^{-1}, \beta] = [\alpha, \beta]^{-1} = [\alpha, \beta^{-1}]$.

3. ACKNOWLEDGMENTS:

The authors thank the research ***council of Mashhad Branch***, (Islamic Azad University) for support.

4. REFERENCES:

[1] M. R. R. Moghadam , H. safa, ”Some Prorerties of Automorphisms of Group,” Ricerche.mat(2010)

[2] D. J. S. Robinson, ‘‘A Course in the Theory of Groups,’’ Springer-Verlag, New York Berlin, (1996).

[3] David M. Riley and P. Shumyatsky, ‘‘Groups with a Positive Law on Commutators,’’ J. of pure and applied algebra, 2005.

****