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MODULO ELEVEN: NUMBER THEORY (11A07; 11A41)

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ABSTRACT

The following paper provides a technique, with the help of which, one can easily and efficiently find remainders when a two or three-digit number is divided by 11. The paper provides logical and simple proofs to the formulae as well as verifies them with the help of examples.

Keywords: *Number theory, modular arithmetic, congruence modulo, modulo 11, division algorithm.*

INTRODUCTION

Modular Arithmetic [1]

When $a = qn + r$, where q is the quotient and r is the remainder upon dividing a by n, we write $a \bmod n = r$.

$3 \bmod 2 = 1, \text{ since } 3 = 1\times 2 + 1$

$6 \bmod 2 = 0, \text{ since } 6 = 2\times 3 + 0$

$11 \bmod 3 = 2, \text{ since } 11 = 3\times 3 + 2$

Definitions and Concepts Used

- *mod 11/modulo 11* implies the **remainder obtained** when a number, here a two or three-digit number, is divided by 11.
- Let m be any multiple of 11, then $m \bmod 11 = 0$.
- *Division Algorithm for integers*: Suppose $b > 0$ and a are integers. Then there exist unique integers q and r such that $a = bq + r$, where $0 \le r < b$. The number q is called the quotient and r is called the remainder. [2]
- The set $Z_n = \{0,1,...n-1\}$ for $n \ge 1$ is a group under addition modulo n. For any $j > 0$ in Z_n the inverse of j is $n - j$. [3]

FORMULATION, PROOFS, and EXAMPLES

Let ab be a two-digit number, where a is the tens digit and b is the ones digit, then:

$$ab \bmod 11 = \begin{cases} b-a & if \quad a \le b \\ 11-(a-b) & if \quad a > b \end{cases}$$

Proof:

i) Given $a \le b$,

ab can be written as :

$$a\times 10 + b \tag{1}$$

Now, adding and subtracting a from (1), we get,

$$a\times 11 + (b-a) \tag{2}$$

Equation (2) modulo eleven gives the remainder as $b-a$.

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ii) Given $a > b$

Equation (2) modulo eleven, in this case, leaves us with $(b-a)\text{mod}11,$ where $b-a<0(\text{since}, a>b).$ The remainder, however, cannot be negative and therefore here, $b-a$ can be viewed as the additive inverse of $a-b,$ which will be $11-(a-b).$

(The set of remainders when a number is divided by 11 is $\{0,1,2,3,4,5,6,7,8,9,10\}$ which is the group Z_{11} with the operation addition modulo 11. The additive inverse of an element j is given by $11-j$).

Examples:

1. Consider the number $24.$ Here, $2<4$ and hence, the remainder is $4-2=2.$
It can be verified by dividing 24 by $11.$ $24=11\times 2+2,$ by division algorithm, which clearly gives 2 as the remainder.

2. Consider the number $42.$ Here, $4>2$ and hence, using the above formula, the remainder would be $11-(4-2)=9.$
It can be verified by dividing 42 by $11.$ $42=11\times 3+9,$ by division algorithm, which clearly gives 9 as the remainder.

Let abc be a three-digit number, where a is the hundreds digit, b is the tens digit and c is the ones digit, then:

$$abc \,\text{mod}11 = \begin{cases} a+c-b & \text{if} \quad a+c\geq b \quad \text{and} \quad a+c-b\leq 11 \\ \left[(a+c)-b\right]-11 & \text{if} \quad a+c\geq b \quad \text{and} \quad a+c-b>11 \\ 11-\left[b-(a+c)\right] & \text{if} \quad a+c<b \end{cases}$$

Proof:

(i) $a+c\geq b;(a+c)-b\leq 11$

abc can be written in the expanded form as:

$$a\times 100+b\times 10+c \tag{3}$$

Adding and subtracting $10a$ and b in (1), we get,

$$110\times a+11\times b+c-10\times a-b \tag{4}$$

Writing $-10\times a$ as $-11\times a+a$ in (2), we get,

$$110\times a+11\times b+c-11\times a+a-b \tag{5}$$

Equation (5) modulo 11, gives the remainder as $a+c-b.$

(ii) $a+c\geq b;(a+c)-b>11$

Reducing equation (5) modulo $11,$ gives $a+c-b.$ Here, since, $a+c-b>11,$ it can be written as:

$$11+\left[(a+c-b)-11\right], \text{ where } \left[(a+c-b)-11\right]<11 \tag{6}$$

(Since, the maximum value of $a+c-b$ can be $18,$ when $a=c=9$ and $b=0.$)

Equation (6) modulo $11,$ gives the remainder as $(a+c-b)-11.$

(iii) $a+c<b$

Again, reducing equation (5) modulo $11,$ we get, $a+c-b.$ Here, $a+c-b<0,$ therefore, it can be viewed as the additive inverse of $b-(a+c),$ where $b-(a+c)>0,$ which is given by $11-\left[b-(a+c)\right].$

These three cases are exhaustive; any three digit number will fit into either of the three cases.

Examples:

1. Consider the number 148 Using the above formula, we get the remainder as $1+8-4=5.$
It can be verified using the division algorithm:
$148=11\times 13+5,$ which clearly gives the remainder as $5.$

2. Consider the number 819 . Using the above formula, since $8+9-1=16>11$, therefore the remainder is, $16-11=5$.
Using division algorithm, we get, $819=11\times 74+5$, which clearly gives the remainder as 5.

3. Consider the number 191. Using the above formula, the remainder will be $11-\left[9-\left(1+1\right)\right]=4$.
Using division algorithm, we get, $191=11\times 17+4$, which clearly gives the remainder as 4.

Let $a0b$ be a three-digit number, where a is the hundreds digit, 0 is the tens digit and b is the ones digit, then:

$$a0b \mod 11=\begin{cases} a+b & if \quad a+b\leq 11 \\ a+b-11 & if \quad a+b>11 \end{cases}$$

This is, in particular, for a three digit number whose tens digit is zero.

The formula for two-digits can also be obtained from the formula for three-digits by equating a to 0.

REFERENCES

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