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NUMERICAL SOLUTIONS OF LAPLACE EQUATION

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ABSTRACT

In this paper, Laplace equation will be solved in rectangular and non-rectangular domains. The solution is done using two methods. The first method is finite difference method with two different operators of Laplace equation (five and nine-points). The second method is finite element method. A comparison took place between the three results (The two result by finite difference and the one taken from finite element method). The best result found from comparison is the nine-points method (the result was more accurate than the other two methods especially in regular domain). The conclusion from this paper is that the nine-point method is the most accurate method.

***Keywords**: Laplace equation;Finite element method; Regular and Irregular domains; Finite diffrence method.*

INTRODUCTION

There are many applications of Laplace equations as study state, Heat flow problems of Dirichlet's and Neuman boundary condition, and other application in physical applications, electric field and so on. [1], [5].

I -First, Laplace operator in the regular domain, of five points formula, it was derived in [5], [10], and the Laplace operator of nine points formula, it was derived in [5], [10], we take example of these methods. Then we solved by finite element method and compare the three results, as shown in the tables below.

The Laplace operator is given by first we use Taylor for regular domain, to derive this operators in regular domain of Laplace equation:

Let $\alpha^i = \frac{d^i}{dx^i}, \beta^i = \frac{d^i}{dY^i}, i = 1, 2, 3...$ Then:

$$u(\mathrm{x+h, y}) = \mathrm{u+h}\,\alpha\,\mathrm{u} + \frac{h^2}{2!}\alpha^2 u + ... \tag{1.1}$$

$$u(\mathrm{x}-h, y) = \mathrm{u-h}\,\alpha\,\mathrm{u} + \frac{h^2}{2!}\alpha^2 u - ... \tag{1.2}$$

$$u(\mathrm{x, y+h}) = \mathrm{u+h}\,\beta\,\mathrm{u} + \frac{h^2}{2!}\beta^2 u + ... \tag{1.3}$$

$$u(\mathrm{x, y-h}) = \mathrm{u-h}\,\beta\,\mathrm{u} + \frac{h^2}{2!}\beta^2 u - .... \tag{1.4}$$

$$Hu = h^2\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right] + \frac{h^4}{12}\left[\frac{d^4u}{dx^4} + \frac{d^4u}{dx4}\right] + \frac{h^6}{360}\left[\frac{d^6u}{dx^6} + \frac{d^6u}{dx^6}\right] + ... \tag{1.5}$$

$$Hu = h^2 \bigtriangledown^2 u + o(\mathrm{h}^4) \tag{1.6}$$

Where $Hu = u(\mathrm{x+h, y}) + u(\mathrm{x-h, y}) + u(\mathrm{x, y+h}) + (\mathrm{x, y-h}) - 4u(x, y)$

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The equation 1.6, is the formula of Laplace operator of five points.

In the same , we can derive the formula of Laplace operator of nine points,[1]

$$ku = 6h^2 \nabla u + o(\mathrm{h}^4) \tag{1.7}$$

where

$$ku = 4\{u(\mathrm{x+h,y}) + u(\mathrm{x+h,y}) + u(\mathrm{x,y+h}) + u(\mathrm{x,y-h})\} + u(\mathrm{x+h,y}) + u(\mathrm{x+h,y}) + u(\mathrm{x,y+h}) + u(\mathrm{x,y-h}) - 20\,\mathrm{u(x,y)}$$

May represents these formulas in equations (1.6) and (1.7) by the stencils respectively as follow [5], [1]:

H =

	1	
1	-4	1
	1	

K =

1	4	1
4	-20	4
1	4	1

III: Implementation: To show the applications of these methods and the method of finite method, we solve the following example.

Example 1.1: Let $\Delta u = 0$, on a square $0 \le x, y \le 1$ with boundary condition $u[x,y]|_{\llcorner} = \mathrm{e}^{(-2x)}\cos(2\,\mathrm{y})$

Solution: a) By taking $h = 1/4$, if we apply the operator H we obtained the linear system of equations corresponding to the figure (1.1) and as follows:

the matrix of five points of square region

$$\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} -B_{20} - B_{22} \\ -B_{19} \\ -B_{18} - B_{16} \\ -B_{23} \\ 0 \\ -B_{15} \\ -B_{10} - B_{24} \\ -B_{11} \\ -B_{14} - B_{12} \end{bmatrix} \tag{1.8}$$

In (1.8) the matrix of five points of square region.
b) If we applying the operator K on the figure (1.1) we obtained the linear the system of equations as shown below:

$$\begin{bmatrix} -20 & 4 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 4 & -20 & 4 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & -20 & 0 & 1 & 4 & 0 & 0 & 0 \\ 4 & 1 & 0 & -20 & 4 & 0 & 4 & 0 & 0 \\ 1 & 4 & 1 & 4 & -20 & 4 & 1 & 4 & 1 \\ 0 & 1 & 4 & 0 & 4 & -20 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 1 & 0 & -20 & 4 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 4 & -20 & 4 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 4 & -20 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} -B_{21} - 4B_{20} - B_{19} - 4B_{22} - B_{23} \\ -B_{20} - 4B_{19} - B_{18} \\ -B_{19} - 4B_{18} - B_{17} - 4B_{16} - B_{15} \\ -B_{24} - 4B_{23} - B_{22} \\ 0 \\ -B_{16} - 4B_{15} - B_{14} \\ -B_{11} - 4B_{10} - B_{25} - 4B_{24} - B_{23} \\ -B_{12} - 4B_{11} - B_{10} \\ -B_{15} - 4B_{14} - B_{13} - 4B_{12} - B_{11} \end{bmatrix} \tag{1.9}$$

In (1.9) the matrix of nine points of square region.

c) If we solve the example 1.1 by finite element method as in figure 1.2, such that, divided the region into triangles, and collected the global matrix for Regularin the table (1.1), [3].

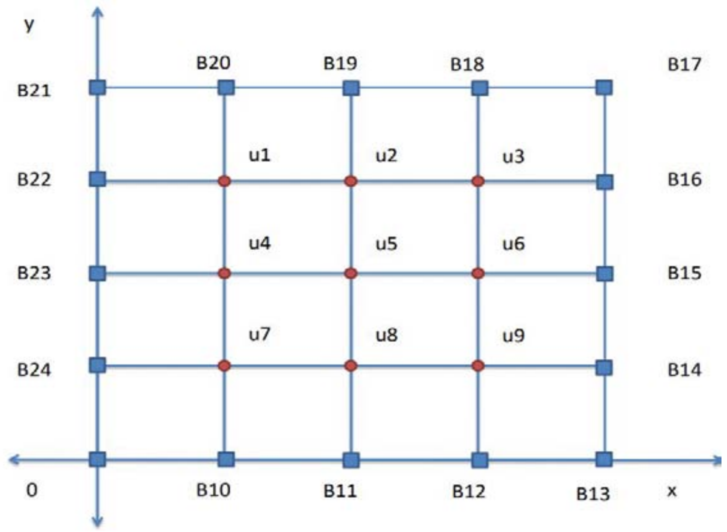

Figure-1.1: The regular region

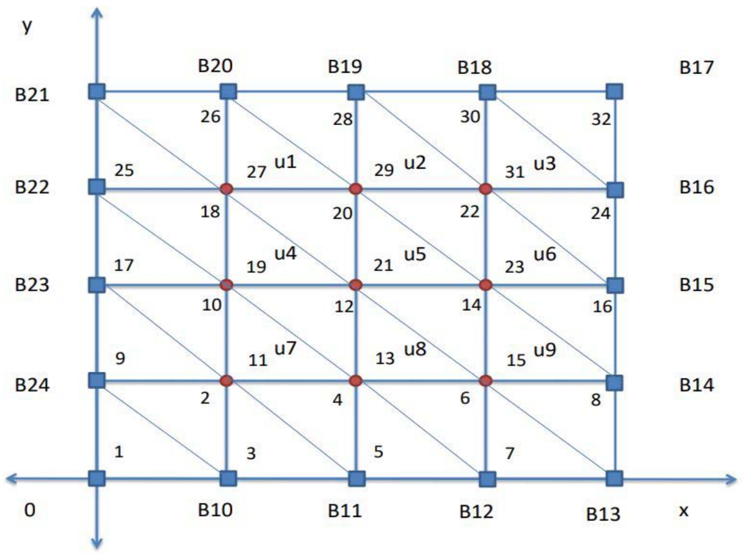

Figure-1.2: Mesh for finite element in regular region

	0	B10	B11	B12	B13	B14	U9	U8	U7	B24	B23	U4	U5	U6	B15	B16	U3	U2	U1	B22	B21	B20	B19	B18	B17
0	1	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B10	- 0.5	2	-0.5	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B11	0	-0.5	2	-0.5	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B12	0	0	-0.5	2	-0.5	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B13	0	0	0	-0.5	1	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B14	0	0	0	0	-0.5	2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
U9	0	0	0	-1	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
U8	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
U7	0	-1	0	0	0	0	0	-1	4	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
B24	0	0	0	0	0	0	0	0	-1	2	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B23	0	0	0	0	0	0	0	0	0	-0.5	2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
U4	0	0	0	0	0	0	0	0	-1	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0
U5	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0
U6	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	-1	0	0	0	0	0	0	0	0
B15	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	2	-0.5	0	0	0	0	0	0	0	0	0
B16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	2	-1	0	0	0	0	0	0	0	-0.5
U3	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-1	4	-1	0	0	0	0	0	-1	0
U2	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0
U1	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	-1	0	0	0
B22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	2	-0.5	0	0	0	0
B21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	1	-0.5	0	0	0
B20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-0.5	2	-0.5	0	0
B19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-0.5	2	-0.5	0
B18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-0.5	2	-0.5
B17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	1

Table- (1.1): Global Matrix for regular

The result (a, b, c) of regular region (Rectangular) it's show in the table (1.2)

Step Length	Maximum Error of H	Maximum Error of K	Maximum Error In Finte Element
¼	2.574297 E0-3	3.455287E-7	2.619269784E-3
1/5	1.809348E0-3	9.518665E-8	
1/8	7.382760E0-4	5.790283E0-9	
1/10	4.7066583E0-4	3.142247E0-9	

Table- (1.2): Result of regular region (Rectangular)

II-Secondly, the Laplace operator in the irregular domain, using five points, [1], [5], [10].and nine-points finite difference formulas, [1], then we solve this example by finite elements method and compare the results as shown in the tables below.

Now, for the non-rectangular region we have for five points

Let $\alpha^{i}=\dfrac{d^{i}}{dx^{i}},\beta^{i}=\dfrac{d^{i}}{dY^{i}},i=1,2,3\ldots$ Then:

$$u(\mathrm{x}+\mathrm{h}_1,\mathrm{y})=\mathrm{u}+\mathrm{h}_1\,\alpha\,\mathrm{u}+\frac{h_1}{2!}\alpha^2u+\ldots. \tag{2.1}$$

$$u(\mathrm{x}-h_3,y)=\mathrm{u}-\mathrm{h}_3\,\alpha\,\mathrm{u}+\frac{h_3}{2!}\alpha^2u-\ldots \tag{2.2}$$

$$u(\mathrm{x},\mathrm{y}+\mathrm{h}_2)=\mathrm{u}+\mathrm{h}_2\,\beta\,\mathrm{u}+\frac{h_2^{\,2}}{2!}\beta^2u+\ldots. \tag{2.3}$$

$$u(\mathrm{x},\mathrm{y}-\mathrm{h}_4)=\mathrm{u}-\mathrm{h}_4\,\beta\,\mathrm{u}+\frac{h_4^{\,2}}{2!}\beta^2u-\ldots. \tag{2.4}$$

$$u_{xx} = \frac{h_3 u_1 + h_1 u_3 - (h_1 + h_3)\, u_0}{\left(\frac{1}{2}\right) h_1\, h_3 (h_1 + h_3)} \tag{2.5a}$$

$$u_{yy} = \frac{h_4 u_2 + h_2 u_4 - (h_2 + h_4)\, u_0}{\left(\frac{1}{2}\right) h_2\, h_4 (h_2 + h_4)} \tag{2.5b}$$

$$\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 = 0; \tag{2.6}$$

$$\alpha_1 = \frac{2}{h^2}\frac{1}{s_1(s_1+s_3)}, \quad \alpha_3 = \frac{2}{h^2}\frac{1}{s_3(s_1+s_3)}$$

$$\alpha_2 = \frac{2}{h^2}\frac{1}{s_2(s_2+s_4)}, \quad \alpha_4 = \frac{2}{h^2}\frac{1}{s_4(s_2+s_4)}$$

$$\alpha_0 = -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \text{where} \quad h_i\, s_i \le h, s_i \le 1, i = 1,2,3,4$$

$$u(x+h_5, y+h_5) = u + h_5(\alpha+\beta)u + \frac{h_5^{\,2}}{2!}(\alpha+\beta)^2 u + ... \tag{2.7}$$

$$u(x-h_6, y+h_6) = u + h_6(-\alpha+\beta)u + \frac{h_6^{\,2}}{2!}(-\alpha+\beta)^2 u + ... \tag{2.8}$$

$$u(x-h_7, y-h_7) = u + h_7(-\alpha-\beta)u + \frac{h_7^{\,2}}{2!}(-\alpha-\beta)^2 u + ... \tag{2.9}$$

$$u(x+h_8, y-h_8) = u + h_8(\alpha-\beta)u + \frac{h_8^{\,2}}{2!}(\alpha-\beta)^2 u + ... \tag{2.10}$$

Find the Laplace operator from the equations (2.1) to (2.4) and multiply by 4, and from (2.7) to (2.10) and multiply by 2, add the two results, we get the following equation.

$$\begin{aligned} \alpha_i u_i &= 6(\alpha^2+\beta^2)u + [4(M3\alpha^3 + N3\beta^3) + 2(D3(\alpha+\beta)^3) + K3(-\alpha+\beta)^3)]u \\ &+ \frac{1}{4!}[4(M4\alpha^4 + N4\beta^4) + 2(D4(\alpha+\beta)^4) + K4(\alpha-\beta)^4)]u \\ &+ \frac{1}{5!}[4(M5\alpha^5 + N5\beta^5) + 2(D5(\alpha+\beta)^5) + K5(-\alpha+\beta)^5)]u \\ &+ \frac{1}{6!}[4(M6\alpha^6 + N6\beta^6) + 2(D6(\alpha+\beta)^6) + K6(\alpha+\beta)^6)]u + \end{aligned} \tag{2.11}$$

Where

$$\alpha_1 = \frac{8}{h^2 s_1(s_1+s_3)}, \quad \alpha_2 = \frac{8}{h^2 s_2(s_2+s_4)}, \quad \alpha_3 = \frac{8}{h^2 s_3(s_3+s_1)}, \quad \alpha_4 = \frac{8}{h^2 s_4(s_4+s_2)}$$

$$\alpha_5 = \frac{8}{h^2 s_5(s_5+s_7)}, \quad \alpha_6 = \frac{8}{h^2 s_6(s_6+s_8)}, \quad \alpha_7 = \frac{8}{h^2 s_7(s_7+s_5)}, \quad \alpha_8 = \frac{8}{h^2 s_8(s_8+s_6)},$$

$$\alpha_0 = \sum_{i=1}^{8} \alpha_i, \, where, h_i = s_i h, 0 \le s_i \le 1, i = 1,2,3...$$

Example 2.1: Find the solution of Laplace equation $\Delta u = 0$ in the first quadrant bounded by the circle $x^2 + y^2 = 1$, as in figure (2.1). and the boundary condition and the exact solution are given by $u[x,y]_{\llcorner} = e^{(-2x)}\cos(2\,y)$.

Solution: By taking the step length $h = 1/4$, and applying the five points operator, and nine points $\overline{K}$ operator, then we solve the system of linear equations, (we used here the method of successive displacement) then , we solve this example by finite element method, we get the global matrix, as in [3], and the results for all above method in the(2.2).

a) Now, apply the five points operators to the example (2.1) (i.e). $\overline{H}$ formula) and we solve the corresponding linear system by iterations, we get the results shown in (2.2).

Finite difference (five points) of the irregular region it's show in (2.12)

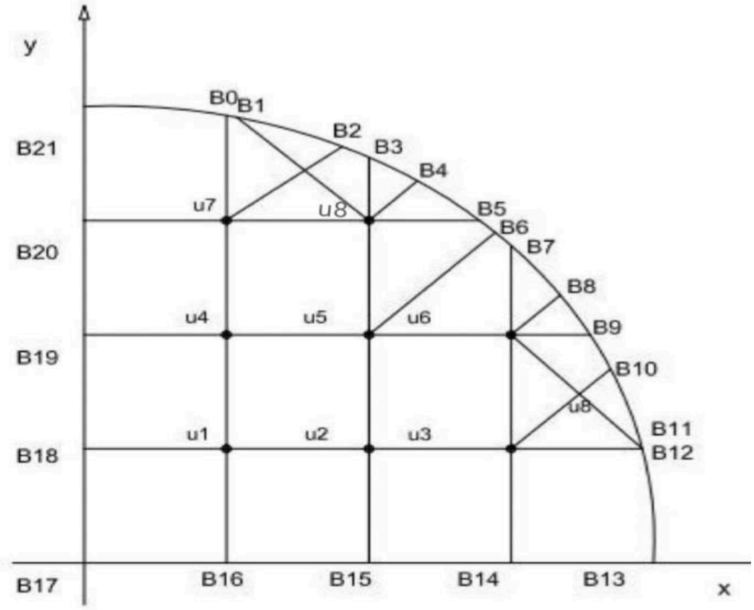

Figure-2.1: Finite difference of the irregular region

$$\begin{bmatrix} -\alpha_0 & \alpha_1 & 0 & \alpha_2 & \alpha_5 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_0 & \alpha_1 & \alpha_6 & \alpha_2 & \alpha_5 & 0 & 0 \\ 0 & \alpha_3 & -\alpha_0 & 0 & \alpha_6 & \alpha_2 & 0 & 0 \\ \alpha_4 & \alpha_8 & 0 & -\alpha_0 & \alpha_1 & 0 & \alpha_2 & \alpha_5 \\ \alpha_7 & \alpha_4 & \alpha_8 & \alpha_3 & -\alpha_0 & \alpha_1 & \alpha_6 & \alpha_2 \\ 0 & \alpha_7 & \alpha_4 & 0 & \alpha_3 & -\alpha_0 & 0 & \alpha_6 \\ 0 & 0 & 0 & \alpha_4 & \alpha_8 & 0 & -\alpha_0 & \alpha_1 \\ 0 & 0 & 0 & \alpha_7 & \alpha_4 & \alpha_8 & \alpha_3 & -\alpha_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} -(\alpha_6 B_{19} + \alpha_3 B_{18} + \alpha_7 B_{17} + \alpha_4 B_{16} + \alpha_8 B_{15}) \\ -(\alpha_7 B_{16} + \alpha_4 B_{15} + \alpha_8 B_{14}) \\ -(\alpha_7 B_{15} + \alpha_4 B_{14} + \alpha_8 B_{13} + \alpha_1 B_{12} + \alpha_5 B_{10}) \\ -(\alpha_6 B_{20} + \alpha_3 B_{19} + \alpha_7 B_{18}) \\ -(\alpha_5 B_6) \\ -(\alpha_8 B_{11} + \alpha\ B_9 + \alpha_5 B_8 + \alpha_2 B_7) \\ -(\alpha_6 B_{21} + \alpha_2 B_0 + \alpha_5 B_2 + \alpha_3 B_{20} + \alpha_7 B_{19}) \\ -(\alpha_6 B_1 + \alpha_2 B_3 + \alpha_5 B_4 + \alpha_1 B_5) \end{bmatrix} \quad (2.12)$$

b) Now, we solve the example (2.1) by nine-points operator we get a linear system of equation as follows. [1]: The Finite difference (nine points) of the irregular region it's show in (2.13)

$$\begin{bmatrix} -2 & 40 & 0 & 4 & 1 & 0 & 0 & 0 \\ 4 & -20 & 4 & 1 & 4 & 1 & 0 & 0 \\ 0 & 4.271261 & -22.26115 & 0 & 1 & 4 & 0 & 0 \\ 4 & 1 & 0 & -20 & 1 & 0 & 4 & 1 \\ 1.093836 & 4 & 1 & 4 & -20.41421 & 4 & 1 & 4 \\ 0 & 1.557775 & 4.861002 & 0 & 5.464102 & -39.10195 & 0 & 1.097168 \\ 0 & 0 & 0 & 4.271261 & 1 & 0 & -22.26115 & 4 \\ 0 & 0 & 0 & 1.557775 & 5.464102 & 1.097168 & 4.861002 & -39.10195 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} -7.8446354 \\ -2.301179 \\ -2.300905846 \\ -3.10952901 \\ -0.050058793 \\ -2.11265603 \\ 0.743938245 \\ 0.943981505 \end{bmatrix} \quad (2.13)$$

We solve this linear system we get the results it is shown in table (2.2)

c) We solve example (2.1) by finite element method as in fig 2.2 and we obtain the global matrixfor non-rectangular region thetable (2.1), then we get the solution by finite element method [3].

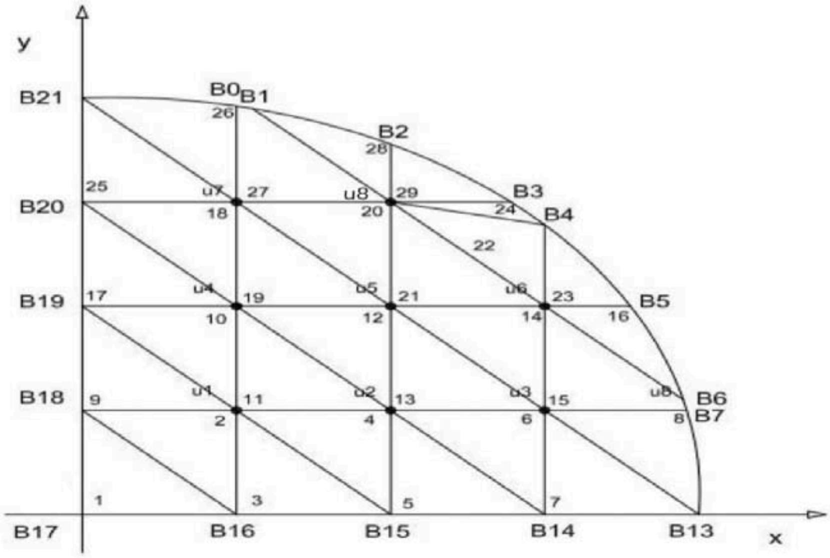

Figure-2.2: Finite element

	B17	B16	B15	B14	B13	B7	U3	U2	U1	B18	B6	B5	U6	U5	U4	B19	B4	B3	U8	U7	B20	B21	B0	B1	B2
B17	1	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B16	-0.5	2	-0.5	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B15	0	-0.5	2	-0.5	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B14	0	0	-0.5	2	-0.5	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B13	0	0	0	-0.5	-1	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B7	0	0	0	0	-0.5	2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
U3	0	0	0	-1	0	-1	4	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
U2	0	0	-1	0	0	0	-1	4	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
U1	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B18	0	0	0	0	0	0	0	0	-1	2	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B6	0	0	0	0	0	0	0	0	0	0	2	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0
B5	0	0	0	0	0	0	0	0	0	0	-0.5	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
U6	0	0	0	0	0	0	-1	0	0	0	0	-1	4	-1	0	0	-1	0	0	0	0	0	0	0	0
U5	0	0	0	0	0	0	0	-1	0	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0
U4	0	0	0	0	0	0	0	0	-1	0	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0
B19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	2	0	0	0	0	0	0	0	0	0
B4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-0.5	0	0	0	0	0	0	0	0	0
B3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	1	-1	0	0	0	0	0	0	0	0
U8	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	0	-1
U7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	-1	0	0
B20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	2	-0.5	0	0	0
B21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	1	-0.5	0	0
B0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-0.5	1	-0.5	0
B1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	1	-0.5
B2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	-0.5	1

Table-(2.1): Global matrix for non- rectangular region

The results (a, b, c) of non-rectangular region it's shown in the table (2.2)

h	Maximum absolute error $\overline{H}$	Maximum absolute error $\overline{K}$	Maximum absolute error by finite element
¼	8.11786-04	3.65823E-06	4.929E-03
1/8	3.123E-04	7.22706E-07	
1/16	1.3479E-04	1.19209E-07	

Table-(2.2): The results of non-rectangular region of three methods

Summary and conclusion:

In this paper we used three method for solving laplace equation with Dirichlet boundary condition, first we use finite difference methods to get laplace operator with five points ,and nine points, we got thesesolutions and compare these to solutions with finite element method , as it's clear in tables (1.2) and (2.2)

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