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ANALYSIS OF MARKOVIAN QUEUEING SYSTEM AND PERFORMANCE MEASURES OF BULK ARRIVAL SYSTEM BY USING FIRST COME FIRST SERVE AND BULK SERVICE

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ABSTRACT

This paper analyzes a single server queueing model with Markov process where customers arrive according to the Poisson process and served with the two types of distributions Exponential distribution and Erlang distribution. According to the service discipline rule FCFS for the first 'w' number of customers are served by the Exponential distribution after that server starts providing service to other customers in batches to reach the system capacity M. We determine the Joint Distribution of the number of customers in the queue and the number with the departing batch at departure epoch by tracking down the number left behind in the queue by departing batch. Our aim is to determine the probability that server is busy and measures the performance of the system, number of customer in the queue, number of customers in the system, waiting time of the customers in the system with the help of theorem.

Keywords*— bulk queue, FCFS, batch service, performance measures, finite waiting capacity*

INTRODUCTION

Queueing theory has many models in which customers arrive to the service station one by one and receive service from server one by one. However, in many real world conditions, customers arrive in groups and they receive service form server in batches. These situations are suitable modeled by bulk-arrival and bulk-service queues which are called bulk queues. Bulk queues have wide range of applications such as transportation, traffic, telecommunication, manufacturing etc. Bulk system must stands for bulk arrival and bulk service i.e. arrival of customers in batches or groups and service of customers with the general bulk service rule and FCFS rule. General bulk service rule defines that server will provide the service in batches with batch size K $(w + 1 \le K \le g)$, if greater than g customers are present in the system then server will provide the service to g customers with batch size g and server will provide the service by first come first serve rule if the number of customers are less or equal to w. queue discipline first come first serve is most frequently used in daily life activities and it is quite different to other disciplines such as last in first out, priority services. Most of the service stations, such as hospitals, barber shop, supermarket where some customers can be served by priority bases. Queueing systems with two classes of subscribers have broad applications in the manufacturing and production systems, distribution and service systems, transportation systems, telecommunication industry, computer and communication systems, etc.

F. Nuts (1967) gave the theory of general bulk service rule first in which customer arrive according to the Poisson process and served in batches according to the general distribution with general bulk service rule. **Bar-Lev *et.al* (2007)** worked on the M/G (m, M) /1 model in which items arrive at the group testing centre according to the Poisson process to be tested and are served in batches with the batch size according to the general distribution, where m and M (>m) are the decision variables where each batch size can be between m and M. They considered that the testing centre has a finite capacity and present an expected profit objective function. They have developed the generating function for the steady-state probabilities of the embedded Markov chain and compute the optimal values of the decision variables (m, M) that maximize the expected profit. **Nair and Neuts (1978)** worked on the steady state probabilities of a class of infinite Markov chains with the bulk queues and stochastic model then present an algorithm by using real arithmetic.

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Singh *et.al* (2017) Work is related to the queuing models with multi-phase service, single server bulk input and the general service time models on the queue theories to analyze the performance of the queuing system. They studied the queuing system with server provides the additional optional service rate apart from the optional service. The server is unreliable and breakdown of the servers can be in the optional service. They examined the queuing system including bulk arrival and queuing service rates by using the general distributions. They use the supplementary variable technique to find the study state behavior and different performance keys of the system to develop the non- Markovian model. They supposed the case of the general distribution of the optional service for delay to repair and time to the bulk arrival queuing model to give the queue size and waiting time. They develop the non-Markovian single server unreliable model with the assumption that the customer joins the system in the batches. **Kumar and Shinde (2018)** discussed on bulk service queueing model. They examined average number of customers in the queue, average number of customers in the system, average waiting time of customers in queue, average waiting time of customers in the system, response time and efficiency of the server corresponding to customers.

We have considered our work in following sections. We consider the model description of Markovian queueing system with bulk arrival with two types of service in section 2. We have defined the following notations used in this work in section 3. In next section 4, we have given the solution of the model and find the probability that the server is busy. In section 5, we have defined the matrix representation of our model. In section 6, we have given the state and proof of some theorems and lemma. In next section 7, we determined the different performances of the Markovian queueing system. In section 8, we have given the conclusion of this work.

2. MODEL DESCRIPTION

Here we have considered a single server queuing system where customers arrive according to the Poisson Process with the arrival rate λ. First we assume that if number of customers is less than ‘w’ then customers are served according to the first come first serve (FCFS) rule. That is, which customer arrives first, he/she will get service first. Second we assume that customers are served in batches according to the ‘‘general bulk service (w, g) rule when the number of customers in the queue will be greater than ‘w’ and less than ‘g’ then server will provide the service to K($w \leq b \leq g$) group of customers in one time. Further we assume that if the number of customers in the queue will be greater than ‘g’ then server will provide the service maximum of ‘g’ customers and remaining of them will have to wait till the next round of service. Third we assume that there is a finite waiting capacity ($M > g$) for the system.

3. NOTATIONS

$dt(0 \leq k \leq w)$: Service time to Kth customer$(0 \leq K \leq w)$
$f_k(x), (x \geq 0), (0 \leq k \leq w)$: Distribution function of the service time to Kth customer
$F_k(x), (x \geq 0, 0 \leq k \leq w)$: Probability density function (pdf)
$F_{k^*}(s)$: Laplace Stieltjes Transformation (L.S.T.)
$f_k(= F_{k^*}(1)(0))$: Mean service time of each customer $t(0 \leq K \leq w)$ where
$F_{k^*}(1)(0)$: Derivative of $f^*(s)$ evaluated at $s = 0$ Define the state of the system at time t.

Similarly for bulk service, we have
$dt(0 \leq k \leq w)$: Service time to ‘b’ customer$(0 \leq K \leq w)$
$q_b(x), (x \geq 0), (0 \leq K \leq w)$: Distribution function of the service time to ‘b’ customer
$Q_b(x), (x \geq 0)$: Probability density function (pdf)
$Q_b(s)$: Laplace Stieltjes Transformation (L.S.T.)
$q_b(= Q_i(1)(0))$: Mean service time of each customer t where
$Q_b(1)(0)$: Derivative of $f^*(s)$ evaluated at $s = 0$ Define the state of the system at time t.
$M_q(t)$: Number of customers in the queue waiting for service
$V_i(t)$: Number of Customers with the server $0 \leq i \leq g$
$Yi(t)$: Remaining service time for i^{th} customer

4. SOLUTION OF MODEL

We define

$$P_{0,0}(t) = \{M_q(t) = 0, V_i(t) = 0\}\, at\ i = 0$$

It may be noted that in state (0, 0) no customer is waiting in the service system.

$$P_{n,1}(t) = \{M_q(t) = n, V_i(t) = 1\}, 1 \leq n \leq w - 1$$

State $(n, 1)$ means that the server is busy with one customer and n $(0 < M)$ customers are waiting in the queue.

$$P_{n,k}(V,t)du = \{M_q(t) = n, V_i(t)\ (0 \leq i \leq n), y < Yi(t) \leq y + dy\}\ (y \geq 0)(0 \leq k \leq w, 0 \leq n \leq M)$$

Find the state (n, k) means that the server is busy with k^{th} customer and n $(0 \le M)$ customers are waiting in the queue.

$$-\lambda P_{0,0} + \sum_{k=1}^{w} P_{0,k}(0)\,(f_k(y)) + \sum_{b=w+1}^{g} P_{0,k}(0) = 0 \quad (1)$$

$$-\lambda P_{n,0} + \lambda P_{n-1,0} + \sum_{k=1}^{w} P_{n,k}(0)\,\big(f_k(y)\big) + \sum_{k=w+1}^{g} P_{n,k}(0) = 0\,;\ 1 \le n \le w \quad (2)$$

$$-\frac{d}{du} P_{0,1}(y) = -\lambda P_{0,0} + \sum_{k=1}^{w} P_{1,k}(0)\, f_1(y) + \sum_{b=w+1}^{g} P_{1,b}(0)\, q_1(y) \quad (3)$$

$$-\frac{d}{du} P_{0,k}(y) = -\lambda P_{0,k}(y) + \sum_{i=1}^{w} P_{k,i}\, f_k(y) + \sum_{i=w+1}^{g} P_{b,i}(0)\, q_b(y);\ 1 \le k \le w, w+1 \le b \le g \quad (4)$$

$$-\frac{d}{du} P_{n,k}(y) = -\lambda P_{n,k}(y) + \lambda P_{n-1,k}(y);\ 1 \le k \le w, 1 \le n \le M-1 \quad (5)$$

$$-\frac{d}{du} P_{n,w}(y) = -\lambda P_{n,w}(y) + \lambda P_{n-1,w}(y) + \sum_{k=1}^{w} P_{n+w,k}(0)\, f_w(y) + \sum_{b=w+1}^{g} P_{n+w,b}(0)\, q_b(y);\quad 1 \le n \le M-w \quad (6)$$

$$-\frac{d}{du} P_{n,b}(y) = -\lambda P_{n,b}(y) + \lambda P_{n-1,b}(y) + \sum_{k=1}^{w} P_{n+g,k}(0)\, f_w(y) + \sum_{b=w+1}^{g} P_{n+g,b}(0)\, q_b(y);\quad M-w+1 \le n \le M-g \quad (7)$$

$$-\frac{d}{du} P_{n,g}(y) = -\lambda P_{n,g}(y) + \lambda P_{n-1,g}(y);\ M-g+1 \le n \le M-1 \quad (8)$$

$$-\frac{d}{du} P_{M,b}(y) = -\lambda P_{M-1,b}(y);\ w+1 \le b \le g \quad (9)$$

Where

$$P_{0,0} = \lim_{t\to 0} P_{0,0}(t)\ \ at\ n = 0$$

$$P_{n,k} = \lim_{t\to\infty} P_{n,k}(y,t);\ 0 \le k \le w, 0 \le n \le M$$

$$P_{n,b} = \lim_{t\to\infty} P_{n,b}(y,t);\ w+1 \le b \le g, 0 \le n \le M$$

Further we have defined the Laplace transformations

$$F_{k^*}(s) = \int_0^\infty e^{-sy}\, dF_k(y) = \int_0^\infty e^{-sy} f_k(y)dy;\ 0 \le k \le w$$

$$Q_{b^*}(s) = \int_0^\infty e^{-sy}\, dQ_b(y) = \int_0^\infty e^{-sy} q_b(y)dy;\ w+1 \le b \le g$$

$$P_{n,k^*}(s) = \int_0^\infty e^{-sy} P_{n,k}(y)dy;\ 0 \le k \le w, 0 \le n \le M$$

$$P_{n,b^*}(s) = \int_0^\infty e^{-sy} P_{n,b}(y)dy;\ w+1 \le b \le g, 0 \le n \le M$$

$$P_{n,k^*}(0) = P_{n,k} = \int_0^\infty P_{n,k}(y)dy;\ 0 \le k \le w, 0 \le n \le M$$

$$P_{n,b^*}(0) = P_{n,b} = \int_0^\infty P_{n,b}(y)dy;\ w+1 \le b \le g, 0 \le n \le M$$

It is easy to show that

$$\int_0^\infty e^{-sy} \frac{d}{dy} P_{n,k}(y)dy = sP_{n,k^*}(s) - P_{n,k}(0);\ 0 \le k \le w, 0 \le n \le M \quad (10)$$

$$\int_0^\infty e^{-sy} \frac{d}{dy} P_{n,b}(y)dy = sP_{n,b^*}(s) - P_{n,b}(0);\ w+1 \le b \le g, 0 \le n \le M \quad (11)$$

Now,

$$P_{0,0} + \sum_{n=0}^{M}\sum_{k=1}^{w} P_{n,k} + \sum_{n=0}^{M}\sum_{b=w+1}^{g} P_{n,b} = 1 \quad (12)$$

Here, equation (12) represents the normalize condition of this model

Equation (1) – (8) are basic equations of the model under consideration which needs take solve in order to get steady state probabilities $P_{0,0}$ and $P_{n,k}$ $\ 0 \le k \le w, 0 \le n \le M$

Now, we take the Laplace transformation of equations (3) – (9) by multiplying e^{-sy} and integrating with respect to y over the limit $0\ to\ \infty$ and using Equation (10) and (11), we obtain

$$(\lambda - s)P_{0,1^*}(s) = \sum_{k=1}^{w} P_{1,k}(0)\, f_{1^*}(s) + \sum_{b=w+1}^{g} P_{1,b}(0)\, q_{1^*}(s) - P_{0,0}(0) \quad (13)$$

$$(\lambda - s)P_{0,k^*}(s) = \sum_{i=1}^{w} P_{k,i}(0)\, f_{k^*}(s) + \sum_{i=w+1}^{g} P_{b,i}(0) q_{b^*}(s) - P_{0,k}(0);\ 1 \le k \le w, w+1 \le b \le g \quad (14)$$

$$(\lambda - s)P_{n,k^*}(s) = \lambda P_{n-1,k^*}(s) - P_{0,k}(0);\ 1 \le k \le w-1, 1 \le n \le M-1 \quad (15)$$

$$(\lambda - s)P_{n,w^*}(s) = \lambda P_{n-1,w^*}(s) + \sum_{k=1}^{w} P_{n+w,k}(0)\, f_{w^*}(s) + \sum_{b=w+1}^{g} P_{n+w,b}(0)\, q_{b^*}(s) - P_{n,b}(0);\quad 1 \le n \le M-w \quad (16)$$

$$(\lambda - s)P_{n,b^*}(s) = \lambda P_{n-1,w^*}(s) + \sum_{k=1}^{w} P_{n+g,k}(0)\, f_{w^*}(s) + \sum_{b=w+1}^{g} P_{n+g,b}(0)\, q_{b^*}(s) - P_{n,b}(0);\quad M-w+1 \le n \le M-b \quad (17)$$

$$(\lambda - s)P_{n,w^*}(s) = \lambda P_{n-1,w}(s) - P_{n,b}(0);\ M-b+1 \le n \le M-1 \quad (18)$$

$$-sP_{M,b^*}(s) = \lambda P_{M-1,b^*}(s) - P_{M,b}(0);\ 0 \le r \le w \quad (19)$$

From equation (1) and (2), we get the following equation

$$\lambda P_{n,0} = \sum_{m=0}^{M}\sum_{k=1}^{w} P_{m,k}(0)\, f_k(y) + \sum_{m=0}^{M}\sum_{b=w+1}^{g} P_{m,b}(0)\,;\ n = 0 \quad (20)$$

By adding equations (13)-(17), we get the general solution of the above equations

$$\sum_{n=0}^{M}\sum_{k=0}^{w} P_{n,k^*}(s) + \sum_{n=0}^{M}\sum_{b=w+1}^{g} P_{n,b^*}(s) = \frac{1-f_w{}^*(s)}{s}\sum_{n=0}^{w} P_{n,k}(0) + \sum_{n=w+1}^{g}\sum_{b=w+1}^{g}\frac{1-q_n{}^*(s)}{s}P_{n,b}(0) +$$

1−qg∗ssn=g+1Mb=w+1gPn,b0

(21)

Taking limit as $s \to 0$ we get the equation

$$1 - P_{0,0} = f_w \sum_{k=0}^{w} P_{n,k}(0) + \sum_{n=1}^{w}\sum_{k=0}^{w} q_n P_{n,b}(0) + q_g \sum_{n=w+1}^{M}\sum_{k=0}^{w} P_{n,b}(0) \tag{22}$$

This equation give the probability that server is busy in providing service to the customer.

5. MATRIX REPRESENTATION

Now we want to determine the Joint Distribution of the number of customers in the queue and the number with the departing batch at departure epoch by using the number left behind in the queue by departing batch.

We have defined the random variable $X(n, k+b),\ n \in w_0$. Here w_0 is a set of whole number.

We have the following assumptions.

If $X(n,k) = 0$, then the server will wait at least one customer arrive in the queue.

If $X(n,k) = P;\ 0 \le P \le w$ Then server will provide service to the customer one by one using FCFS rule.

If $X(n,k) > w$, Then server will provide the service to customers in batches.

If $X(n,k) >, \tau\ (w+1 \le \tau \le g)$, Then all the τ customers are taken for service with the batch of size τ.

Next we have defined, $Y(n,k),\ n \in w_0$ is the number of customers that arrive during the service period of K^{th} batch.

Now observing the state of system at two consecutive departure epoch n^{th} and $n+1^{th}$ we have following system

$$X(n+1,k) = \begin{cases} Y(n+1,0); & if\ X(n,P) = 0\ for\ P \in M_{0,w} \quad, k = 0 \\ Y(n+1,1); & if\ X(n,P) = 1\ for\ P \in M_{0,w} \quad, k = 1 \\ \vdots & \\ Y(n+1,w); & if\ X(n,P) = w\ for\ P \in M_{0,w} \quad, k = w \\ Y(n+1,w+1); & if\ X(n,P) = w+1\ for\ P \in M_{0,w} \quad, b = w+1 \\ \vdots & \\ Y(n+1,w+1); & if\ X(n,P) = g\ for\ P \in M_{0,w} \quad, b = g \\ X(n,P) - w + Y(n+1,g); & if\ X(n,k) = g+1 \dots N\ for\ l \in M_{0,w} \quad, b = g \\ 0; & otherwise \end{cases} \tag{23}$$

The discrete time process$X(n,k)$, $n \in w_0$ and $k \in M_{0,w}$ constitute a two dimensional Markov Chain $\{(i,0),(i,1)\dots(i,w)\}$ where $M \in \{0,1,2\dots\text{N}\}$

Here $P = \{P_{ij}\}$ is the one step transition probability matrix of order $(M+1)(w+1) \times (M+1)(w+1)$ where each P_{ij} is a matrix of order$(w+1)(w+1)$. The row of each matrix P_{ij} are indexed $(i,1),(i,2)\dots(i,w)$ and the column are indexed as $(i,1),(i,2)\dots(i,w),\ i,j \in M$

Now P_{ij} is given by

$$P_{ij} = \begin{cases} A_j{}^{(i)}; & i = 0, 0 \le j \le M-1 \\ A_M{}^{(i)}; & i = 0,\ \ j = M \\ A_j{}^{(i+1)}; & 1 \le i \le w, 0 \le j \le M-1 \\ A_M{}^{(i+1)}; & 1 \le i \le w, j = M \\ A_{j-(i-w)}^{(w+1)}; & w+1 \le i \le g, i-w \le j \le M-1 \\ A_{M-(i-w)}^{(w+1)}; & w+1 \le i \le g, j = M \\ A_{j-(i-g)}^{(g+1)}; & g+1 \le i \le M, i-g \le j \le M-1 \\ A_{M-(i-g)}^{(g+1)}; & g+1 \le i \le M, j = M \\ 0; & otherwise \end{cases} \tag{24}$$

Where 0 is a null matrix of order $(g-w)(g-w)$.

$$P = \begin{bmatrix} A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & \cdots & A_{M-w-1}^{(0)} & A_{M-b}^{(0)} & \cdots & A_{M-1}^{(0)} & \overline{A}_M^{(0)} \\ A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & \cdots & A_{M-w-1}^{(0)} & A_{M-b}^{(0)} & \cdots & A_{M-1}^{(0)} & \overline{A}_M^{(0)} \\ A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & \cdots & A_{M-w-1}^{(0)} & A_{M-b}^{(0)} & \cdots & A_{M-1}^{(0)} & \overline{A}_M^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & \cdots & A_{M-w-1}^{(0)} & A_{M-w}^{(0)} & \cdots & A_{M-1}^{(0)} & \overline{A}_M^{(0)} \\ A_0^{(2)} & A_1^{(2)} & A_2^{(2)} & \cdots & A_{M-w-1}^{(2)} & A_{M-w}^{(2)} & \cdots & A_{M-1}^{(2)} & \overline{A}_M^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_0^{(b-a)} & A_1^{(b-a)} & A_2^{(w)} & \cdots & A_{M-b-1}^{(w)} & A_{M-b}^{(w)} & \cdots & A_{M-1}^{(w)} & \overline{A}_M^{(w)} \\ A_0^{(w+1)} & A_1^{(w+1)} & A_2^{(w+1)} & \cdots & A_{M-w-1}^{(w+1)} & A_{M-w}^{(w+1)} & \cdots & A_{M-1}^{(w+1)} & \overline{A}_M^{(w+1)} \\ 0 & A_0^{(w+1)} & A_1^{(w+1)} & \cdots & A_{M-g-2}^{(w+1)} & A_{M-g-1}^{(w+1)} & \cdots & A_{M-1}^{(w+1)} & A_M^{(w+1)} \\ 0 & 0 & A_0^{(w+1)} & \cdots & A_{M-g-3}^{(w+1)} & A_{M-g-2}^{(w+1)} & \cdots & A_{M-1}^{(w+1)} & A_M^{(w+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_0^{(w+1)} & A_1^{(w+1)} & \cdots & A_g^{(w+1)} & A_{g+1}^{(w+1)} \\ 0 & 0 & 0 & \cdots & 0 & A_0^{(w+1)} & \cdots & A_{g-1}^{(w+1)} & A_g^{(w+1)} \end{bmatrix}$$

In the matrix "P", Row are indexed as 0, 1, 2, . . . M and column are indexed as 0, 1, 2, . . . M. Here we use the various service time distributions. First we have used the Erlang distribution and second we have used the Poisson distribution.

Let $P_{n,b^\sim}$ be the probability that there are 'n' $(0 \le n \le M)$ customers in the system after the departure of b $(w+1 \le b \le g)$ customers. Once we get the joint distribution of customers in the queue and number of customers departing, then we can easily find the other distributions such as distribution of number of customers in the queue $(P_{n^\sim})$ $(0 \le n \le M)$ and distribution of number of customers departing $(Q_{k^\sim})$ .then we have

$$P_{n^\sim} = \sum_{k=0}^{w} P_{n,b+k^\sim} \quad (0 \le n \le M, w+1 \le b \le g, 0 \le k \le w) \tag{25}$$

$$Q_{k^\sim} = \sum_{n=0}^{M} P_{n,b^\sim} \quad (w+1 \le b \le g) \tag{26}$$

6. JOINT DISTRIBUTION OF NUMBER OF CUSTOMERS IN THE QUEUE AND THE NUMBER OF CUSTOMERS WITH THE SERVER

In this section, we observed the following relation

$$P_{n,k^\sim} = \frac{P_{n,k(0)}}{\sum_{m=0}^{n}\sum_{k=0}^{w} P_{m,k(0)}}, \quad (0 \le r \le w), (0 \le n \le m) \tag{27}$$

We have assumed that

The value of $\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}$ is given by

$$\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)} = \frac{1-P_{0,0}}{h} \tag{28}$$

Now from equation (19), we have the following

$$1 - P_{0,0} = f_w \sum_{k=0}^{w} P_{n,k}(0) + \sum_{n=1}^{w}\sum_{k=0}^{w} q_n P_{n,b}(0) + q_g \sum_{n=w+1}^{N}\sum_{k=0}^{w} P_{n,b}(0)$$

Where $h = f_w P_{n^+} + \sum_{n=1}^{w} q_n P_{n^+} + q_g \sum_{n=w+1}^{m} P_{n^+}$

Proof: By dividing equation (22) by $\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}$ and using equation (25) and (26)

$$\frac{1-P_{0,0}}{\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}} = \frac{f_w \sum_{k=0}^{w} P_{n,k(0)}}{\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}} + \frac{\sum_{n=1}^{w}\sum_{k=0}^{w} q_n P_{n,k(0)}}{\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}} + \frac{q_g \sum_{n=w+1}^{m}\sum_{k=0}^{w} P_{n,k(0)}}{\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}}$$

$$\frac{1-P_{0,0}}{\sum_{n=0}^{m}\sum_{k=0}^{w} P_{n,k(0)}} = f_w P_{w^\sim} + q_n \sum_{n=1}^{w} P_{n^\sim} + q_g \sum_{n=w+1}^{m} P_{n^\sim} \tag{29}$$

Theorem: The state probabilities $\{P_{n,0}, P_{n,b}\}$ and $\{P_{n,b^\sim}, P_{n^\sim}\}$ are related as

$$P_{n,0} = E^{-1}\left(P_{0^\sim} - \sum_{t=0}^{n} P_{t,0^\sim}\right), 0 \le n \le M-1 \tag{30}$$

$$P_{n,k} = E^{-1}\left(P_{k^\sim} - \sum_{t=0}^{n} P_{t,k^\sim}\right), 0 \le b \le g, 0 \le n \le M-1 \tag{31}$$

$$P_{n,w} = E^{-1}\left(\sum_{t=w}^{\min(w+n,m)} P_{t^\sim} - \sum_{t=0}^{n} P_{t,w^\sim}\right), 0 \le n \le M-1 \tag{32}$$

Where: $E = \lambda h + P_{0^\sim}$ and $h = f_0 P_{n^\sim} + \sum_{n=1}^{w} f_n P_{n^\sim} + f_w \sum_{n=w+1}^{m} P_{n^\sim}$

$$P_{N,x} = \sum_{k=0}^{w} P_{M,k} = 1 - P_{n,0} - \sum_{n=0}^{m-1}\sum_{k=0}^{w} P_{n,k}$$

Proof: Dividing equation (1) by $\sum_{n=0}^{M}\sum_{k=0}^{w} P_{m,k(0)}$

$$\frac{h(\lambda P_{0,0})}{1-P_{0,0}} = \sum_{k=1}^{w} q_b(y)P_{0,k^{\sim}}$$

$$\lambda h P_{0,0} = (1-P_{0,0})(f_k(y)P_{0^{\sim}}$$

$$\frac{P_{0,0}}{P_{0^{\sim}}} = \frac{(1-P_{0,0})(f_k(y)}{\lambda h} \quad (33)$$

$$\frac{\lambda h P_{n,0}}{(1-P_{0,0})} = \sum_{k=1}^{w}(f_k(y)P_{m,k^{\sim}}$$

$$\lambda h P_{n,0} = \left(1-P_{0,0}\right)(f_k(y)P_{m^{\sim}} \quad (34)$$

$$P_{n,0} = \frac{(1-P_{0,0})(f_k(y)P_{m^{\sim}}}{\lambda h}$$

$$P_{n,0} = \frac{P_{0,0}P_{m^{\sim}}}{P_{0^{\sim}}} \quad (35)$$

Now using equation (35) in (33), we get

$$P_{0,0} = \frac{P_{0^{\sim}}}{\lambda h + P_{0,0}} \quad (36)$$

$$\lambda P_{0,1} = \sum_{b=w+1}^{g} P_{1,k}(0)f_1 - P_{0,0}(0) \quad (37)$$

$$\lambda P_{0,k} = \sum_{i=w+1}^{g} P_{k,i}(0)f_k - P_{0,k}(0),\ \ 1 \le r \le w \quad (38)$$

$$\lambda P_{n,k} = \lambda P_{n-1,k} - P_{n,k}(0),\ \ \ 1 \le k \le w-1, 1 \le n \le M-1 \quad (39)$$

$$\lambda P_{n,w} = \lambda P_{n-1,w} + \sum_{k=1}^{w} P_{n+w,k}f_w - P_{n,w}(0), 1 \le n \le M-w \quad (40)$$

$$\lambda P_{n,w} = \lambda P_{n-1,w} - P_{n,w}(0), M-g+1 \le n \le M-1 \quad (41)$$

Similarly

$$P_{0,1} = \frac{1-P_{0,0}}{\lambda h}\left(\sum_{k=1}^{w} P_{1,k^{\sim}}f_1 - P_{0,0^{\sim}}\right) \quad (42)$$

$$P_{0,b} = \frac{1-P_{0,0}}{\lambda h}\left(P_{b^{\sim}} - P_{0,n^{\sim}}\right) \quad (43)$$

We know that

$$1-P_{0,0} = \frac{\lambda h}{\lambda h + P_{0^{\sim}}} \quad (44)$$

$$P_{0,1} = \frac{P_{1^{\sim}} - P_{0,0^{\sim}}}{\lambda h + P_{0^{\sim}}} \quad (45)$$

$$P_{0,k} = \frac{P_{k^{\sim}} - P_{0,n^{\sim}}}{\lambda h + P_{0^{\sim}}} \quad (46)$$

Applying a simple procedure

$$\lambda P_{n,0} = \lambda P_{n-1,0} - P_{n,0}(0), 1 \le k \le w-1, 1 \le n \le M-1$$

$$P_{n,0} = \frac{\sum_{m=0}^{M}\sum_{k=0}^{w} P_{m,b}}{\lambda}\left[P_{0^{\sim}} - \sum_{m=0}^{n} P_{m,0^{\sim}}\right]$$

$$P_{n,0} = \frac{1-P_{0,0}}{\lambda h}\left[P_{0^{\sim}} - \sum_{m=0}^{n} P_{m,0^{\sim}}\right]$$

$$P_{n,0} = \frac{\left[P_{0^{\sim}} - \sum_{m=0}^{n} P_{m,0^{\sim}}\right]}{\lambda h + P_{0,0}},\ \ 1 \le n \le M-1 \quad (47)$$

$$P_{n,k} = \frac{\left[P_{k^{\sim}} - \sum_{m=0}^{n} P_{m,k^{\sim}}\right]}{\lambda h + P_{0,0}},\ \ \ \ 1 \le k \le w-1, 1 \le n \le M-1 \quad (48)$$

After combining these equations we get result (30)

$$P_{n,0} = \frac{1}{\lambda h + P_0^{\sim}}\left[P_1^{\sim} - P_{0,0}^{\sim} + P_0^{\sim} - \sum_{m=0}^{N} P_{m,0}^{\sim}\right]$$

Similarly we get equation (31) after combining the result (46) and (48)

Now we have to find the term $P_{M,b}\ for\ 0 \le k \le w$

We know that

$$-SP_{M,b}^{*} = \lambda P_{M-1,b}^{*}(s) - P_{M,0}(0);\ \ \ \ 0 \le k \le w$$

$$P_{M,b}^{*}(s) = -\lambda P_{M-1,b}^{*(1)}(s)$$

Put $s = 0$

$$P_{N,b} = -\lambda P_{N-1,b}^{*(1)}(0)$$

To find $P_{N-1,w}^{*(1)}$ Differentiate equation (15), (16) and (18) with respect to S, we get

$$(\lambda - S)P_{n,k}^{*}(S) = \lambda P_{n-1,k}^{*(1)}(S) - P_{0,k}(0);\ \ 1 \le k \le w-1,\ \ 1 \le n \le M-1$$

$$(\lambda - s)P_{n,k^{*}}(s) = \lambda P_{n-1,w}^{*(1)}(s) + \sum_{k=1}^{w} P_{n+w,k}(0)\, f_w^{*(1)}(s) + \sum_{b=w+1}^{g} P_{n+w,b}(0)\, q_b^{*(1)}(s);\ 1 \le n \le M-w$$

$$(\lambda - s)P_{n,w^{*}}(s) = \lambda P_{n-1,w}^{1}(s);\ N-b+1 \le n \le M-1$$

By putting S = 0 in equation (15), (16) and (18), we get

$$\lambda P_{n,k}(0) = P^{1}_{n-1,k}(0);\ 1 \le k \le w-1,\ \ 1 \le n \le M-1 \tag{49}$$

$$\lambda P_{n,k}(0) = \lambda P_{n-1,w} + \lambda P^{1}_{n-1,w}(0); \quad M-w+1 \le n \le M-1 \tag{50}$$

$$\lambda P_{n,k}(0) = \lambda P^{1}_{n-1,w}(0) + \sum_{k=1}^{w} P_{n+w,k}(0)\, f^{(1)}_{w} + \sum_{b=w+1}^{g} P_{n+w,b}(0)\, q^{(1)}_{b};\ 1 \le n \le M-1 \tag{51}$$

From (16) similarly

$$-P_{M,w} = \lambda P_{M-1,w}(0);\ 0 \le r \le w \tag{52}$$

Now we can obtain $P_{n,r}$ from the equation (49), (51) and (52)

7. PERFORMANCES MEASURES

The distribution of numbers of customers in our system $P_{M,x}$

$$P_{M,x} = \begin{cases} \sum_{k=1}^{w} P_{n,k;\ 0 \le n \le M} \\ \sum_{b=w-1}^{g} P_{n,b;\ 0 \le n \le M} \end{cases}$$

The distribution of number of customer undergoing service with the server $P^{\emptyset}_{n}$

$$P^{\emptyset}_{n} = \begin{cases} P_{n,1}; & 1 \le n \le w \\ \sum_{b=w-1}^{g} P_{n,b}; & w+1 \le n \le g \end{cases}$$

The distribution of number of customers in the queue P^{Δ}_{n}

$$P^{\Delta}_{n} = \begin{cases} P_{n,0}; & when\ no\ queue \\ \sum_{n=1}^{w} P_{n,k}; & 1 \le n \le M \\ \sum_{b=w-1}^{g} P_{n,b}; & 1 \le n \le M \end{cases}$$

Distribution of number of customers in the system (including the number of customers with the server).

8. CONCLUSION

We have considered the Markovian queueing system with single server finite queue where arrival rate of customers follow the passion distribution and service rate of customers follow the exponential distribution. Our aim in his work is to decrease the congestion which can be done by controlling either arrival-rates or service-rates which is the most reviewed problem in the general life situation. By using this model we can find the rate of busy period of server and determine the customer's satisfaction by introducing two types of services, FCFS service and bulk service. We have analyzed the joint distribution of the number of customers in the queue and the number with the server, and distributions of the number of customers in the queue, in the system, and the number with the server and also we have determined the various performance measures such as the average number of customers in the queue. We have analyzed that these two types of services are more beneficial to decrease the congestion rather than considering the only one service. This model has lots of application in manufacturing, computer-communication network and telecommunication systems.

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