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BEST PROXIMITY POINT THEOREM FOR (Ψ, Φ) – WEAK CONTRACTIONS

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ABSTRACT

The aim of this paper is to extend the result which was proved by ovidiv (2010) and to show the existence and uniqueness of the best proximity point for (Ψ, Φ) –weak contractions.

Keywords: *Best proximity point, p-property, $((\Psi, \Phi))$-weak contractions, generalised contractions, complete metric space.*

INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space. A mapping $T: X \to X$ is a contraction if there exists a constant $k(0,1)$ such that $d(Tx, Ty) \leq kd(x, y)$holds for any $x, y \in X$.

If X is complete, then every contraction has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point X(the banach contraction principle obviously every contraction is a continuous function.

A mapping T: X → X is a Φ-weak contraction if for each $x, y \in X$, there exists a function $\Phi: [0, \infty) \to [0, \infty)$ such that Φ is positive on $[0, \infty)$, $\Phi(0) = 0$ and $d(Tx, Ty) \leq d(x, y) - \Phi(d(x, y))$.

The concept of the weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997. They defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed points. Rhodes [6] showed that most results of [1] are true for any banach space. Also Rhodes proved the following generalization of the Banach contraction principle.

Theorem: 1 Let (X, d) be a nonempty complete metric space and let $T: X \to X$ be a Φ-weak contraction on X. If Φ is a continuous and nondecreasing function with$\Phi(t) > 0$ for all $t > 0$ and $\Phi(0) = 0$, then T has a unique fixed point.
Every contraction is a Φ-weak contraction $\Phi(t) = kt$where $0 < k < 1$.

Dutta and Choudhury [3] proved the following generalization of theorem 1.

Theorem 2: Let (X, d) be a nonempty complete metric space and let $T: X \to X$ be a self-mapping satisfying the inequality $\Psi(d(Tx, Ty)) \leq \Psi(d(x, y)) - \Phi(d(x, y))$, where Ψ,Φ: $[0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with$\Psi(t) = \Phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Doric [4] generalized theorem 2.

Theorem 3: Let (X, d) be a nonempty complete metric space and let $T: X \to X$ be a self-mapping satisfying the inequality $\Psi(d(Tx, Ty)) \leq \Psi(M(x, y)) - \Phi(M(x, y))$, for any $x, y \in X$, where M is given by $M(x, y) = max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{d(x,Ty)+d(Ty,y)}{2}\}$.

a) $\Psi: [0, \infty) \to [0, \infty)$ is a continuous monotone non-decreasing function with $\Psi(t) = 0$ if and only if $t = 0$,
b) $\Phi: [0, \infty) \to [0, \infty)$is a lower semi-continuous function with $\Phi(t) = 0$ if and only if $t = 0$.

Then T has a fixed point.

Ovidiu [5] extended the result proved by Doric [4], Rhodes [6] and Choudhary [3].

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Theorem 4: Let (X, d) be a nonempty complete metric space and $T: X \to X$be a mapping satisfying for all $x, y \in X \Psi(d(Tx, Ty)) \le \Psi(M(x, y)) - \Phi(M(x, y))$, where

a)$\Psi: [0, \infty) \to [0, \infty)$ is a monotone non-decreasing function with $\Psi(t) = 0$ if and only if $t = 0$,

b) $\Phi: [0, \infty) \to [0, \infty)$ is a function with $\Phi(t) = 0$ if and only if $t = 0$ and $\lim_{n\to\infty} \Phi(a_n) > 0$ if $\lim_{n\to\infty} a_n = a > 0$,

c) $\Phi(a) > \Psi(a) - \Psi(a-)$for any $a > 0$, where $\Psi(a-)$ is the left limit of Ψ at a.

Then T has a unique fixed point.

Definition 5: [2] Let A, B be nonempty subsets of a metric space X. A map $T: A \to B$is said to be a generalised weakly contracting mapping if for all $x, y \in A$, then

$$\Psi(d(Tx, Ty)) \le \Psi(M(x, y)) - \Phi(max\{d(x, y), d(y, Ty) - d(A, B)\})$$

where$M(x, y) = max\{d(x, y),\ d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), \frac{1}{2}[d(x, Ty) + d(y, Ty)] - d(A, B)\}$.

Theorem 6: [7] Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T: A \to B$ be a weakly contractive mapping satisfying $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the p-property.Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

MAIN RESULTS

Theorem 7: Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T: A \to B$ be such that $T(A_0) \subseteq B_0$. Suppose

$$\Psi(d(Tx, Ty)) \le \Psi(M(x, y)) - \Phi(max\{d(x, y), d(y, Ty) - d(A, B)\}) \tag{1}$$

where

a) $\Psi: [0, \infty) \to [0, \infty)$ is a monotone non-decreasing function with $\Psi(t) = 0$ if and only if $t = 0$,

b) $\Phi: [0, \infty) \to [0, \infty)$is a lower semi-continuous function with $\Phi(t) = 0$ if and only if $t = 0$.

Furthermore the pair (A, B) has the p-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.

Proof: Choose $x_0 \in A$.

Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$.

Analogously, regarding the assumption, $Tx_1 \in T(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Recursively, we obtain a sequence (x_n) in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in N \tag{2}$$

Claim: $d(x_n, x_{n+1}) \to 0$

If $x_N = x_{N+1}$, then x_N is a best proximity point.

By the p-property, we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$$

Hence we assume that $x_n \ne x_{n+1}$ for all $n \in N$.

Since $d(x_{n+1}, Tx_n) = d(A, B)$, from (1), we have for all $n \in N$.

$$\begin{aligned}
&\Psi(d(x_{n+1}, x_{n+2})) = \Psi(d(Tx_n, Tx_{n+1})) \\
&\le \Psi(max\,\{d(x_n, x_{n+1}), d(x_n, Tx_n) - d(A, B), \quad d(x_{n+1}, Tx_{n+1}) - d(A, B), \\
&\qquad \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] - d(A, B)\}) - \Phi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B)\}) \\
&\le \Psi(max\,\{d(x_n, x_{n+1}), d(x_n, Tx_n) - d(A, B), \quad d(x_{n+1}, Tx_{n+1}) - d(A, B), \\
&\qquad \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] - d(A, B)\}) - \Phi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B)\}) \\
&\le \Psi(max\,\{d(x_n, x_{n+1}), d(x_n, Tx_n) - d(A, B), \quad d(x_{n+1}, Tx_{n+1}) - d(A, B), \\
&\qquad \frac{1}{2}(d(x_n, Tx_{n+1})) - d(A, B)\}) - \Phi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B)\})
\end{aligned}$$

Since $\frac{1}{2}(d(x_n, Tx_{n+1})) - d(A, B) \le \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, Tx_{n+1})) - d(A, B)$

$$\begin{aligned}
&\le max\,\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A, B)\} \\
d(x_n, Tx_n) - d(A, B) &\le d(x_n, x_{n+1}), d(x_{n+1}, Tx_n) - d(A, B) \\
&= d(x_n, x_{n+1})
\end{aligned}$$

It follow that

$$\Psi(d(Tx_n, Tx_{n+1}) \le \Psi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A,B)\}) - \Phi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1}) - d(A,B)\})$$

$$\Psi\big(d(x_{n+1}, x_{n+2})\big) \le \Psi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})\Phi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \quad (3)$$

Suppose that $d(x_n, x_{n+1}) \le d(x_{n+1}, x_{n+2})$ for some positive integer n.
Then from (3), we have

$$\Psi(d(x_{n+1}, x_{n+2})) \le \Psi(d(x_{n+1}, x_{n+2})) - \Phi(d(x_{n+1}, x_{n+2})).$$

that is $\Phi(d(x_{n+1}, x_{n+2})) \le 0$ which implies that $d(x_{n+1}, x_{n+2}) = 0$, contradicting our assumption.

Therefore $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for any $n \in N$ and hence $\{d(x_n, x_{n+1})\}$ is monotone decreasing sequence of nonnegative real numbers, hence there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

In the view of the fact from (3), for any $n \in N$, we have

$$\Psi(d(x_{n+1}, x_{n+2})) \le \Psi(d(x_n, x_{n+1})) - \Phi(d(x_n, x_{n+1})),$$

Taking the limit as $n \to \infty$ in the above inequality, and using the conditions of Ψ and Φ we have

$$\Psi(r) \le \Psi(r) - \Phi(r) \text{ which implies } \Phi(r) = 0$$

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ (4)

Next we show that (x_n) is a Cauchy sequence.

If otherwise there exists $\varepsilon > 0$, for which we can find two sequences of positive integers (m_k) and (n_k) such that for all positive integers $m_k > n_k > k$, $d(x_{m_k}, x_{n_k}) \ge \varepsilon$ and $d(x_{m_k}, x_{n_{k-1}}) < \varepsilon$.

Now $\varepsilon \le d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k})$,
that is $\varepsilon \le d(x_{m_k}, x_{n_k}) < \varepsilon + d(x_{n_{k-1}}, x_{n_k})$

Taking the limit as $k \to \infty$ in the above inequality and using (4) we have

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon \quad (5)$$

Again $$d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}).$$

Taking the limit as $k \to \infty$ in the above inequalities and using (4) and (5) we have

$$\lim_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+!}}) = \varepsilon \quad (6)$$

Again $d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$ and $d(x_{m_k}, x_{n_{k+1}}) \le d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$

Letting $k \to \infty$ in the above inequalities and using (4) and (5) we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) = \varepsilon \quad (7)$$

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \varepsilon \quad (8)$$

For $x = x_{m_k}$, $y = y_{m_k}$ we have

$$d(x_{m_k}, Tx_{m_k}) - d(A,B) \le d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, Tx_{m_k}) - d(A,B)$$
$$= d(x_{m_k}, x_{m_{k+1}})$$

Similarly $d(x_{n_k}, Tx_{n_k}) - d(A,B) = d(x_{n_k}, x_{n_{k+1}})$.

Also $d(x_{m_k}, Tx_{n_k}) - d(A,B) = d(x_{m_k}, x_{n_{k+1}})$ and $d(x_{n_k}, Tx_{m_k}) - d(A,B) = d(x_{n_k}, x_{m_{k+1}})$.

From (1) we have

$$\Psi(d(x_{m_{k+1}}, x_{n_{k+1}})) = \Psi(d(Tx_{m_k}, Tx_{n_k}))$$
$$\le \Psi(max\,\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}) - d(A,B), \quad d(x_{n_k}, Tx_{n_k}) - d(A,B), \frac{1}{2}[d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})] - d(A,B)\}) - \Phi(max\,\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, Tx_{n_k}) - d(A,B)\})$$
$$\le \Psi(max\,\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), \quad d(x_{n_k}, x_{n_{k+1}}), \frac{1}{2}[d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})]\}) - \Phi(max\,\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_{k+1}})\})$$

It follows that

$$\Psi(d(Tx_{m_k},Tx_{n_k})) \leq \Psi(max\,\{d(x_{m_k},x_{n_k}),d(x_{n_k},Tx_{n_{k+1}}),$$
$$\frac{1}{2}[d(x_{m_k},x_{n_{k+1}})+d(x_{n_k},x_{m_{k+1}})]\}) - \Phi(max\,\{d(x_{m_k},x_{n_k}),d(x_{n_k},Tx_{n_{k+1}})\})$$
$$\Psi(d(x_{m_{k+1}},Tx_{n_{k+1}}) \leq \Psi(max\,\{d(x_{m_k},x_{n_k}),d(x_{n_k},x_{n_{k+1}})\}) - \Phi(max\,\{d(x_{m_k},x_{n_k}),d(x_{n_k},Tx_{n_{k+1}})\})$$

From (4), (5), (7) and (8) and letting $k\to\infty$ in the above inequalities and using the conditions of Ψ and Φ, we have $\Psi(\varepsilon)\leq\Psi(\varepsilon)-\Phi(\varepsilon)$ which is contradiction by virtue of property Φ.

Hence (x_n) is a Cauchy sequence.

Since $(x_n)\subset A$ and A is a closed subset of the complete metric space (X,d), there exists x^* in A such that $x_n\to x^*$.

Putting $x=x_n$ and $y=x^*$ in (1) and since

$$d(x_n,Tx^*)\leq d(x_n,x^*)+d(x^*,Tx_n) \text{ and}$$
$$d(x^*,Tx_n)\leq d(x^*,Tx^*)+d(Tx^*,Tx_n)$$

We have

$$\Psi(d(x_{n+1},Tx^*)-d(A,B)\leq\Psi d(Tx_n,Tx^*))$$
$$\leq\Psi(max\,\{d(x_n,x^*),d(x_n,Tx_{n+1}),d(x^*,Tx^*)-d(A,B),$$
$$\frac{1}{2}[d(x_n,Tx^*)+d(x^*,Tx_n)]-d(A,B)\})-\Phi(max\,\{d(x_n,x^*),d(x^*,Tx^*)-d(A,B)\})$$

Taking the limit as $n\to\infty$ in the above inequalities and using the conditions of $\Psi$and Φ,we have

$$\Psi(d(x^*,Tx^*)-d(A,B))\leq\Psi(d(x^*,Tx^*)-d(A,B))-\Phi(d(x^*,Tx^*)-d(A,B))$$

which implies that $d(x^*,Tx^*)=d(A,B)$

Hence x^* is a best proximity point of T.

For the uniqueness

Let p and q be two best proximity point and suppose that $p\neq q$,
Then putting $x=p$ and $y=q$ in (1) we obtain

$$\Psi(d(T_p,T_q))\leq\Psi(max\{d(p,q),d(p,T_p)-d(A,B),d(q,T_q)-d(A,B),\frac{1}{2}[d(p,T_p)+d(q,T_q)]-d(A,B)\})$$
$$-\Phi(max\{d(p,q),d(q,T_q)-d(A,B)\})$$

that is $\Psi(d(p,q))\leq\Psi(d(p,q))-\Phi(d(p,q))$

which is contradiction by virtue of a property Φ.
There $p=q$
This completes the proof.

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