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NUMERICAL SOLUTIONS OF LAPLACE EQUATION<br>SAJA J. KAHLAF ${ }^{\mathbf{1}} \boldsymbol{\&}$ ALI A. MHASSIN*2<br>1,2Department of Mathematics, College of Education for PureScience, University of Anbar, Anbar, Iraq.

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#### Abstract

In this paper, Laplace equation will be solved in rectangular and non-rectangular domains. The solution is done using two methods. The first method is finite difference method with two different operators of Laplace equation (five and nine-points). The second method is finite element method. A comparison took place between the three results (The two result by finite difference and the one taken from finite element method). The best result found from comparison is the nine-points method (the result was more accurate than the other two methods especially in regular domain). The conclusion from this paper is that the nine-point method is the most accurate method.


Keywords: Laplace equation;Finite element method; Regular and Irregular domains; Finite diffrence method.

## INTRODUCTION

There are many applications of Laplace equations as study state, Heat flow problems of Dirichlet's and Neuman boundary condition, and other application in physical applications, electric field and so on. [1], [5].

I -First, Laplace operator in the regular domain, of five points formula, it was derived in [5], [10], and the Laplace operator of nine points formula, it was derived in [5], [10], we take example of these methods. Then we solved by finite element method and compare the three results, as shown in the tables below.

The Laplace operator is given by first we use Taylor for regular domain, to derive this operators in regular domain of Laplace equation:
Let $\quad \alpha^{i}=\frac{d^{i}}{d x^{i}}, \beta^{i}=\frac{d^{i}}{d Y^{i}}, i=1,2,3 \ldots$ Then:

$$
\begin{equation*}
u(\mathrm{x}+\mathrm{h}, \mathrm{y})=\mathrm{u}+\mathrm{h} \alpha \mathrm{u}+\frac{h^{2}}{2!} \alpha^{2} u+\ldots \tag{1.1}
\end{equation*}
$$

$$
\begin{equation*}
u(\mathrm{x}-h, y)=\mathrm{u}-\mathrm{h} \alpha \mathrm{u}+\frac{h^{2}}{2!} \alpha^{2} u-\ldots \tag{1.2}
\end{equation*}
$$

$$
\begin{equation*}
u(\mathrm{x}, \mathrm{y}+\mathrm{h})=\mathrm{u}+\mathrm{h} \beta \mathrm{u}+\frac{h^{2}}{2!} \beta^{2} u+\ldots \tag{1.3}
\end{equation*}
$$

$$
\begin{equation*}
u(\mathrm{x}, \mathrm{y}-\mathrm{h})=\mathrm{u}-\mathrm{h} \beta \mathrm{u}+\frac{h^{2}}{2!} \beta^{2} u-\ldots \tag{1.4}
\end{equation*}
$$

$$
\begin{equation*}
H u=h^{2}\left[\frac{d^{2} u}{d x^{2}}+\frac{d^{2} u}{d y^{2}}\right]+\frac{h^{4}}{12}\left[\frac{d^{4} u}{d x^{4}}+\frac{d^{4} u}{d x 4}\right]+\frac{h^{6}}{360}\left[\frac{d^{6} u}{d x^{6}}+\frac{d^{6} u}{d x^{6}}\right]+\ldots \tag{1.5}
\end{equation*}
$$

$$
\begin{equation*}
H u=h^{2} \nabla^{2} u+o\left(h^{4}\right) \tag{1.6}
\end{equation*}
$$

Where $H u=u(\mathrm{x}+\mathrm{h}, \mathrm{y})+u(\mathrm{x}-\mathrm{h}, \mathrm{y})+u(\mathrm{x}, \mathrm{y}+\mathrm{h})+(\mathrm{x}, \mathrm{y}-\mathrm{h})-4 u(x, y)$

[^0]The equation 1.6, is the formula of Laplace operator of five points.
In the same, we can derive the formula of Laplace operator of nine points,[1]

$$
\begin{equation*}
k u=6 h^{2} \nabla u+o\left(h^{4}\right) \tag{1.7}
\end{equation*}
$$

where

$$
\begin{aligned}
k u= & 4\{u(\mathrm{x}+\mathrm{h}, \mathrm{y})+u(\mathrm{x}+\mathrm{h}, \mathrm{y})+u(\mathrm{x}, \mathrm{y}+\mathrm{h})+u(\mathrm{x}, \mathrm{y}-\mathrm{h})\}+u(\mathrm{x}+\mathrm{h}, \mathrm{y})+u(\mathrm{x}+\mathrm{h}, \mathrm{y}) \\
& +u(\mathrm{x}, \mathrm{y}+\mathrm{h})+u(\mathrm{x}, \mathrm{y}-\mathrm{h})-20 \mathrm{u}(\mathrm{x}, \mathrm{y})
\end{aligned}
$$

May represents these formulas in equations (1.6) and (1.7) by the stencils respectively as follow [5], [1]:
$H=$

|  | 1 |  |
| :---: | :---: | :---: |
| 1 | -4 | 1 |
|  | 1 |  |


$K=$| 1 | 4 | 1 |
| :---: | :---: | :---: |
| 4 | -20 | 4 |
| 1 | 4 | 1 |

III: Implementation: To show the applications of these methods and the method of finite method, we solve the following example.

Example 1.1: Let $\Delta u=0$, on a square $0 \leq x, y \leq 1$ with boundary condition $u[x, y]_{\llcorner }=\mathrm{e}^{(-2 x)} \cos (2 y)$
Solution: a) By taking $h=1 / 4$, if we apply the operator H we obtained the linear system of equations corresponding to the figure (1.1) and as follows:
the matrix of five points of square region

$$
\left(\begin{array}{ccccccccc}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0  \tag{1.8}\\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{array}\right)\left[\begin{array}{c}
u_{1} \\
u_{2} \\
u_{3} \\
u_{4} \\
u_{5} \\
u_{6} \\
u_{7} \\
u_{8} \\
u_{9}
\end{array}\right]=\left[\begin{array}{l}
-B_{20}-B_{22} \\
-B_{19} \\
-B_{18}-B_{16} \\
-B_{23} \\
0 \\
-B_{15} \\
-B_{10}-B_{24} \\
-B_{11} \\
-B_{14}-B_{12}
\end{array}\right]
$$

In (1.8) the matrix of five points of square region.
b) If we applying the operator K on the figure (1.1) we obtained the linear the system of equations as shown below:

$$
\left[\begin{array}{ccccccccc}
-20 & 4 & 0 & 4 & 1 & 0 & 0 & 0 & 0  \tag{1.9}\\
4 & -20 & 4 & 1 & 4 & 1 & 0 & 0 & 0 \\
0 & 4 & -20 & 0 & 1 & 4 & 0 & 0 & 0 \\
4 & 1 & 0 & -20 & 4 & 0 & 4 & 0 & 0 \\
1 & 4 & 1 & 4 & -20 & 4 & 1 & 4 & 1 \\
0 & 1 & 4 & 0 & 4 & -20 & 0 & 1 & 4 \\
0 & 0 & 0 & 4 & 1 & 0 & -20 & 4 & 0 \\
0 & 0 & 0 & 1 & 4 & 1 & 4 & -20 & 4 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 4 & -20
\end{array}\right]\left[\begin{array}{c}
u_{1} \\
u_{2} \\
u_{3} \\
u_{4} \\
u_{5} \\
u_{6} \\
u_{7} \\
u_{8} \\
u_{9}
\end{array}\right]=\left[\begin{array}{c}
-B_{21}-4 B_{20}-B_{19}-4 B_{22}-B_{23} \\
-B_{20}-4 B_{19}-B_{18} \\
-B_{19}-4 B_{18}-B_{17}-4 B_{16}-B_{15} \\
-B_{24}-4 B_{23}-B_{22} \\
0 \\
-B_{16}-4 B_{15}-B_{14} \\
-B_{11}-4 B_{10}-B_{25}-4 B_{24}-B_{23} \\
-B_{12}-4 B_{11}-B_{10} \\
-B_{15}-4 B_{14}-B_{13}-4 B_{12}-B_{11}
\end{array}\right]
$$

In (1.9) the matrix of nine points of square region.
c) If we solve the example 1.1 by finite element method as in figure 1.2, such that, divided the region into triangles, and collected the global matrix for Regularin the table (1.1), [3].


Figure-1.1: The regular region


Figure-1.2: Mesh for finite element in regular region

|  | 0 | B10 | B11 | B12 | B13 | B14 | U9 | U8 | U7 | B24 | B23 | U4 | U5 | U6 | B15 | B16 | U3 | U2 | U1 | B22 | B21 | B20 | B19 | B18 | B17 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | $-0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B10 | 0.5 | 2 | $-0.5$ | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B11 | 0 | $-0.5$ | 2 | -0.5 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B12 | 0 | 0 | $-0.5$ | 2 | $-0.5$ | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B13 | 0 | 0 | 0 | $-0.5$ | 1 | $-0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B14 | 0 | 0 | 0 | 0 | -0.5 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U9 | 0 | 0 | 0 | -1 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U8 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U7 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | -0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B23 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| U5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U6 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | $-0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 |
| U3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 |
| U2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | -1 | 0 | 0 |
| U1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | -1 | 0 | 0 | 0 |
| B22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | $-0.5$ | 0 | 0 | 0 | 0 |
| B21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 | 1 | -0.5 | 0 | 0 | 0 |
| B20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | -0.5 | 2 | -0.5 | 0 | 0 |
| B19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | $-0.5$ | 2 | $-0.5$ | 0 |
| B18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | -0.5 | 2 | -0.5 |
| B17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 | 1 |

Table- (1.1): Global Matrix for regular
The result ( $\mathbf{a}, \mathbf{b}, \mathbf{c}$ ) of regular region (Rectangular) it's show in the table (1.2)

| Step Length | Maximum Error of H | Maximum Error of K | Maximum Error In Finte <br> Element |
| :---: | :---: | :---: | :---: |
| $1 / 4$ | $2.574297 \mathrm{E} 0-3$ | $3.455287 \mathrm{E}-7$ | $2.619269784 \mathrm{E}-3$ |
| $1 / 5$ | $1.809348 \mathrm{E} 0-3$ | $9.518665 \mathrm{E}-8$ |  |
| $1 / 8$ | $7.382760 \mathrm{E} 0-4$ | $5.790283 \mathrm{E} 0-9$ |  |
| $1 / 10$ | $4.7066583 \mathrm{E} 0-4$ | $3.142247 \mathrm{E} 0-9$ |  |

Table- (1.2): Result of regular region (Rectangular)
II-Secondly, the Laplace operator in the irregular domain, using five points, [1], [5], [10].and nine-points finite difference formulas, [1], then we solve this example by finite elements method and compare the results as shown in the tables below.

Now, for the non-rectangular region we have for five points
Let $\quad \alpha^{i}=\frac{d^{i}}{d x^{i}}, \beta^{i}=\frac{d^{i}}{d Y^{i}}, i=1,2,3 \ldots$ Then:

$$
\begin{equation*}
u\left(\mathrm{x}+\mathrm{h}_{1}, \mathrm{y}\right)=\mathrm{u}+\mathrm{h}_{1} \alpha \mathrm{u}+\frac{h_{1}}{2!} \alpha^{2} u+\ldots \tag{2.1}
\end{equation*}
$$

$$
\begin{equation*}
u\left(\mathrm{x}-h_{3}, y\right)=\mathrm{u}-\mathrm{h}_{3} \alpha \mathrm{u}+\frac{h_{3}}{2!} \alpha^{2} u-\ldots \tag{2.2}
\end{equation*}
$$

$$
\begin{equation*}
u\left(\mathrm{x}, \mathrm{y}+\mathrm{h}_{2}\right)=\mathrm{u}+\mathrm{h}_{2} \beta \mathrm{u}+\frac{h_{2}^{2}}{2!} \beta^{2} u+\ldots \tag{2.3}
\end{equation*}
$$

$$
\begin{equation*}
u\left(\mathrm{x}, \mathrm{y}-\mathrm{h}_{4}\right)=\mathrm{u}-\mathrm{h}_{4} \beta \mathrm{u}+\frac{\mathrm{h}_{4}^{2}}{2!} \beta^{2} u-\ldots \tag{2.4}
\end{equation*}
$$

$$
\begin{align*}
& u_{x x}=\frac{h_{3} u_{1}+h_{1} u_{3}-\left(\mathrm{h}_{1}+\mathrm{h}_{3}\right) \mathrm{u}_{0}}{\left(\frac{1}{2}\right) \mathrm{h}_{1} h_{3}\left(\mathrm{~h}_{1}+\mathrm{h}_{3}\right)}  \tag{2.5a}\\
& u_{y y}=\frac{h_{4} u_{2}+h_{2} u_{4}-\left(\mathrm{h}_{2}+\mathrm{h}_{4}\right) \mathrm{u}_{0}}{\left(\frac{1}{2}\right) \mathrm{h}_{2} h_{4}\left(\mathrm{~h}_{2}+\mathrm{h}_{4}\right)}  \tag{2.5b}\\
& \alpha_{0} u_{0}+\alpha_{1} u_{1}+\alpha_{2} u_{2}+\alpha_{3} u_{3}+\alpha_{4} u_{4}=0 ;  \tag{2.6}\\
& \alpha_{1}=\frac{2}{h^{2}} \frac{1}{s_{1}\left(s_{1}+s_{3}\right)}, \quad \alpha_{3}=\frac{2}{h^{2}} \frac{1}{s_{3}\left(s_{1}+s_{3}\right)} \\
& \alpha_{2}=\frac{2}{h^{2}} \frac{1}{s_{2}\left(s_{2}+s_{4}\right)}, \alpha_{4}=\frac{2}{h^{2}} \frac{1}{s_{4}\left(s_{2}+s_{4}\right)} \\
& \alpha_{0}=-\left(\alpha_{1}+\alpha_{2}+\alpha_{3}+\alpha_{4}\right), \text { where } \quad \mathrm{h}_{i} s_{i} \leq h, s_{i} \leq 1, i=1,2,3,4 \\
& u\left(\mathrm{x}+\mathrm{h}_{5}, \mathrm{y}+\mathrm{h}_{5}\right)=\mathrm{u}+\mathrm{h}_{5}(\alpha+\beta) \mathrm{u}+\frac{h_{5}^{2}}{2!}(\alpha+\beta)^{2} u+\ldots .  \tag{2.7}\\
& u\left(\mathrm{x}-\mathrm{h}_{6}, \mathrm{y}+\mathrm{h}_{6}\right)=\mathrm{u}+\mathrm{h}_{6}(-\alpha+\beta) \mathrm{u}+\frac{h_{6}{ }^{2}}{2!}(-\alpha+\beta)^{2} u+\ldots .  \tag{2.8}\\
& u\left(\mathrm{x}-\mathrm{h}_{7}, \mathrm{y}-\mathrm{h}_{7}\right)=\mathrm{u}+\mathrm{h}_{7}(-\alpha-\beta) \mathrm{u}+\frac{h_{7}^{2}}{2!}(-\alpha-\beta)^{2} u+\ldots .  \tag{2.9}\\
& u\left(\mathrm{x}+\mathrm{h}_{8}, \mathrm{y}-\mathrm{h}_{8}\right)=\mathrm{u}+\mathrm{h}_{8}(\alpha-\beta) \mathrm{u}+\frac{h_{8}^{2}}{2!}(\alpha-\beta)^{2} u+\ldots . \tag{2.10}
\end{align*}
$$

Find the Laplace operator from the equations (2.1) to (2.4) and multiply by 4, and from (2.7) to (2.10) and multiply by 2, add the two results, we get the following equation.

$$
\begin{align*}
& \left.\alpha_{i} u_{i}=6\left(\alpha^{2}+\beta^{2}\right) \mathrm{u}+\left[4\left(\mathrm{M} 3 \alpha^{3}+\mathrm{N} 3 \beta^{3}\right)+2\left(\mathrm{D} 3(\alpha+\beta)^{3}\right)+\mathrm{K} 3(-\alpha+\beta)^{3}\right)\right] \mathrm{u} \\
& \left.+\frac{1}{4!}\left[4\left(\mathrm{M} 4 \alpha^{4}+\mathrm{N} 4 \beta^{4}\right)+2\left(\mathrm{D} 4(\alpha+\beta)^{4}\right)+\mathrm{K} 4(\alpha-\beta)^{4}\right)\right] \mathrm{u} \\
& \left.+\frac{1}{5!}\left[4\left(\mathrm{M} 5 \alpha^{5}+\mathrm{N} 5 \beta^{5}\right)+2\left(\mathrm{D} 5(\alpha+\beta)^{5}\right)+\mathrm{K} 5(-\alpha+\beta)^{5}\right)\right] \mathrm{u}  \tag{2.11}\\
& \left.+\frac{1}{6!}\left[4\left(\mathrm{M} 6 \alpha^{6}+\mathrm{N} 6 \beta^{6}\right)+2\left(\mathrm{D} 6(\alpha+\beta)^{6}\right)+\mathrm{K} 6(\alpha+\beta)^{6}\right)\right] \mathrm{u}+
\end{align*}
$$

Where

$$
\begin{aligned}
& \alpha_{1}=\frac{8}{h^{2} s_{1}\left(s_{1}+s_{3}\right)}, \quad \alpha_{2}=\frac{8}{h^{2} s_{2}\left(s_{2}+s_{4}\right)}, \quad \alpha_{3}=\frac{8}{h^{2} s_{3}\left(s_{3}+s_{1}\right)}, \quad \alpha_{4}=\frac{8}{h^{2} s_{4}\left(s_{4}+s_{2}\right)} \\
& \alpha_{5}=\frac{8}{h^{2} s_{5}\left(s_{5}+s_{7}\right)}, \quad \alpha_{6}=\frac{8}{h^{2} s_{6}\left(s_{6}+s_{8}\right)}, \quad \alpha_{7}=\frac{8}{h^{2} s_{7}\left(s_{7}+s_{5}\right)}, \quad \alpha_{8}=\frac{8}{h^{2} s_{8}\left(s_{8}+s_{6}\right)}, \\
& \alpha_{0}=\sum_{i=1}^{8} \alpha_{i} \text {, where, } h_{i}=s_{i} h, 0 \leq s_{i} \leq 1, i=1,2,3 \ldots
\end{aligned}
$$

Example 2.1: Find the solution of Laplace equation $\Delta u=0$ in the first quadrant bounded by the circle $x^{2}+y^{2}=1$, as in figure (2.1). and the boundary condition and the exact solution are given by $u[x, y]_{L}=\mathrm{e}^{(-2 x)} \cos (2 \mathrm{y})$.

Solution: By taking the step length $h=1 / 4$, and applying the five points operator, and nine points $\overline{\mathrm{K}}$ operator, then we solve the system of linear equations, (we used here the method of successive displacement) then , we solve this example by finite element method, we get the global matrix, as in [3], and the results for all above method in the( 2.2).
a) Now, apply the five points operators to the example (2.1) (i.e). $\bar{H}$ formula) and we solve the corresponding linear system by iterations, we get the results shown in (2.2).

Finite difference (five points) of the irregular region it's show in (2.12)


Figure-2.1: Finite difference of the irregular region

$$
\left.\left.\left.\left[\begin{array}{cccccccc}
-\alpha_{0} & \alpha_{1} & 0 & \alpha_{2} & \alpha_{5} & 0 & 0 & 0  \tag{2.12}\\
\alpha_{3} & -\alpha_{0} & \alpha_{1} & \alpha_{6} & \alpha_{2} & \alpha_{5} & 0 & 0 \\
0 & \alpha_{3} & -\alpha_{0} & 0 & \alpha_{6} & \alpha_{2} & 0 & 0 \\
\alpha_{4} & \alpha_{8} & 0 & -\alpha_{0} & \alpha_{1} & 0 & \alpha_{2} & \alpha_{5} \\
\alpha_{7} & \alpha_{4} & \alpha_{8} & \alpha_{3} & -\alpha_{0} & \alpha_{1} & \alpha_{6} & \alpha_{2} \\
0 & \alpha_{7} & \alpha_{4} & 0 & \alpha_{3} & -\alpha_{0} & 0 & \alpha_{6} \\
0 & 0 & 0 & \alpha_{4} & \alpha_{8} & 0 & -\alpha_{0} & \alpha_{1} \\
0 & 0 & 0 & \alpha_{7} & \alpha_{4} & \alpha_{8} & \alpha_{3} & -\alpha_{0}
\end{array}\right] \right\rvert\, \begin{array}{c}
u_{1} \\
u_{2} \\
u_{3} \\
u_{4} \\
u_{5} \\
u_{6} \\
u_{7} \\
u_{8}
\end{array}\right]=\left\lvert\, \begin{array}{c}
-\left(\alpha_{6} B_{19}+\alpha_{3} B_{18}+\alpha_{7} B_{17}+\alpha_{4} B_{16}+\alpha_{8} B_{15}\right) \\
-\left(\alpha_{7} B_{16}+\alpha_{4} B_{15}+\alpha_{8} B_{14}\right) \\
-\left(\alpha_{7} B_{15}+\alpha_{4} B_{14}+\alpha_{8} B_{13}+\alpha_{1} B_{12}+\alpha_{5} B_{10}\right) \\
-\left(\alpha_{6} B_{20}+\alpha_{3} B_{19}+\alpha_{7} B_{18}\right) \\
-\left(\alpha_{5} B_{6}\right) \\
-\left(\alpha_{8} B_{11}+\alpha B_{9}+\alpha_{5} B_{8}+\alpha_{2} B_{7}\right) \\
-\left(\alpha_{6} B_{21}+\alpha_{2} B_{0}+\alpha_{5} B_{2}+\alpha_{3} B_{20}+\alpha_{7} B_{19}\right) \\
-\left(\alpha_{6} B_{1}+\alpha_{2} B_{3}+\alpha_{5} B_{4}+\alpha_{1} B_{5}\right)
\end{array}\right.\right\rfloor
$$

b) Now, we solve the example (2.1) by nine-points operator we get a linear system of equation as follows. [1]:

$$
\begin{align*}
& \text { The Finite difference (nine points) of the irregular region it's show in (2.13) } \\
& {\left[\begin{array}{cccccccc}
-2 & 40 & 0 & 4 & 1 & 0 & 0 & 0 \\
4 & -20 & 4 & 1 & 4 & 1 & 0 & 0 \\
0 & 4.271261 & -22.26115 & 0 & 1 & 4 & 0 & 0 \\
4 & 1 & 0 & -20 & 1 & 0 & 4 & 1 \\
1.093836 & 4 & 1 & 4 & -20.41421 & 4 & 1 & 4 \\
0 & 1.557775 & 4.861002 & 0 & 5.464102 & -39.10195 & 0 & 1.097168 \\
0 & 0 & 0 & 4.271261 & 1 & 0 & -22.26115 & 4 \\
0 & 0 & 0 & 1.557775 & 5.464102 & 1.097168 & 4.861002 & -39.10195
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3} \\
u_{4} \\
u_{5} \\
u_{6} \\
u_{7} \\
u_{8}
\end{array}\right]=\left[\begin{array}{c}
-7.8446354 \\
-2.301179 \\
-2.300905846 \\
-3.10952901 \\
-0.050058793 \\
-2.11265603 \\
0.743938245 \\
0.943981505
\end{array}\right]} \tag{2.13}
\end{align*}
$$

We solve this linear system we get the results it is shown in table (2.2)
c) We solve example (2.1) by finite element method as in fig 2.2 and we obtain the global matrixfor nonrectangular region thetable (2.1), then we get the solution by finite element method [3].


Figure-2.2: Finite element

|  | B17 | B16 | B15 | B14 | B13 | B7 | U3 | U2 | U1 | B18 | B6 | B5 | U6 | U5 | U4 | B19 | B4 | B3 | U8 | U7 | B20 | B21 | B0 | B1 | B2 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| B17 | 1 | -0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B16 | -0.5 | 2 | -0.5 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B15 | 0 | -0.5 | 2 | -0.5 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B14 | 0 | 0 | -0.5 | 2 | -0.5 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B13 | 0 | 0 | 0 | -0.5 | -1 | $0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B7 | 0 | 0 | 0 | 0 | -0.5 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U3 | 0 | 0 | 0 | -1 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U2 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | $0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | $0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $0.5$ | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U6 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| U4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 |
| B19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | $-0.5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $0.5$ | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| U8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | 0 | 0 | 0 | -1 |
| U7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | -1 | 4 | -1 | 0 | -1 | 0 | 0 |
| B20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 2 | -0.5 | 0 | 0 | 0 |
| B21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -0.5 | 1 | $0.5$ | 0 | 0 |
| B0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | $-0.5$ | 1 | $0.5$ | 0 |
| B1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $0.5$ | 1 | $0.5$ |
| B2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | $0.5$ | 1 |

Table-(2.1): Global matrix for non- rectangular region
The results (a, b, c) of non-rectangular region it's shown in the table (2.2)

| h | Maximum absolute error $\bar{H}$ | Maximum absolute error $\bar{K}$ | Maximum absolute error by <br> finite element |
| :---: | :---: | :---: | :---: |
| $1 / 4$ | $8.11786-04$ | $3.65823 \mathrm{E}-06$ | $4.929 \mathrm{E}-03$ |
| $1 / 8$ | $3.123 \mathrm{E}-04$ | $7.22706 \mathrm{E}-07$ |  |
| $1 / 16$ | $1.3479 \mathrm{E}-04$ | $1.19209 \mathrm{E}-07$ |  |

Table-(2.2): The results of non-rectangular region of three methods

## Summary and conclusion:

In this paper we used three method for solving laplace equation with Dirichlet boundary condition, first we use finite difference methods to get laplace operator with five points, and nine points, we got thesesolutions and compare these to solutions with finite element method , as it's clear in tables (1.2) and (2.2)

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