International Journal of Mathematical Archive-11(8), 2020, 21-27
Available online through www.ijma.info ISSN 2229 – 5046

CASE STUDY OF TOP ANGLE OF AN OBTUSE TRIANGLE WITH DIFFERENCE OF TWO ANGLE 90°

RAJBALI YADAV*

Ordnance Factory Khamaria Jabalpur, (M.P.), India.-482005.

(Received On: 21-07-20; Revised & Accepted On: 27-07-20)

ABSTRACT

In this study it is proved that in a triangle if difference of two angles are 90° & top angle is divided in two parts (α1 & α2), and separation line divide base in two equal parts then difference of, Cosine & Sine ratio of angle(α2 & α1)and Cosine & Sine ratio of angle (α1 & α2) is a Constant.

Keywords: *Differentiation, Maximum area, Obtuse triangle.*

1. INTRODUCTION

In present (m+ n) cot Θ = m cotα1 – n cotα2 is relation among separated angles (α1 & α2) and separation line inclination angleΘ . This study establish relation between separated angles (α1 & α2) and says in a triangle, if difference of two angles are 90̊ and separation line divide base in two equal parts then *difference of, Cosine & Sine ratio of angle (α2 & α1) and Cosine & Sine ratio of angle (α1 & α2) is a Constant .*

2. OBTUSE TRIANGLE WITH DIFFERENCE OF TWO ANGLE 90°

2.1 In a triangle when difference of two angle is 90̊ and base of triangle is divided in two equal parts by line drawn from top of the triangle (Figure 2) then
'Difference of, Cosine and Sine ratio of second & first (α2 & α1) separated angles and Cosine & Sine ratio of first & second (α1 & α2) separated angles, is a Constant'.

It can be written as,

$$\frac{cos\alpha 2}{sin\alpha 1} - \frac{cos\alpha 1}{sin\alpha 2} = Constant = 2$$

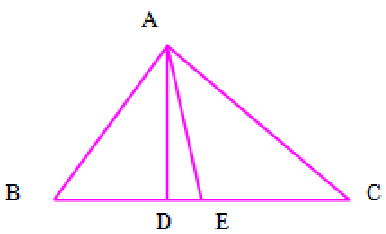

Figure-1

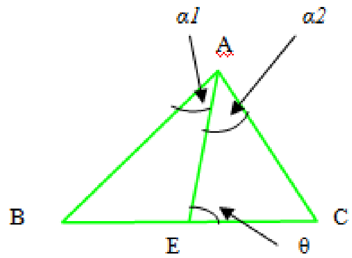

Figure-2

Abbreviation & various Steps used in this paper:

α1 = First separated angle
α2 = Second separated angle
E is mid of BC line
AB = c , BC =a , AC = b
Angle ABC = β, Angle BAC = α , Angle ACB =C

Corresponding Author: Rajbali Yadav*
Ordnance Factory Khamaria Jabalpur, (M.P.), India.-482005.

For max. area of Δ ADE in ΔABC

1) $a = \frac{(b^2-c^2)}{\sqrt{(b^2+c^2)}} a$

2) $\alpha + 2\beta = 270^\circ$ Or $\beta - C = 90^\circ$

3) $Cos\alpha = -Sin2\beta = Sin2C = \frac{2bc}{b^2+c^2}$

4) $BC\ Sin(\alpha + 2\beta) + 2DE\ Sin\ \alpha = 0$

5) $AB^2 + AC^2 = (2DE)^2$

6) $2DE \times AD = AB \times AC$

7) $\frac{cos\alpha 1}{sin\alpha 2} - \frac{cos\alpha 2}{sin\alpha 1} = 2.0$

2.1 Proof.

in Δ ABC (Figure1)
E is mid of BC

D is perpendicular at at BC
BD = (a/2-DE) , DC = (a/2+DE)

In Δ ADB
$AD^2 = c^2 - (a/2-DE)^2$ (1)

In Δ ADC
$AD^2 = b^2 - (a/2+DE)^2$ (2)

After solving (1) & (2)
$c^2 - (a/2-DE)^2 = b^2 - (a/2+DE)^2$
$c^2 - (a/2)^2 - DE^2 + DE \times a = b^2 - (a/2)^2 - DE^2 - DE \times a$
$c^2 + DE \times a = b^2 - DE \times a$

$$DE = \frac{b^2-c^2}{2a}$$

Put the value of DE in equation (2)

$$AD^2 = b^2 - \left[\frac{a}{2} + \frac{(b^2-c^2)}{2a}\right]^2 \quad \text{OR} \quad AD = \frac{\sqrt{(2ab)^2 - (a^2+b^2-c^2)^2}}{2a}$$

Area of Δ ADE (A1) = ½ AD× DE

$$= \frac{1}{2} \times \frac{\sqrt{(2ab)^2 - (a^2+b^2-c^2)^2}}{2a} \times \frac{(b^2-c^2)}{2a}$$

$$A1 = \frac{(b^2-c^2)}{8a^2} \times \sqrt{\left[(2ab)^2 - (a^2+b^2-c^2)^2\right]} \quad (3)$$

$$A1^2 = \frac{(b^2-c^2)^2}{64a^4} \times \left[(2ab)^2 - (a^2+b^2-c^2)^2\right]$$

For being area of Δ ADE Max. Or Mini.

$$\frac{dA1}{da} = \frac{dA1}{db} = \frac{dA1}{dc} = 0$$

Take $dA1/da = 0$

$$2A1\frac{dA1}{da} = \frac{1}{64}\frac{d}{da}\frac{(b^2-c^2)^2}{a^4} \times \left[(2ab)^2 - (a^2+b^2-c^2)^2\right]$$

$$128A1\frac{dA1}{da} = (b^2-c^2)^2 \frac{d}{da}\frac{1}{a^4} \times \left[(2ab)^2 - (a^2+b^2-c^2)^2\right]$$

$$= (b^2-c^2)^2 \frac{d}{da}[4b^2/a^2 - (1 + b4/a4 + 2b^2/a^2 + c4/a4 - 2c^2/a^2 - 2b^2c^2/a4]$$

$$\frac{d}{da}\left[4b^2/a^2-(1+b4/a4+2b^2/a^2+c4/a4-2c^2/a^2-2b^2c^2/a4\right]=0 \quad \text{put } \frac{dA1}{da}=0$$

$-8b^2/a3 + 4b\mathbf{4}/a\mathbf{5} + 4b^2/a3 + 4c\mathbf{4}/a\mathbf{5} - 4c^2/a3 - 8b^2c^2/a\mathbf{5} = 0$

$-2b^2 + b\mathbf{4}/a^2 + b^2 + c\mathbf{4}/a^2 - c^2 - 2b^2c^2/a^2 = 0$

$b\mathbf{4}/a^2 + c\mathbf{4}/a^2 - 2b^2c^2/a^2 = b^2 + c^2$

$[(b^2- c^2)/a]^2 = b^2 + c^2$

$\mathbf{a = b^2 - c^2/\sqrt{(b^2 + c^2)}}$ For Max. area of Δ ADE (4)

Put the value of **a** in eqn. (3)

$$A1=\frac{(b^2-c^2)}{8a^2}\sqrt{\left[(2ab)^2-\left(a^2+b^2-c^2\right)^2\right]}$$

$$A1max=\frac{\left(b^2-c^2\right)}{\frac{8\left(b^2-c^2\right)^2}{\left(b^2+c^2\right)}}\times\frac{\sqrt{4\left(b^2-c^2\right)^2b^2-\left[\left(b^2-c^2\right)^2+b^2-c^2\right]^2}}{b^2+c^2}$$

$$=\frac{\left(b^2+c^2\right)}{8\left(b^2-c^2\right)}\times\frac{\sqrt{4\left(b^2-c^2\right)^2b^2-\left[b4+c4-2b^2c^2+b4-c4\right]^2}}{\left(b^2+c^2\right)^2}$$

$$=\frac{\left(b^2+c^2\right)}{8\left(b^2-c^2\right)}\times\frac{\sqrt{4\left(b^2-c^2\right)^2b^2}}{b^2+c^2}-\frac{\left[2b^2\left(b^2-c^2\right]^2\right.}{\left(b^2+c^2\right)^2}$$

$$=\frac{\left(b^2+c^2\right)}{8\left(b^2-c^2\right)}\times\frac{\sqrt{4b^2\left(b^2-c^2\right)^2\left(b^2+c^2\right)-\left[2b^2\left(b^2-c^2\right]^2\right.}}{\left(b^2+c^2\right)^2}$$

$$=\frac{\left(b^2+c^2\right)\times 2b\left(b^2-c^2\right)}{8\left(b^2-c^2\right)\times\left(b^2+c^2\right)}\times\sqrt{\left(b^2+c^2-b^2\right)}$$

$$A1max=\frac{b\times c}{4}$$

2.2 According to Cosine rule

$$\operatorname{Cos}\alpha=\frac{\left(b^2+c^2-a^2\right)}{2bc}$$

Put the value of **a**

$$\operatorname{Cos}\alpha=\frac{b^2+c^2-\frac{\left(b^2-c^2\right)^2}{\left(b^2+c^2\right)}}{2bc}=\frac{b4+c4+2b^2c^2-b4-c4+2b^2c^2}{2bc\left(b^2+c^2\right)}=\frac{4b^2c^2}{2bc\left(b^2+c^2\right)}$$

$$\operatorname{Cos}\alpha=\frac{2bc}{\left(b^2+c^2\right)}$$

Add 1 both side

$$\operatorname{Cos}\alpha+1=\left(2bc+b^2+c^2\right)/\left(b^2+c^2\right)$$

$$\operatorname{Cos}\alpha+1=\left(b+c\right)^2/\left(b^2+c^2\right)$$

Equation (2) can also be written as

$[a/(b - c)]^2 = (b + c)^2/(b^2 + c^2)$

Hence

$\operatorname{Cos}\alpha + 1 = [a/(b - c)]^2$

$a/(b - c) = \sqrt{(1 + \operatorname{Cos}\alpha)}$

$= \sqrt{(1 + 2\operatorname{Cos}^2\alpha/2 - 1)}$

$$=\frac{\sqrt{2}\operatorname{Cos}\alpha/2}{a} \; or \; \frac{\left(b-c\right)}{\sqrt{2}\operatorname{Cos}\alpha/2}=1$$

Using Sine rule

$$\frac{(\operatorname{Sin}\beta-\operatorname{Sin}C)}{\operatorname{Sin}\alpha}=\frac{1}{\sqrt{2}\operatorname{Cos}\alpha/2} \qquad C=180^\circ-\left(\alpha+\beta\right)$$

$$\text{Sin}\beta - \text{Sin}\left[180 - (\alpha + \beta)\right] = \frac{\text{Sin}\alpha}{\sqrt{2}\ \text{Cos}\ \alpha/2}$$

$$\text{Sin}\beta - \text{Sin}(\alpha + \beta) = \frac{2\text{Sin}(\alpha/2)\text{Cos}(\alpha/2)}{\sqrt{2}\ \text{Cos}\ \alpha/2}$$

$\text{Sin}\beta - \text{Sin}(\alpha + \beta) = \sqrt{2}\ \text{Sin}\ \alpha/2$

$2\text{Cos}[(\beta+\alpha+\beta)/2]\text{Sin}[(\beta-\alpha-\beta)/2] = \sqrt{2}\ \text{Sin}(\alpha/2)$

Using (SinC –SinD) formula

$$\frac{-2\text{Cos}(\alpha + 2\beta)}{2} = \sqrt{2}$$

$$\text{Cos}\frac{(\alpha + 2\beta)}{2} = -\frac{1}{\sqrt{2}} = \text{Cos}(180^\circ - 45^\circ)$$

$\alpha + 2\beta = 270^\circ$

$\alpha + \beta + \beta = 270^\circ$

$180 - C + \beta = 270^\circ$ or $\beta - C = 90^\circ$

$\text{Cos}\alpha = 2bc/(b^2 + c^2)$

And

$\alpha + 2\beta = 270^\circ$ Or $\alpha = 270^\circ - 2\beta$

$\text{Cos}(270^\circ - 2\beta) = 2bc/(b^2 + c^2)$

$$-Sin2\beta = \frac{2bc}{(b^2 + c^2)}$$

$\beta = 90 + C$

$$-\text{Sin}2(90 + C) = \frac{2bc}{(b^2 + c^2)} = -\text{Sin}(180 + 2C) = \frac{2bc}{(b^2 + c^2)}$$

$$Sin2C = \frac{2bc}{(b^2 + c^2)}$$

$$Cos\alpha = Sin2\beta = Sin2C = \frac{2bc}{b^2 + c^2}$$

In figure (1)

E is mid of BC line

$$\text{Area of } \Delta \text{ ADE } (A1) = \frac{(b^2 - c^2)}{8a^2} \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}$$

And

$$\text{Area of } \Delta \text{ ABC } (A) = \frac{1}{4} \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}$$

Dividing A1/A

$$A1/A = \frac{(b^2 - c^2)/8a^2 \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}}{1/4 \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}}$$

$A1/A = (b^2 - c^2)/2a^2$ (5)

According to figure 1

$$A1 = \frac{1}{2} \times DE \times AD$$

$$A = \frac{1}{2} \times BC \times AD$$

Hence A1/A =DE/BC, Put the value of A1/A in equation no. (5)

$DE/BC = (b^2 - c^2)/2a^2$ (6)

Using Sine rule

$$\frac{2DE}{BC} = \frac{\text{Sin}^2\beta - \text{Sin}^2C}{\text{Sin}^2\alpha}$$

$$= \frac{(\text{Sin}\beta + \text{Sin}C)\ (\text{Sin}\beta - \text{Sin}C)}{\text{Sin}^2\alpha}$$

$$= \frac{[\text{Sin}\beta + \text{Sin}(180^\circ - \alpha + \beta)][(\text{Sin}\beta - \text{Sin}(180^\circ - \alpha + \beta)]}{\text{Sin}^2\alpha} \qquad C = 180^\circ - (\alpha + \beta)$$

$$= \frac{[\text{Sin}\beta + \text{Sin}(\alpha + \beta)][\text{Sin}\beta - \text{Sin}(\alpha + \beta)]}{\text{Sin}^2\alpha}$$

$$= \frac{2\text{Sin}\left[(\alpha + 2\beta)/2\right]\text{Cos}(\alpha/2)\times - 2\text{Cos}\left[(\alpha + 2\beta)/2\right]\text{Sin}(\alpha/2)}{\text{Sin}^2\alpha}$$ use SinC +SinD and SinC - SinD formula

$$= \frac{2\text{Sin}\left[(\alpha + 2\beta)/2\right]\text{Cos}\left[(\alpha + 2\beta)/2\right]\times 2\,\text{Sin}(\alpha/2)\text{Cos}(\alpha/2)}{\text{Sin}^2\alpha}$$

$$= -\text{Sin}[(\alpha + 2\beta) \times \text{Sin}\alpha \times / \text{Sin}^2\alpha$$

2DE/BC = - Sin[(α + 2β)/ Sinα

2DE Sinα = - BC Sin[(α + 2β)

BC Sin[(α + 2β) + 2DE Sinα = 0

By previous relation it is proved that for maximum area of Δ ADE

α + 2β = 270˚

Put this value in above equation

BC Sin 270˚+ 2DE Sinα = 0 , - BC +2DE Sinα = 0

Sinα =BC/2DE as BC =a and DE = a1 So, Sinα = a/2a1 (7)

According to equation no. (5) 2DE/BC = $(b^2 - c^2)/a^2$

Sinα = BC/2DE

= $1/(b^2 - c^2)/a^2$

Sinα = $a^2/(b^2 - c^2)$ (8)

According to equation no. (4)

$$a^2 = \frac{(b^2 - c^2)^2}{b^2 + c^2}$$

put the value of a^2 in equation no.(8)

$$\text{Sin}\alpha = \frac{\frac{(b^2 - c^2)^2}{(b^2 + c^2)}}{(b^2 - c^2)}$$

$$\text{Sin}\alpha = \frac{b^2 - c^2}{b^2 + c^2} \qquad (9)$$

$b^2 + c^2$

According to equation no. (7) Sinα = a/2a1

Put the value of a from equation no4 and Sinα from equation no. (9) in equation no. (7)

$$\frac{(b^2 - c^2)}{(b^2 + c^2)} = \frac{\frac{b^2 - c^2}{\sqrt{(b^2 + c^2)}}}{2a1}$$

$$\frac{(b^2 - c^2)}{(b^2 + c^2)} = \frac{b^2 - c^2}{2a1 \times \sqrt{(b^2 + c^2)}}$$

$$2a1 = \frac{(b^2 + c^2)}{\sqrt{(b^2 + c^2)}}$$

$(2a1)^2 = b^2 + c^2$ OR $(2DE)^2 = AC^2 + AB^2$ As, AB = c AC =b & DE = a1

$AC^2 + AB^2 = (2DE)^2$ (10)

As we are knowing Max. area of ΔADE = bc/4

In figure 1 area of ΔADE = 1/2×AD×DE

Hence
1/2×AD×DE = bc/4
AD = bc/2DE Or bc/2a1, DE= a1
And a 1=a / 2Sinα

$$\text{Hence} \quad AD = \frac{bc}{2a/2\mathrm{Sin}\alpha} = \frac{bc\,\mathrm{Sin}\alpha}{a} \tag{11}$$

In eqn. no. (4) & (9)

$$a = \frac{(b^2 - c^2)}{\sqrt{(b^2+c^2)}} \text{and } \mathrm{S\,in}\alpha = \frac{(b^2 - c^2)}{(b^2+c^2)}$$

Put the value of a & Sinα in eqn. no. (11)

$$AD = \frac{bc \times \frac{(b^2 - c^2)}{(b^2-c^2)}}{\sqrt{(b^2+c^2)}} \quad \text{Or} \quad AD = \frac{bc}{\sqrt{(b^2+c^2)}}$$

$$AD^2 = \frac{(bc)^2}{(b^2+c^2)} \quad \text{weknow } b^2+c^2 = (2a1)^2$$

$$AD^2 = \frac{(bc)^2}{(2a1)^2}$$

$$2a1 \times AD = b \times c \quad Or \quad 2DE \times AD = AB \times AC \tag{12}$$

We know,

$$Cos\alpha = -Sin2\beta = Sin2C = \frac{2bc}{b^2 + c}$$

And
$b^2 + c^2 = (2DE)^2$, bc = 2AD × DE

$$Cos\alpha = -Sin2\beta = Sin2C = \frac{2bc}{b^2 + c} = \frac{4AD \times DE}{4DE^2} = \frac{AD}{DE}$$

In figure 2, AD/DE= TanΘ

Hence
Cosα =- Sin2β = Sin2C = TanΘ

In figure 2
(CE+BE) CotΘ =CE Cotα1 - BE Cotα2
BC CotΘ =BC/2(Cotα1 - Cotα2) As BE = CE= BC/2
2CotΘ =Cotα1 - Cotα2 TanΘ = Cosα
2/TanΘ =Cotα1 - Cotα2
2/Cosα =Cotα1 - Cotα2

$$\frac{2}{Cos\alpha} = \frac{Cos\alpha 1}{Sin\alpha 1} - \frac{Cos\alpha 2}{Sin\alpha 2}$$

$$= \frac{Sin\alpha 2 Cos\alpha 1 - Cos\alpha 2 Sin\alpha 1}{Sin\alpha 1 Sin\alpha 2} = \frac{Sin(\alpha 2 - \alpha 1)}{Sin\alpha 1 Sin\alpha 2}$$

2 Sinα1Sinα2 = Cosα Sin(α2- α1)
2 Sinα1Sin(α –α1) = Sin(α- 2α1) Cosα As α2= α-α1 and α = α1 +α2

Cos(α1-α+α1) – Cos(α1 +α –α1) =1/2[sin(α -2α1+α) +sin(α-2α1-α) Using 2SinASinB & 2SinACosB Formula

Cos(2α1-α) – Cosα = 1/2[sin2(α -α1) - sin2α1]

Cos(2α1-α) – Cosα = 1/2[2 sin(α -α1)cos(α-α1) - 2sinα1cosα1] using, Sin2α = 2sinacosa

Cos(2α1-α) – Cosα = sin(α -α1)cos(α-α1) - sinα1cosα1

Cosα = Cos(2α1-α) +sinα1cosα1- sin(α -α1)cos(α-α1) \

Cosα = Cos (2α1-α1-α2) + sinα1cosα1- sinα2cosα2, As α- α1 =α2

$$\text{Cos}\alpha = \text{Cos}(\alpha 1 - \alpha 2) + \frac{\sin 2\alpha 1}{2} - \frac{\sin 2\alpha 2}{2}$$

Cosα = Cos(α1-α2) +1/2(sin2α1 - sin2α2)

$$= \text{Cos}(\alpha 1 - \alpha 2) + \frac{2\cos(\alpha 1 + \alpha 2)\sin(\alpha 1 - \alpha 2)}{2}$$

= Cos(α1-α2) + cos(α1+α2) sin(α1 - α2)

= Cos(α1-α2) + cosα sin(α1 - α2)

Cosα = Cos(α1-α2) - cosα sin(α2 – α1) As sin(-α) = -sinα

Cosα + cosα sin(α2 – α1) = Cos(α1-α2)

Cosα [1 + sin(α2 – α1)] = Cos(α2-α1)

$$\text{Cos}\alpha = \frac{\text{Cos}(\alpha 2 - \alpha 1)}{1 + \sin(\alpha 2 - \alpha 1)}$$

$$1 + \sin(\alpha 2 - \alpha 1) = \frac{\text{Cos}(\alpha 2 - \alpha 1))}{\text{Cos}\alpha} = \frac{\text{Cos}(\alpha 2 - \alpha 1)}{\text{Cos}(\alpha 2 + \alpha 1)}$$

$$\sin(\alpha 2 - \alpha 1) = \frac{\text{Cos}(\alpha 2 - \alpha 1)}{\text{Cos}(\alpha 2 + \alpha 1)} - 1$$

$$= \frac{\text{Cos}(\alpha 2 - \alpha 1) - \text{c os}(\alpha 2 + \alpha)1}{\text{Cos}(\alpha 2 + \alpha 1)} \text{ using (Cos C-Cos D) formula}$$

$$= \frac{2 \text{ in}[s(\alpha 2 - \alpha 1 + \alpha 2 + \alpha 1)/2]\sin[(\alpha 2 + \alpha 1 - \alpha 2 + \alpha 1)/2]}{\text{Cos}(\alpha 2 + \alpha 1)}$$

$$\sin(\alpha 2 - \alpha 1) = \frac{2\sin\alpha 2 \times \sin\alpha 1}{\text{Cos}(\alpha 2 + \alpha 1)}$$

2sin(α2 – α1)× Cos(α2+α1)= 2× 2sinα2×sinα1

sin(α2 – α1+α+ α1) + sin(α2+α1-α2-α1)= 2× 2sinα2×sinα1

sin2α2 - sin2α1 = 2× 2sinα2×sinα1

2sinα2cosα2 - 2sinα1cosα1 = 2× 2sinα2×sinα1

sinα2cosα2 - sinα1cosα1 = 2sinα2×sinα1

$$\frac{\sin\alpha 2\cos\alpha 2}{\sin\alpha 2 \times \sin\alpha 1} - \frac{\sin\alpha 1\cos\alpha 1}{\sin\alpha 2 \times \sin\alpha 1} = 2$$

$$\frac{cos\alpha 2}{sin\alpha 1} - \frac{cos\alpha 1}{\sin\alpha 2} = 2$$

3. REFERENCES

1. Rajbali Yadav, Right angle triangle with application of two circles on the base of triangle, International Journal of Mathematical Archive- 11(7), 2020, 17-22.
2. S.L LONEY plain trigonometry part 1, product formulae page77.
3. www.askiitians.com

Source of support: Nil, Conflict of interest: None Declared.