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THE RELIABILITY ANALYSIS OF M/G/1 RETRIAL QUEUEING SYSTEM FOR THIRD OPTIONAL SERVICE SUBJECT TO SERVER BREAKDOWN AND REPAIR

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ABSTRACT

In this paper we are analysing the reliability analysis of third optional service for a single server retrial queueing system subject to server breakdowns occurs randomly at any phase of service. Arrival of passengers follows the Poisson's distribution law with parameter λ, while the servers serving their services *(parameter μ)* using exponential distribution law on the basis of FCFS discipline. This exponential distribution rate comes to fail at second phase and third optional phase of service. We also derive the joint probability distribution for the server, the probability generating function of the stationary queue size distribution, some important performance measures and reliability analysis with a numerical example. We obtain the transient and the steady state solutions for both queueing and reliability measures, by using a supplementary variable method.

Keywords – *The reliability analysis, optional service, server breakdown and repair, M/G/1 queue, stationary queue size distribution and reliability index.*

1. INTRODUCTION

Queueing models with optional service are the new upcoming requirements in queueing systems. These systems are mostly suitable for passport sevakendra, banking service, telecommunications, operating systems etc. It is necessary to check the reliability analysis of these models based on different variations. We are also moving towards third optional service respect to server breakdown and repair. The first and second are the essential services for all the arriving customers. These types of systems are known as multiphase queueing systems. K.C.Madan [11] studied an M/G/1 queue with second optional service in which first essential service time follows a general distribution but second optional service is assumed to be exponentially distributed. Medhi [12] generalized this model by considering that the second optional is also governed by a general distribution. D.H.Shi [2] gives the new method for the calculation of mean number of failures and availability of server for a repairable system. W. Li, D. Shi, X. Chao, [3] were the first to make paper in the segment of reliability analysis of M/G/1 queueing system with server breakdowns and vacations. Y. Tang, [6] represents the single server M/G/1 queueing system subject to breakdowns – their reliability with queueing problems. O. Kella, [10] was the first to provide the optimization control of vacation scheme in an M/G/1 queue. This optimization control is helpful for the solution of mean number of failures in the system. Gautam Chaudhary [13] to [16] works for Bernoulli vacation, delayed repair, multiple vacation policy for two phases of service respect to server breakdown and repair. D.R. Cox, [17] gives the most suitable tool of supplementary variables for non Markovian stochastic process.

The remaining overview of the paper is as follows –In point 2, we represent the description of mathematical model and assumptions. Point 3 stands for the model solution and derivations of the stationary distribution of the queue size for the server. Point 4 stands for the reliability analysis of the system and derivation of availability of server for steady state and number of failures in the system. Point 5 stands for numerical illustration to know the effect of reliability factor of the system. Finally conclusion is drawn in last one. To derive the probability generating function for queue size distribution at different phases of service, we apply the supplementary variable technique by introducing one or more supplementary variables.

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2. THE MATHEMATICAL MODEL AND ASSUMPTIONS

The assumptions of this model are: Let us suppose an M/G/1 retrial queueing system. In this system arrivals of customers are according to a Poisson process with arrival rate 'λ'. It is a single server model. The system serves according to first come, first serve (FCFS) service discipline. All the arriving customers join the queue and serves the essential preliminary first phase and second phase of regular service. Let *B(v)* and *b(v)*respectively be the probability distribution function and density function of the first service times with mean rate $1 / \mu_1$ and $\mu_1(x)$ is the hazard rate function, and let *C(v)* and *c(v)*respectively be the probability distribution function and density function of the second service times with mean rate $1 / \mu_2$ and $\mu_2(x)$ is the hazard rate function. When the second service of a customer is completed then customer can apply for third optional service with probability *p* or he can leave the system permanently with probability *1 – p*. On leaving the system if there is any customer at the head of the queue joins the server and apply for first essential service. The third service times are consider to be exponentially distributed with mean service rate $1 / \mu_3$. Let us suppose for first essential service the server has exponential distribution with mean $1 / \alpha_1$. For second and third service the server fails with an exponential distribution of rate α_2 and α_3. While serving the customers the server can break down at any instant, and if it happens then the server is sent for repair. The distributions of the repair time for all the three phases are arbitrarily distributed with the probability distribution functions $G_i(x)$, for $i = 1,2,3$, with probability density functions $g_i(x)$, means are $1 / \beta_i$ for $i = 1,2,3$ and $\beta_i(x)$ are the hazard rate function for $i = 1,2,3$. When the server is repair it again restarts its service to serve the customers, where the service time is cumulative. All the process involved in the system are consider to be mutually exclusive to each other.

Let the total number of customers in the system is N(t) at time t. By using a supplementary variable method, we obtain the transient and the steady state solutions for both queueing and reliability measures. To solve under a Markov process, let X(t) = the elapsed service time of the customer which is served, Y(t) = the elapsed repair time of the customer for failed server at time t, Q(t) = probability of server is at idle state at time t, $P_n^{(1)}(t;x)dx$ = probability of n customers in the queue at time t, excluding the one which is serve, the elapsed service time of a customer between x and $x + dx$ during first essential service, $P_n^{(2)}(t;x)dx$ = probability of n customers in the queue at time t, the elapsed service time of a customer between x and $x + dx$ during second essential service, $P_n^{(3)}(t)dx$ = probability of n customers in the queue during third optional service. $S_n^{(i)}(t,x;y)dy$ = is the joint probability of n customers at time t for elapsed service time under service is equal to x, and server is repaired with elapsed repair time between y and $y + dy$, where $i = 1,2,3$ for first, second essential service and third optional service respectively. To obtain a bivariate Markov process $\{N(t), X(t)\}$at any instant t, system can be characterised by the random variables *N(t), X(t)* and *Y(t)*.From time range *t*to $t + dt$and letting $\Delta t \to 0$, the system of forward equations for $n = 0, 1, 2….$ are governed as follows –

$$\left(\frac{d}{dt}+\lambda\right)Q(t)=\mu_3 P_0^{(3)}(t)+(1-p)\left[\int_0^\infty P_0^{(1)}(t,x)\mu_1(x)dx+\int_0^\infty P_0^{(2)}(t,x)\mu_2(x)dx\right] \tag{2.1}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_1(x)+\lambda+\alpha_1\right]P_n^{(1)}(t,x)=\lambda P_{n-1}^{(1)}(t,x)+\int_0^\infty S_n^{(1)}(t,x;y)\beta_1(y)dy \tag{2.2}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_2(x)+\lambda+\alpha_2\right]P_n^{(2)}(t,x)=\lambda P_{n-1}^{(2)}(t,x)+\int_0^\infty S_n^{(2)}(t,x;y)\beta_2(y)dy \tag{2.3}$$

$$\left[\frac{\partial}{\partial t}+\mu_3(x)+\lambda+\alpha_3\right]P_n^{(3)}(t)=\lambda P_{n-1}^{(3)}(t)+p\left[\int_0^\infty P_n^{(1)}(t,x)\mu_1(x)dx+\int_0^\infty P_n^{(2)}(t,x)\mu_2(x)dx\right]$$
$$+\int_0^\infty S_n^{(3)}(t,y)\beta_3(y)dy \tag{2.4}$$

$$\left[\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\lambda+\beta_1(y)\right]S_n^{(1)}(t,x;y)=\lambda S_{n-1}^{(1)}(t,x;y) \tag{2.5}$$

$$\left[\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\lambda+\beta_2(y)\right]S_n^{(2)}(t,x;y)=\lambda S_{n-1}^{(2)}(t,x;y) \tag{2.6}$$

$$\left[\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\lambda+\beta_3(y)\right]S_n^{(3)}(t,y)=\lambda S_{n-1}^{(3)}(t,y) \tag{2.7}$$

Under the boundary conditions,

$$P_n^{(1)}(t;0) = (1-p)\left[\int_0^\infty P_{n+1}^{(1)}(t,x)\mu_1(x)dx + \int_0^\infty P_{n+1}^{(2)}(t,x)\mu_2(x)dx\right] + \mu_3(x)P_{n+1}^{(3)}(t)\text{, } n \geq 1 \tag{2.8}$$

$$P_0^{(1)}(t;0) = (1-p)\left[\int_0^\infty P_1^{(1)}(t,x)\mu_1(x)dx + \int_0^\infty P_1^{(2)}(t,x)\mu_2(x)dx\right] + \mu_3(x)P_1^{(3)}(t) + \lambda Q(t)\text{, } n\geq 1 \tag{2.9}$$

$$S_n^{(1)}(t,x;0) = \alpha_1 P_n^{(1)}(t;x) \tag{2.10}$$

$$S_n^{(2)}(t,x;0) = \alpha_2 P_n^{(2)}(t;x) \tag{2.11}$$

$$S_n^{(3)}(t;0) = \alpha_3 P_n^{(3)}(t) \tag{2.12}$$

The set of above equations (2.1) – (2.12) are solved together under the normalising condition,

$$1 = Q(t) + \sum_{n=0}^{\infty}\left[\begin{array}{l} P_n^{(3)}(t) + \int_0^\infty P_n^{(1)}(t,x)dx + \int_0^\infty P_n^{(2)}(t,x)dx + \int_0^\infty S_n^{(3)}(t,y)dy + \int_0^\infty\int_0^\infty S_n^{(1)}(t,x,y)dxdy \\ \qquad + \int_0^\infty\int_0^\infty S_n^{(2)}(t,x,y)dxdy \end{array}\right]$$

with initial condition Q(0) =1, with any fixed *t, x, y*.

3. THE MODEL SOLUTION

Let us introduce the following PGFs,

$$P_v^{(1)}(t,x,z) = \sum_{n=0}^{\infty} z^n P_n^{(1)}(t,x)\text{, } P_v^{(2)}(t,x,z) = \sum_{n=0}^{\infty} z^n P_n^{(2)}(t,x)\text{, } P_v^{(3)}(t,z) = \sum_{n=0}^{\infty} z^n P_n^{(3)}(t)$$

$$S_v^{(1)}(t,x,y,z) = \sum_{n=0}^{\infty} z^n S_n^{(1)}(t,x,y)\text{ , } S_v^{(2)}(t,x,y,z) = \sum_{n=0}^{\infty} z^n S_n^{(2)}(t,x,y)\text{ , } S_v^{(3)}(t,y,z) = \sum_{n=0}^{\infty} z^n S_n^{(3)}(t,y)$$

Using definition of Laplace transform *f(r)*,

$$\widehat{f}(r) = \int_0^\infty e^{-rt} f(t)dt$$

and define $\xi_i(r,z) = r + \lambda - \lambda z + \alpha_i - \alpha_i \widehat{g}_i(r + \lambda - \lambda z)$, for *i = 1,2,3*

Theorem 3.1: The Laplace – Stieltjes transforms and probability for stationary queue size distribution is given by -

$$\widehat{P}_v^{(1)}(r,x,z) = \frac{[\xi_2(r,z)+\mu_2][(r+\lambda-\lambda z)\widehat{Q}(r)-1]}{[\widehat{c}(\xi_1(r,z))-z][\xi_2(r,z)+\mu_2] - p\widehat{c}\,\xi_1(r,z)\xi_2(r,z)}\exp\{-\xi_1(r,z)x\}(1-B_1(x))$$

$$\widehat{P}_v^{(2)}(r,x,z) = \frac{[\xi_3(r,z)+\mu_3][(r+\lambda-\lambda z)\widehat{Q}(r)-1]}{[\widehat{c}(\xi_2(r,z))-z][\xi_2(r,z)+\mu_2] - p\widehat{c}\,\xi_1(r,z)\xi_2(r,z)}\exp\{-\xi_2(r,z)x\}(1-B_2(x))$$

$$\widehat{P}_v^{(3)}(r,x,z) = \frac{p\widehat{c}\,\xi_1(r,z)[(r+\lambda-\lambda z)\widehat{Q}(r)-1]}{[\widehat{c}(\xi_1(r,z))-z][\xi_2(r,z)+\mu_2] - p\widehat{c}\,\xi_1(r,z)\xi_2(r,z)}$$

$$\widehat{Q}(r) = \frac{1}{r+\lambda-\lambda z_r}$$

$$S_v^{(1)}(r,x,y,z) = \alpha_1 \widehat{P}_v^{(1)}(r,x,z)\exp\{-(r+\lambda-\lambda z)y\}(1-G_1(y))$$

$$S_v^{(2)}(r,x,y,z) = \alpha_2 \widehat{P}_v^{(2)}(r,x,z)\exp\{-(r+\lambda-\lambda z)y\}(1-G_2(y))$$

$$S_v^{(3)}(r,y,z) = \alpha_3 \widehat{P}_v^{(3)}(r,z)\exp\{-(r+\lambda-\lambda z)y\}(1-G_3(y))$$

where z_r is the root of the equation,

$$x = \widehat{c}\,\xi_1(r,x) - \frac{p\xi_2(r,x)\widehat{c}\,\xi_1(r,x)}{\xi_2(r,x)+\mu_2}\text{ inside } |z| = 1, \text{Re}(r) > 0.$$

Proof: Taking Laplace transform from equation (2.1) – (2.7) with respect to t we have

$$(r+\lambda)\widehat{Q}(r) - 1 = \mu_3 \widehat{P}_0^{(3)}(r) + (1-p)\left[\int_0^\infty \widehat{P}_0^{(1)}(r,x)\mu_1(x)dx + \int_0^\infty \widehat{P}_0^{(2)}(r,x)\mu_2(x)dx\right] \tag{3.1}$$

$$\frac{\partial}{\partial x}\widehat{P}_n^{(1)}(r,x)+(r+\mu_1(x)+\lambda+\alpha_1)\widehat{P}_n^{(1)}(r,x)=\lambda\widehat{P}_{n-1}^{(1)}(r,x)+\int_0^\infty \widehat{S}_n^{(1)}(r,x;y)\beta_1(y)dy \quad (3.2)$$

$$\frac{\partial}{\partial x}\widehat{P}_n^{(2)}(r,x)+(r+\mu_2(x)+\lambda+\alpha_2)\widehat{P}_n^{(2)}(r,x)=\lambda\widehat{P}_{n-1}^{(2)}(r,x)+\int_0^\infty \widehat{S}_n^{(2)}(r,x;y)\beta_2(y)dy \quad (3.3)$$

$$\left[r+\mu_3(x)+\lambda+\alpha_3\right]\widehat{P}_n^{(3)}(r)=\lambda\widehat{P}_{n-1}^{(3)}(r)+p\left[\int_0^\infty \widehat{P}_n^{(1)}(r,x)\mu_1(x)dx+\int_0^\infty \widehat{P}_n^{(2)}(r,x)\mu_2(x)dx\right]$$
$$+\int_0^\infty \widehat{S}_n^{(3)}(r,x;y)\beta_3(y)dy \quad (3.4)$$

$$\frac{\partial}{\partial y}\widehat{S}_n^{(1)}(r,x;y)+(s+\lambda+\beta_1(y)\widehat{S}_n^{(1)}(r,x;y)=\lambda\widehat{S}_{n-1}^{(1)}(r,x;y) \quad (3.5)$$

$$\frac{\partial}{\partial y}\widehat{S}_n^{(2)}(r,x;y)+(s+\lambda+\beta_2(y)\widehat{S}_n^{(2)}(r,x;y)=\lambda\widehat{S}_{n-1}^{(2)}(r,x;y) \quad (3.6)$$

$$\frac{\partial}{\partial y}\widehat{S}_n^{(3)}(r,y)+(s+\lambda+\beta_3(y)\widehat{S}_n^{(3)}(r,y)=\lambda\widehat{S}_{n-1}^{(3)}(r,y) \quad (3.7)$$

Similarly taking Laplace transform from equation (2.8) – (2.12) with respect to t we have

$$\widehat{P}_n^{(1)}(r;0)=(1-p)\left[\int_0^\infty \widehat{P}_{n+1}^{(1)}(r,x)\mu_1(x)dx+\int_0^\infty \widehat{P}_{n+1}^{(2)}(r,x)\mu_2(x)dx\right]+\mu_3(x)\widehat{P}_{n+1}^{(3)}(r),\ n\geq 1 \quad (3.8)$$

$$\widehat{P}_0^{(1)}(r;0)=(1-p)\left[\int_0^\infty \widehat{P}_1^{(1)}(r,x)\mu_1(x)dx+\int_0^\infty \widehat{P}_1^{(2)}(r,x)\mu_2(x)dx\right]+\mu_3(x)\widehat{P}_1^{(3)}(r)+\lambda\widehat{Q}(r),n\geq 1 \quad (3.9)$$

$$\widehat{S}_n^{(1)}(r,x;0)=\alpha_1\widehat{P}_n^{(1)}(r;x) \quad (3.10)$$

$$\widehat{S}_n^{(2)}(r,x;0)=\alpha_2\widehat{P}_n^{(2)}(r;x) \quad (3.11)$$

$$\widehat{S}_n^{(3)}(r;0)=\alpha_3\widehat{P}_n^{(3)}(r) \quad (3.12)$$

On multiplying equation (3.2) by $\sum_{n=0}^{\infty} z^n$, using probability generating function defined in section 3 and on simplifying we have,

$$\frac{\partial}{\partial x}\widehat{P}_v^{(1)}(r,x,z)=-(r+\alpha_1+\mu_1+\lambda-\lambda z)\widehat{P}_v^{(1)}(r,x,z)+\int_0^\infty \widehat{S}_v^{(1)}(r,x,y,z)\beta_1(y)dy \quad (3.13)$$

Again applying similar operations from equation (3.4) – (3.12), we have

$$\frac{\partial}{\partial y}\widehat{S}_v^{(1)}(r,x,y,z)=-(s+\lambda-\lambda z+\beta_1(y))\widehat{S}_v^{(1)}(r,x,y,z) \quad (3.14)$$

$$\frac{\partial}{\partial y}\widehat{S}_v^{(2)}(r,x,y,z)=-(s+\lambda-\lambda z+\beta_2(y))\widehat{S}_v^{(2)}(r,x,y,z) \quad (3.15)$$

$$\frac{\partial}{\partial y}\widehat{S}_v^{(3)}(r,y,z)=-(s+\lambda-\lambda z+\beta_3(y))\widehat{S}_v^{(3)}(r,y,z) \quad (3.16)$$

$$\left[r+\mu_3(x)+\lambda-\lambda z+\alpha_3\right]\widehat{P}_v^{(3)}(r,z)=p\left[\int_0^\infty \widehat{P}_v^{(1)}(r,x,z)\mu_1(x)dx+\int_0^\infty \widehat{P}_v^{(2)}(r,x,z)\mu_2(x)dx\right]$$
$$+\int_0^\infty \widehat{S}_v^{(3)}(r,y,z)\beta_3(y)dy \quad (3.17)$$

$$z\widehat{P}_v^{(1)}(r,0,z)=(1-p)\left[\int_0^\infty \widehat{P}_v^{(1)}(r,x,z)\mu_1(x)dx+\int_0^\infty \widehat{P}_v^{(2)}(r,x,z)\mu_2(x)dx\right]+\mu_3(x)\widehat{P}_v^{(3)}(r,z)$$
$$-(1-p)\left[\int_0^\infty \widehat{P}_0^{(1)}(r,x)\mu_1(x)dx+\int_0^\infty \widehat{P}_0^{(2)}(r,x)\mu_2(x)dx\right]+\mu_3(x)\widehat{P}_0^{(3)}(r)+\lambda z\widehat{Q}(r) \quad (3.18)$$

$$\widehat{S}_v^{(1)}(r,x,0,z)=\alpha_1\widehat{P}_v^{(1)}(r;x,z) \quad (3.19)$$

$$\widehat{S}_v^{(2)}(r,x,0,z)=\alpha_1\widehat{P}_v^{(2)}(r;x,z) \quad (3.20)$$

$$\widehat{S}_v^{(3)}(r;0,z)=\alpha_3\widehat{P}_v^{(3)}(r,z) \quad (3.21)$$

On rewriting equation (3.1)

$$-\left[\mu_3\widehat{P}_0^{(3)}(r)+(1-p)\left[\int_0^\infty \widehat{P}_0^{(1)}(r,x)\mu_1(x)dx+\int_0^\infty \widehat{P}_0^{(2)}(r,x)\mu_2(x)dx\right]\right]=1-(r+\lambda)\widehat{Q}(r) \tag{3.22}$$

Using equation (3.22) in equation (3.18) then,

$$z\widehat{P}_v^{(1)}(r,0,z)=1+(1-p)\left[\int_0^\infty \widehat{P}_v^{(1)}(r,x,z)\mu_1(x)dx+\int_0^\infty \widehat{P}_v^{(2)}(r,x,z)\mu_2(x)dx\right]+\mu_3(x)\widehat{P}_v^{(3)}(r,z)$$
$$+(\lambda z-\lambda-r)\widehat{Q}(r) \tag{3.23}$$

Now solving equations (3.11), (3.12) and (3.13)

$$\widehat{S}_v^{(1)}(r,x,y,z)=\widehat{S}_v^{(1)}(r,x,0,z)\exp[-(s+\lambda-\lambda z)y](1-G_1(y)) \tag{3.24}$$

$$\widehat{S}_v^{(2)}(r,x,y,z)=\widehat{S}_v^{(2)}(r,x,0,z)\exp[-(s+\lambda-\lambda z)y](1-G_2(y)) \tag{3.25}$$

$$\widehat{S}_v^{(3)}(r,y,z)=\alpha_3\widehat{P}_v^{(3)}(r,z)\exp[-(s+\lambda-\lambda z)y](1-G_3(y)) \tag{3.26}$$

Put value of (3.24) in equation (3.13) becomes

$$\frac{\partial}{\partial x}\widehat{P}_v^{(1)}(r,x,z)=-(r+\alpha_1+\mu_1+\lambda-\lambda z)\widehat{P}_v^{(1)}(r,x,z)+\alpha_1\widehat{P}_v^{(1)}(r,x,z)\int_0^\infty \exp[-(s+\lambda-\lambda z)y](1-G_1(y))\beta_1(y)dy$$

on solving above equation,

$$\widehat{P}_v^{(1)}(r,x,z)=\widehat{P}_v^{(1)}(r,0,z)\exp\{-\xi_1(r,z)x\}(1-B(x)) \tag{3.27}$$

also,

$$\int_0^\infty \widehat{P}_v^{(1)}(r,x,z)\mu_1(x)dx+\int_0^\infty \widehat{P}_v^{(2)}(r,x,z)\mu_2(x)dx=\widehat{P}_v^{(1)}(r,0,z)\widehat{c}\{\xi_1(r,z)\} \tag{3.28}$$

Using equation (3.21), (3.26) and (3.28) in equation (3.17) then,

$$[r+\mu_3(x)+\lambda-\lambda z+\alpha_3]\widehat{P}_v^{(3)}(r,z)=p\widehat{P}_v^{(1)}(r,0,z)\widehat{c}\{\xi_1(r,z)\}+\alpha_3\widehat{P}_v^{(3)}(r,z)\int_0^\infty \exp[-(s+\lambda-\lambda z)y](1-G_3(y))\beta_3(y)dy$$

$$[r+\mu_3(x)+\lambda-\lambda z+\alpha_3]\widehat{P}_v^{(3)}(r,z)=p\widehat{P}_v^{(1)}(r,0,z)\widehat{c}\{\xi_1(r,z)\}+\alpha_3\widehat{P}_v^{(3)}(r,z)\widehat{g}_3(r+\lambda-\lambda z)$$

$$\widehat{P}_v^{(3)}(r,z)=\frac{p\widehat{P}_v^{(1)}(r,0,z)\widehat{c}\{\xi_1(r,z)\}}{\xi_3(r,z)+\mu_3} \tag{3.29}$$

Integrating (3.27) with respect to x by parts then,

$$\widehat{P}_v^{(1)}(r,z)=\widehat{P}_v^{(1)}(r,0,z)\left[\frac{1-\widehat{c}\{\xi_1(r,z)\}}{\xi_1(r,z)}\right] \tag{3.30}$$

On combining (3.19), (3.24) and (3.30)

$$\widehat{S}_v^{(1)}(r,z)=\alpha_1\frac{1-\widehat{c}\{\xi_1(r,z)\}}{\xi_1(r,z)}\frac{1-\widehat{g}_1(r+\lambda-\lambda z)}{(r+\lambda-\lambda z)}\widehat{P}_v^{(1)}(r,0,z) \tag{3.31}$$

$$\widehat{S}_v^{(2)}(r,z)=\alpha_2\frac{1-\widehat{c}\{\xi_2(r,z)\}}{\xi_2(r,z)}\frac{1-\widehat{g}_2(r+\lambda-\lambda z)}{(r+\lambda-\lambda z)}\widehat{P}_v^{(1)}(r,0,z) \tag{3.32}$$

$$\widehat{S}_v^{(3)}(r,z)=\alpha_3\frac{1-\widehat{c}\{\xi_3(r,z)\}}{\xi_3(r,z)+\mu_3}\frac{1-\widehat{g}_3(r+\lambda-\lambda z)}{(r+\lambda-\lambda z)}\widehat{P}_v^{(1)}(r,0,z) \tag{3.33}$$

Using eq (3.23) and (3.29)

$$\widehat{P}_v^{(1)}(r,x,z)=\frac{[\xi_2(r,z)+\mu_2][(r+\lambda-\lambda z)\widehat{Q}(r)-1]}{[\widehat{c}(\xi_1(r,z))-z][\xi_2(r,z)+\mu_2]-p\widehat{c}\,\xi_1(r,z)\xi_2(r,z)}\exp\{-\xi_1(r,z)x\}(1-B_1(x))$$

Similarly for $\widehat{P}_v^{(2)}(r,x,z)$ and $\widehat{P}_v^{(3)}(r,x,z)$.

Theorem 3.2: For the steady state system,

(1) The probability for the idle state of server, $Q = 1-\rho_1\left(1+\frac{\alpha_1}{\beta_1}\right)-\rho_2\left(1+\frac{\alpha_2}{\beta_2}\right)-p\rho_3\left(1+\frac{\alpha_3}{\beta_3}\right)$

(2) The probability for the busy state of server, $P = \rho_1+\rho_2+p\rho_3$

(3) The probability for the repair state of server, $R = \rho_1\frac{\alpha_1}{\beta_1}+\rho_2\frac{\alpha_2}{\beta_2}+p\rho_3\frac{\alpha_3}{\beta_3}$ where $\rho_i = \frac{\lambda}{\mu_i}$

Proof: By Tauberian property,

Now let $\hat{P}_v(r,z) = \hat{P}_v^{(1)}(r,z)+\hat{P}_v^{(2)}(r,z)+\hat{P}_v^{(3)}(r,z)+\hat{R}_v^{(1)}(r,z)+\hat{R}_v^{(2)}(r,z)+\hat{R}_v^{(3)}(r,z)$ represents the probability generating function for number in the queue then applying results of theorem (3.1),

$$\hat{P}_v(r,z) = \frac{\left[p\xi_3(r,z)\hat{c}\{\xi_2(r,z)+p\xi_2(r,z)\hat{c}\{\xi_1(r,z)\}+(\xi_2(r,z)+\mu_2)(1-\hat{c}\{\xi_1(r,z)\}\right]\left[\hat{Q}(r)-1/r+\lambda-\lambda z\right]}{(\xi_3(r,z)+\mu_3)\hat{c}\,\xi_2(r,z)-(\xi_2(r,z)+\mu_2)(\hat{c}\{\xi_1(r,z)\}-z)-p\xi_2(r,z)\hat{c}\{\xi_1(r,z)\}} \tag{3.34}$$

Multiplying above equation by r and taking $r \to 0$ and applying Tauberian property,

$$P_v(z) = \lim_{r\to 0} r\hat{P}_v(r,z) = \frac{\left[p\xi_3(0,z)\hat{c}\{\xi_2(0,z)+p\xi_2(0,z)\hat{c}\{\xi_1(0,z)\}+(\xi_2(0,z)+\mu_2)(1-\hat{c}\{\xi_1(0,z)\}\right]Q}{(\xi_3(0,z)+\mu_3)\hat{c}\,\xi_2(0,z)-(\xi_2(0,z)+\mu_2)[\hat{c}\{\xi_1(0,z)-z\}]-p\xi_2(0,z)\hat{c}\{\xi_1(0,z)\}}$$

Set $z = 1$ and apply L- Hospital rule, on simplifying

$$P_v(1) = \lim_{z\to 1} P_v(z) = \frac{\left[-p\lambda\left(1+\frac{\alpha_3}{\beta_3}\right)-\lambda\mu_2\left(1+\frac{\alpha_2}{\beta_2}\right)/\mu_1-\lambda\mu_2\left(1+\frac{\alpha_1}{\beta_1}\right)/\mu_1\right]Q}{p\lambda\left(1+\frac{\alpha_3}{\beta_3}\right)+\mu_2\left[\lambda\left(1+\frac{\alpha_2}{\beta_2}\right)/\mu_1-1\right]+\mu_2\left[\lambda\left(1+\frac{\alpha_1}{\beta_1}\right)/\mu_1-1\right]}$$

 Q + $P_v(1)$ = 1, we have

$$Q = 1-\rho_1\left(1+\frac{\alpha_1}{\beta_1}\right)-\rho_2\left(1+\frac{\alpha_2}{\beta_2}\right)-p\rho_3\left(1+\frac{\alpha_3}{\beta_3}\right)$$

where $\rho_1\left(1+\frac{\alpha_1}{\beta_1}\right)+\rho_2\left(1+\frac{\alpha_2}{\beta_2}\right)+p\rho_3\left(1+\frac{\alpha_3}{\beta_3}\right)<1$, emerges for stability condition.

$$P = \lim_{z\to 1}\lim_{r\to 0} r[\hat{P}_v^{(1)}(r,z)+\hat{P}_v^{(2)}(r,z)+\hat{P}_v^{(3)}(r,z)]$$

$$R = \lim_{z\to 1}\lim_{r\to 0} r[\hat{R}_v^{(1)}(r,z)+\hat{R}_v^{(2)}(r,z)+\hat{R}_v^{(3)}(r,z)]$$

are direct calculations.

Theorem 3.3: The Laplace Steiltjes transform for the moment generating function $\sum_{n=0}^{\infty} P_n z^n$ is,

$$\hat{P}_v(r,z) = \frac{\left[p\xi_3(r,z)\hat{c}\{\xi_2(r,z)+p\xi_2(r,z)\hat{c}\{\xi_1(r,z)\}+(\xi_2(r,z)+\mu_2)(1-\hat{c}\{\xi_1(r,z)\}\right]}{(\xi_3(r,z)+\mu_3)\hat{c}\,\xi_2(r,z)-(\xi_2(r,z)+\mu_2)(\hat{c}\{\xi_1(r,z)\}-z)-p\xi_2(r,z)\hat{c}\{\xi_1(r,z)\}}\left[\frac{1}{r+\lambda-\lambda z_r}-\frac{1}{r+\lambda-\lambda z}\right]$$

Proof: From theorem (3.1) and equation (3.34), we have the result.

4. THE RELIABILITY ANALYSIS

In this section, the availability of system, their failure frequency, failure time of server etc are analysed. Let
M(t) = availability of system at time t (the probability of system working at time t), then the following results obtained.

Theorem 4.1: The Laplace Steiltjes transform of M(t) is given by,

$$\hat{M}\ (r) = \frac{1}{r+\lambda-\lambda z_r}+\frac{(r/r+\lambda-\lambda z_r)-1)\left[p\xi_3(r,1)\hat{c}\{\xi_2(r,1)+p\xi_2(r,1)\hat{c}\{\xi_1(r,1)\}+(\xi_2(r,1)+\mu_2)(1-\hat{c}\{\xi_1(r,1\}\right]}{(\xi_3(r,1)+\mu_3)\hat{c}\,\xi_2(r,1)-(\xi_2(r,1)+\mu_2)(\hat{c}\{\xi_1(r,1)\}-1)-p\xi_2(r,1)\hat{c}\{\xi_1(r,1)\}}$$

Corollary 1: The availability of the server for steady state is given by,

$$A = 1 - \rho_1 \frac{\alpha_1}{\beta_1} - \rho_2 \frac{\alpha_2}{\beta_2} - p\rho_3 \frac{\alpha_3}{\beta_3}$$

Corollary 2: The failure frequency of the server for steady state is given by,

$$M_f = \rho_1\alpha_1 + \rho_2\alpha_2 + p\rho_3\alpha_3$$

where z_r is the root of the equation (3.1) inside $|z| = 1, \operatorname{Re}(r) > 0.$

Proof: From theorem (3.1) and result $\widehat{M}(r) = \widehat{Q}(r) + \widehat{P}_v^{(1)}(r,1) + \widehat{P}_v^{(2)}(r,1) + \widehat{P}_v^{(3)}(r,1)$,
We have the result.

Theorem 4.2: The Laplace Steiltjes transform of $M_1(t)$, $M_2(t)$ and $M_3(t)$ is given by,

$$\widehat{M}_1(r) = \frac{\alpha_1(\xi_2(r,1)+\mu_2)[r\widehat{Q}(r)-1]}{(\xi_3(r,1)+\mu_3)\hat{c}\,\xi_2(r,1) - (\xi_2(r,1)+\mu_2)(\hat{c}\{\xi_1(r,1)\}-1) - p\xi_2(r,1)\hat{c}\{\xi_1(r,1)\}} \cdot \frac{1-\hat{c}\{\xi_1(r,1)\}}{\xi_1(r,1)}$$

$$\widehat{M}_2(r) = \frac{\alpha_2(\xi_3(r,1)+\mu_3)[r\widehat{Q}(r)-1]}{(\xi_3(r,1)+\mu_3)\hat{c}\,\xi_2(r,1) - (\xi_2(r,1)+\mu_2)(\hat{c}\{\xi_1(r,1)\}-1) - p\xi_2(r,1)\hat{c}\{\xi_1(r,1)\}} \cdot \frac{1-\hat{c}\{\xi_2(r,1)\}}{\xi_2(r,1)}$$

$$\widehat{M}_3(r) = \frac{p\alpha_3\xi_2(r,1)\hat{c}\{\xi_1(r,1)\}[r\widehat{Q}(r)-1]}{(\xi_3(r,1)+\mu_3)\hat{c}\,\xi_2(r,1) - (\xi_2(r,1)+\mu_2)(\hat{c}\{\xi_1(r,1)\}-1) - p\xi_2(r,1)\hat{c}\{\xi_1(r,1)\}}.$$

Proof: Using [2], and $\widehat{M}_i(r) = \sum_{n=0}^{\infty}\int_0^{\infty} \alpha_i \widehat{P}_n^{(i)}(r,x)dx = \alpha_i \widehat{P}_v^{(i)}(r,1)\ \ for\ i = 1,2,3.$
The result follows by theorem (3.1).

Theorem 4.3: The Laplace Steiltjes transform of R(t) is given by,

$$\widehat{R}(r) = \frac{1}{r+\lambda-\lambda z_r} + \frac{p(r+\alpha_2)\hat{c}(r+\alpha_2) + p(r+\alpha_1)\hat{c}(r+\alpha_1) + (1-\hat{c}(r+\alpha_1))(r+\alpha_3+\mu_3)}{\left[(\hat{c}(r+\alpha_1)-1)(r+\alpha_3+\mu_3) - p(r+\alpha_2)(r+\alpha_3)\hat{c}(r+\alpha_1)\right]} \cdot \left[\frac{r}{r+\lambda-\lambda z_r} - 1\right]$$

Proof: Apply the result

$$\widehat{R}(r) = \widehat{Q}(r) + \lim_{z\to 1^-}\left[\int_0^{\infty} \widehat{P}_v^{(1)}(r,x,z)dx + \int_0^{\infty} \widehat{P}_v^{(2)}(r,x,z)dx\right] + \lim_{z\to 1^-} \widehat{P}_v^{(3)}(r,z)$$

The result follows.

5. NUMERICAL EXAMPLE

Now, we are demonstrating the effect of failure rates of α_i *(for i =1, 2, 3)* on some of the performance measures of the system in steady state.

Let us suppose $p = 0.2$, $\lambda = 0.50$, $\mu_1 = 1.50$, $\mu_2 = 1.25$, $\mu_3 = 1.00$, $\beta_1 = 0.15$, $\beta_2 = 0.10$, $\beta_3 = 0.05$ and allow the failure rates α_i *(for i= 2, 3)* varies from 0 to 0.08.

The availability of the server for steady state is given by,

$$A = 1 - \rho_1 \frac{\alpha_1}{\beta_1} - \rho_2 \frac{\alpha_2}{\beta_2} - p\rho_3 \frac{\alpha_3}{\beta_3}$$

The failure frequency of the server for steady state is given by,

$$M_f = \rho_1\alpha_1 + \rho_2\alpha_2 + p\rho_3\alpha_3 \text{ where } \rho_i = \frac{\lambda}{\mu_i}$$

The result of this setup is given in table 1, where the server availability for the steady state A and failure frequency M_f for steady state are calculated using the above data mention.

It is observed that higher the value of α_i *(for i=2, 3)* results in low server availability and high failure frequency.

α_1	α_2	α_3	β_1	β_2	β_3	A	M_f
0	0	0				1	0
0.10	0.05	0	0.15	0.10	0.05	0.582	0.053
0.10	0.05	0.02	0.15	0.10	0.05	0.542	0.055
0.10	0.05	0.04	0.15	0.10	0.05	0.502	0.057
0.10	0.05	0.06	0.15	0.10	0.05	0.432	0.059
0.10	0.05	0.08	0.15	0.10	0.05	0.422	0.061
0.10	0	0.05	0.15	0.10	0.05	0.682	0.038
0.10	0.02	0.05	0.15	0.10	0.05	0.602	0.046
0.10	0.04	0.05	0.15	0.10	0.05	0.522	0.054
0.10	0.06	0.05	0.15	0.10	0.05	0.442	0.062
0.10	0.08	0.05	0.15	0.10	0.05	0.362	0.070
0.10	0.08	0.08	0.15	0.10	0.05	0.49	0.06

Table-6.1: Effect of Reliability factor α_2 and α_3.

6. CONCLUSION

This paper, we are analysing the M/G/1 single server retrial queueing system and observe the reliability analysis impact on third optional service respect to server breakdown and repair. By using random variables for supplementary variable technique based on Markov chain model, we derive the transient and stationary queueing model, and the reliability measures for steady state with numerical illustration. It is clear from section 5 that on increasing the phases of service then there is a great impact on system performance. The chances of server breakdown and low server availability also increases due to increase in number of service phases. Optional service are customer dependent but they affect the system reliability. For future work, it can be analysed by taking more number of servers in the same model.

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