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RELATION BETWEEN DOMINATION NUMBER, SEIDEL ENERGY OF GRAPH AND RANK

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ABSTRACT

In this paper we find few bounds which relate domination number of book graph, and Seidel energy of book graph and rank of the incident matrix of the book graph and pose some open problems for further research. The motivation for this paper is to establish a link between there parameters.

Keywords: *Seidel Matrix, Domination, Incidence Matrix, Rank.*

I. INTRODUCTION

By a graph G = (V, E) we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [3]. Graphs have various special patterns like path, cycle, star, complete graph, bipartite graph, complete bipartite graph, regular graph, strongly regular graph etc. For the definitions of all such graphs we refer to Harry [5]. The study of Cross product of graph was initiated by Imrich [9]. For structure and recognition of Cross Product of graph we refer to Imrich [8]. In literature, the concept of domination in graphs was introduced by Claude Berge in 1958 and Oystein Ore in [1962] by [14]. For review of domination and its related parameters we refer to Acharya *et.al* [1979] and Haynes *et.al* [1998a, 1998b] [6 & 7]. The concepts of perfect domination was introduced by Cockayne *et.al.* [1993]. [2]. For more details we refer to Dejter and Pujol [1995], Fellows and Hoover [1991)], Yen and Lee [1996] [2 & 4]. The split domination in graphs was introduced by Kulli & Janakiram [13]. They defined the split dominating set, the split domination number and obtained several interesting results regarding the split domination number of some standard graphs. They have also obtained relations of split domination number with the other parameters such as domination number, connected domination number, vertex covering number etc., These are all refer to Kulli & Janakiram.

J. Liu and B. Liu defined the Seidel energy graph in generalization for Laplacian energy and analyze. For the Seidel energy bound using the rank of the sediel matrix and extended the concept of energy to Hermite matrix. Willem H. Harmers in Seidel switching and graph energy investigates that Seidel switching changes the spectrum, but not the energy and present an infinite family of example with maximal energy. In this chapter the seidel matrix [15] which is the generalization of the adjacency matrix is analyzed for sharp bound of the Seidel energy for the connected and disconnected graph. In this paper we find few bounds which relate domination number of book graph , and Seidel energy of book graph and rank of the incident matrix of the book graph and pose some open problems for further research. The motivation for this paper is to establish a link between there parameters. This is the chapter motivated from the Paper [1]. CLEMENTS BRAND, NORBERT SEITER, Eigen values and determination in graph, Mathematic Slovaca, Vol.46 (1996). No. 1.1,33 – 39, where the authors had found the relation between largest eigen value of the Laplacian matrix and Domination number.

II. PRELIMINARIES

Definition 2.1: Seidel Matrix:
In [48], Haemers defined the Seidel energy $\mathrm{SE(G)}$ of a simple connected graph *G* as the sum of the absolute values of eigen values of the seidel matrix of *G*. Here, $\mathrm{SE(G)}$ where

$$S(G) = [s_{ij}] = \begin{cases} -1 & \text{If } v_i \text{ and } v_j \text{ are adjacent and } i \neq j \\ 1 & \text{If } v_i \text{ and } v_j \text{ are non adjacent and } i \neq j \\ 0 & if\ \ i = j \end{cases}$$

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Definition 2.2: Rank of the matrix
The number of non zero row in the reduced form of a matrix A is called the rank of A denoted rank (A) or $\rho(A)$. Rank of the matrix is the number of linearly independent columns. A matrix always represents a linear transformation between two vector spaces. From the rank of the matrix. We come to know several properties above this linear transformation. Rank of the matrix equals the dimension of the linear manifold spanned by vectors $x_1, x_2, x_3, \ldots . x_k$

Definition 2.3: Incidence Matrix
Incidence Matrix Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an $n \times m$ matrix $A = a_{ij}$ whose n rows correspond to the n vertices and the m columns correspond to m edges such that

$$a_{ij} = \begin{cases} 1 & if\ jth\ edge\ m_j\ is\ incident\ on\ the\ ith\ vertex \\ 0 & other\ wise \end{cases}$$

It is also called vertex-edge incidence matrix and is denoted by A(G).

Definition 2.4: Domination
A set D ⊆ V is called a dominating set if every vertex in ∀D is adjacent to some vertex of D. Notice that D is a dominating set if and only if N[D] = V. The domination number of G, denoted as γ = γ(G), is the cardinality of a smallest dominating set of V . We call a smallest dominating set a γ-set.

Definition 2.5: Book Graph
Book graph is the Cartesian product of star S_{m+1} and path P_2. Book graph is defined by B_m

Theorem 2.5: [11] The domination number of book graph is

$$\gamma(B_m) = 2$$

Theorem 2.6: [11] The split domination number of book graph is

$$\gamma^s(B_m) = 2$$

Theorem 2.7: [12] The Perfect domination number of book graph is

$$\gamma^s(B_m) = 2$$

III Main Result:

3.1 The Seidel matrix of Book Graph B_m

$$S(B_m) = \left[\begin{array}{ccccccc|ccccccc}
0 & -1 & -1 & -1 & -1 & \cdots -1 & 1 & -1 & 1 & 1 & 1 & 1 \cdots & 1 & 1 \\
-1 & 0 & 1 & 1 & 1 \cdots & 1 & 1 & 1 & -1 & 1 & 1 & 1 \cdots & 1 & 1 \\
-1 & 1 & 0 & 1 & 1 \cdots & 1 & 1 & 1 & 1 & -1 & 1 & 1 \cdots & 1 & 1 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
-1 & 1 & 1 & 1 & 1 \cdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 \cdots & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 \cdots & 1 & 0 & 1 & 1 & 1 & 1 & 1 \cdots & 1 & -1 \\
\hline
-1 & 1 & 1 & 1 & 1 & \cdots 1 & 1 & 0 & 1 & 1 & 1 & 1 \cdots & 1 & 1 \\
1 & -1 & 1 & 1 & 1 \cdots & 1 & 1 & -1 & 0 & 1 & 1 & 1 \cdots & 1 & 1 \\
1 & 1 & -1 & 1 & 1 \cdots & 1 & 1 & -1 & 1 & 0 & 1 & 1 \cdots & 1 & 1 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & 1 \cdots & -1 & 1 & -1 & 1 & 1 & 1 & 1 \cdots & 0 & 1 \\
1 & 1 & 1 & 1 & 1 \cdots & 1 & -1 & -1 & 1 & 1 & 1 & 1 \cdots & 1 & 0
\end{array}\right]$$

$$S(B_m) = \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right]$$

Where A and B are

$$A = \begin{bmatrix} 0 & -1 & -1 & -1 & -1 & \cdots & -1 & 1 \\ -1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ -1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ -1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ -1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ -1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

3.2 Incidence Matrix of the book graph

$$I(B_m) = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ - & - & - & - & & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Table 3.3: Book Graph B_m

G	$\gamma(G)$	$\rho(G)$	Eigen values	$SE(G)$
B_3	2	7	$3, 1+2\sqrt{3}, 1-2\sqrt{3}, -3, -3, -1, 1, 1$	18.9282
B_4	2	9	$5, 5, -3, -3, -3, -3, -1, 1, 1, 1$	26
B_5	2	11	$7, 1+2\sqrt{5}, 1-2\sqrt{5}$, -3, -3, -3,-3,-1,1,1,1,1	32.944
B_6	2	14	$9, 1+2\sqrt{6}, 1-2\sqrt{6}$, -3, -3, -3,-3,-3,-1,1,1,1,1,1	39.7978
B_7	2	16	$11, 1+2\sqrt{7}, 1-2\sqrt{7}$, -3, -3, -3,-3,-3,-3,-1,1,1,1,1,1,1	46.58300
B_8	2	18	$13, 1+2\sqrt{8}, 1-2\sqrt{8}$, -3, -3, -3,-3,-3,-3,-3,-1,1,1,1,1,1,1,1,	53.315
$B_{9.}$	2	20	$15, 1+2\sqrt{9}, 1-2\sqrt{9}$, -3, -3, -3,-3,-3,-3,-3,-3,-1,1,1,1,1,1,1,1,1,	60
B_{10}	2	22	$17, 1+2\sqrt{9}, 1-2\sqrt{9}$, -3, -3, -3,-3,-3,-3,-3,-3,-3,-1,1,1,1,1,1,1,1,1,1	66.6494
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B_m	2	$2m+1$ $where\ m = 3,4,5$ $2m+2$ $where\ m \geq 6$	$(2m-3), 1+2\sqrt{m}, 1-2\sqrt{m}, (m-1)\times -3$ $-1, (m-1)\times 1.$	$6m+4\sqrt{m}-6$

Lemma 3.4: [14] The Seidel Eigen values $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots\dots\dots\dots \lambda_n$ of the Seidel matrix of the book graph B_m satisfies the following relations.

1. $\sum_{i=1}^{n} \lambda_1 = 0$
2. $\sum_{i=1}^{n} \lambda_i^2 = n(n-1)$

Lemma 3.5: The Eigen value of the Book graph is

$$(2m-3), 1+2\sqrt{m}, 1-2\sqrt{m}, (m-1)\times -3, -1, (m-1)\times 1.$$

Lemma 3.6: The energy of Seidel matrix of book graph is

$$SE(B_m) = 6m + 4\sqrt{m} - 6$$

Theorem 3.7: Let G be a book graph B_m is a cross product of star S_{m+1} and path P_2, $SE(G)$ is the Seidel energy of the graph G, $I(G)$ is the incident matrix of graph G. $\rho(G) = Rank\ I(G)$, $\gamma(G)$ is the domination number of G then

$$\gamma(G) = \left\lceil \frac{SE(G)}{Rank\ I(G)} \right\rceil - 1$$

Proof: The proof can be done in two methods
a) Direct methods
b) Mathematical Induction
a) Direct method

From theorem 3.2.1 $\gamma(G) = 2$, where G is the book graph B_m

From table 1 $SE(B_m) = 6m + 4\sqrt{m} - 6$, where $m \geq 3$

$$Rank\ I(G) = \begin{cases} 2m+1 & where\ m = 3,4,5 \\ 2m+2 & where\ m > 5 \end{cases}$$

From the table 1 we get
Case-1: $m = 3,4,5$

$$\frac{SE(G)}{Rank\ I(G)} = \frac{6m+4\sqrt{m}-6}{2m+1} \geq 3\text{--------------} (1)$$

Case-2: $m > 5$

$$\frac{SE(G)}{Rank\ I(G)} = \frac{6m+4\sqrt{m}-6}{2m+2} \geq 3\text{-------------} (2)$$

From equation (1) and (2)

$$\frac{SE(G)}{Rank\ I(G)} > 3$$

$$\left\lceil \frac{SE(G)}{Rank\ I(G)} \right\rceil - 1 = 2$$

(b) Mathematical Induction to prove that

$$\gamma(G) = \left\lceil \frac{SE(G)}{Rank\ I(G)} \right\rceil - 1$$

For B_3 $\gamma(G) = \left\lceil \frac{18.9282}{7} \right\rceil - 1 = 2$

For B_4 $\gamma(G) = \left\lceil \frac{26}{9} \right\rceil - 1 = 2$

Let LHS=RHS for $n = k$ ie

$$\gamma(G_k) = \left\lceil \frac{SE(G_k)}{Rank\ I(G_k)} \right\rceil - 1$$

to prove that LHS=RHS for $n = k+1$,

i.e. $\gamma(G_{k+1}) = \left\lceil \frac{SE(G_{k+1})}{Rank\ I(G_{k+1})} \right\rceil - 1$

we know that $SE(G) \leq SE(G_{k+1})$

Rank $I(G_k) \leq Rank\ I(G_{k+1})$

By Inspection $2Rank\ I(G_{k+1}) > SE(G_{k+1})$

Hence we conclude LHS=RHS for all n

Hence

$$\gamma(G) = \left\lceil \frac{SE(G)}{Rank\ I(G)} \right\rceil - 1$$

Corollary: From [11 & 12] theorem 2.6, 2.7, 2.8 $\gamma(B_m) = 2$

$$\gamma(B_m) = \gamma_{pt}(B_m) = \gamma^s(B_m) = \left\lceil \frac{SE(G)}{Rank\ I(G)} \right\rceil - 1$$

IV. CONCLUSION

1) $\gamma(B_m) = \gamma_{pt}(B_m) = \gamma^s(B_m) = \left\lceil \frac{SE(G)}{Rank\ \ I(G)} \right\rceil - 1$

V. FEW OPEN PROBLEMS

1) It can be show that for theorem 3.7 book graph we can get better results for specific cases of n number of vertices being odd, even, prime etc.
2) The relation between there parameter can be extended to other cross product of graph and other type of Domination.
3) Similarly bounds i.e relation between domination number, Seidel energy of graph and Rank of the incient matrix could be extended to any cross product of two graph and other relates topics.

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