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STRUCTURES OF SIMPLE SEMIRINGS

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ABSTRACT

In this paper we determined some characteristics of simple semiring and also proved some results on simple semirings which was introduced by Golan [1].

PRELIMINARIES

A triple (S, +, .) is called a semiring if (S, +) is a semigroup; (S, .) is semigroup; a (b + c) = ab + ac and (b + c) a = ba + ca for every a, b, c in S. (S, +) is said to be band if a + a = a for all a in S. A (S, +) semigroup is said to be rectangular band if a + b + a = a for all a, b in S. A semigroup (S, .) is said to be a band if $a = a^2$ for all a in S. A semigroup (S, .) is said to be rectangular band if aba = a.

Definition 1.1: A semigroup (S, .) is said to be left (right) singular if ab = a (ab = b) for all a, b in S.

Definition 1.2: A semigroup (S, +) is said to be left (right) singular if a + b = a (a + b = b) for all a, b in S.

Definition 1.3: A semiring (S, +, .) is said to be zero square semiring if $x^2 = 0$ for all x in S.

Definition 1.4: An element 'a' of 'S' is called E - inverse if there is an element 'x' of S such that ax + ax = ax,
i.e ax ε E (S), where E (S) is the set of all idempotent elements of S.

Definition 1.5: A semigroup 'S' is called an E - inverse semigroup if every element of S is an E- inverse.

Definition 1.6: A semigroup (S, +) is said to be left regular if aba = ab.

Definition 1.7: A viterbi semiring is a semiring in which S is additively idempotent and multiplicatively subidempotent. i.e., a + a = a and $a + a^2 = a$ for all a in S.

Definition 1.8: A semiring (S, +) is said to be Additively Idempotent Semiring if a + a = a for all a in S.

Definition 1.9: [3] A semiring S is called simple if a + 1 = 1 + a = 1 for any $a \in S$.

Theorem 1.10: Let (S, +,·) be a simple semiring then following are true.
(i) ab + a = a = a + ab (ii) ab + a + ab = a (iii) a + ab + a = a(iv) $a^2 + a = a = a + a^2$

Proof: Since (S, +,·) be a simple semiring b + 1 = 1 for every b in (S, +,·) ⇒ a.(b + 1) = a.1 ⇒ ab + a = a.

Similarly, a + ab = a.
ii) ab + a = a ⇒ ab + a.1 =a ⇒ ab + a(1 + b) = a ⇒ ab + a + ab = a
iii) a + ab + a = a(1 + b) + a = a.1 + a = a + a = a(1 + 1) = a.1 = a ⇒ a + ab + a = a
iv) a = a ⇒ a.1 = a ⇒ a(a + 1) = a ⇒ a^2+ a = a.

Similarly, $a + a^2 = a$. therefore, $a^2 + a = a = a + a^2$

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Theorem 1.11: Let (S, +,·) be a simple semiring then (S, +) is a band.

Proof: Since (S, +, ·) be a simple semiring b + 1 = 1 for every b in (S, +, ·) ⇒ a. (b + 1) = a.1 ⇒ ab + a = a, for all a in S ⇒ a.1 + a = a (taking b = 1) ⇒ a + a = a∴ (S, +) is a b and.

Theorem 1.12: Let (S, +, ·) be a simple semiring then (S, +, ·) is viterbi semiring.

Proof: From the theorem 1.10, S satisfies $a^2 + a = a = a + a^2$.

From the theorem 2, (S, +) is a band.

Therefore, S is viterbi semiring.

Theorem 1.13: Let S be a simple semiring. If (S, +) is a cancellative then (i) (S, ·) is a band. (ii) (S, ·) is a rectangular band.

Proof: Since From the theorem 1.10, $a^2 + a = a \Rightarrow a^2 + a = a + a \Rightarrow a^2 = a$ ((S, +) is cancellative) ⇒(S, ·) is a band.

Since from the theorem 1, a + ab = a⇒ (a + ab)a = a.a⇒ $a^2 + aba = a^2 \Rightarrow a + aba = a \Rightarrow a + aba = a + a \Rightarrow aba = a$. ((S, +) is cancellative) ⇒ (S, ·) is a rectangular band.

Theorem 1.14: If S is a simple semiring and (S, .) is a left singular then (S, +) is a band.

Proof: From the theorem 1.10, a + ab = a. Since (S, .) is left singular implies ab = a ⇒ a + a = a ⇒ (S, +) is a b and

Example 1.15:

+	a	2a
a	a	a
2a	a	2a

.	a	2a
a	a	a
2a	2a	2a

Theorem 1.16: If S is a simple semiring and (S, +) is a right singular semigroup, then (S, +) is a rectangular band.

Proof: From the theorem 1.10, a + ab = a, for all a, bin S ⇒ a + ab + b = a + b ⇒ a + ab + b=b (∵ (S, +) is a rightsingular) ⇒ a + ab + b + a = b + a ⇒ a + ab + b + a= a (∵ (S, +) is a right singular) ⇒ a + b + a= a. Hence (S, +) is a rectangular band.

Theorem 1.17: If S is a zero square and simple semiring where 0 is the additive identity in S then aba = 0 and bab = 0 for all a, b in S.

Proof: **a** + ab = a for all a, b in S, from theorem1.10, ⇒ $a^2 + aba = a^2$ ⇒ 0 + aba=0 (∵ S is a zero square semiring, $a^2 = 0$) ⇒ aba = 0

Also, b + ba = b for all b, a in S ⇒ $b^2 + bab = b^2$ ⇒ 0 + bab = 0 (∵ S is a zero square semiring, $b^2 =0$) ⇒bab = 0. Hence, aba = 0 and bab = 0.

Theorem 1.18: Let S be a simple Semiring.

(i) If (S, .) is left regular semigroup and (S, .) is commutative then S is an E – inversesemigroup.
(ii) If (S, .) is band, then S is an E – inversesemigroup.

Proof:
(i) From theorem1.10, a + ab = a for all a, b in S
⇒ (a + ab) b = ab ⇒ $ab + ab^2 = ab$ ⇒ $aba + ab^2a = aba$ ⇒ ab + a.bb.a= ab(∵ S is leftregular)⇒ ab + (bab) a = ab ((S, .) is commutative) ⇒ ab + baa = ab ⇒ ab + aba=ab
⇒ab + ab=ab(∵ S is leftregular)⇒S is an E – inverse semigroup.

ii) From theorem 1.10, a + ab = a for all a, b in S
⇒ (a + ab) b = ab ⇒ $ab + ab^2 = ab$ ⇒ ab + ab=ab ((S, .) is band)
⇒ S is an E – inverse semigroup.

Theorem 1.19: If S is a Simple Semiring with additive identity 0 then ab = o for all a, b in S when (S, +) is cancellative.

Proof: From theorem 1.10, a + ab = a for all a, b in S
$\Rightarrow$ a + a + ab = a + a $\Rightarrow$ a + a + ab = a + a + 0 $\Rightarrow$ ab = 0 ($\because$(S, +) is cancellative)

Theorem 1.20: If a, b, c and d are elements of a simple semiring S satisfying a + c = b and b + d = a and (S, +) is commutative, then a = b.

Proof: If S is a Simple Semiring, i.e, a = a + aNow, a = a + b+ d ($\because$ a = b +d) = a + a + c+d ($\because$ b = a +c) = a + c+ d ($\because$ a = a +a)
= b + d + c+ d ($\because$ a = b +d) = b + d + d+ c($\because$ (S, +) is Commutative)

= b + d+ c ($\because$ d = d +d) = a + c =b ($\because$ b = a + c)

Theorem 1.21: If S is a Simple Semiring then a^n+ 1 = 1 for every a in S.

Proof: Let S be a simple semiring then we have a + 1 = 1 for every a in S. If n = 1 then proof is obvious.

If n = 2 then a^2 + 1 = aa + 1 = aa + a + 1 = a(a + 1) + 1 = a.1 + 1 = a + 1 = 1.

If n = 2 then the statement is true.

Assume that the statement is true for n = k the a^k + 1 = 1.

We have to prove that the statement is true for n = k + 1.

Consider a^{K+1} + 1 =a^Ka + 1 = a^ka + a + 1 = a(a^k + 1) + 1 = a.1 + 1 = a + 1 = 1.

Hence the result is true for n = k + 1.

Therefore, If S is a Simple Semiring then a^n+ 1 = 1 for every a in S.

Theorem 1.22: If S is a Simple Semiring then ab+ 1 = 1 for every a, b in S.

Proof: If S is a Simple Semiring then a+ 1 = 1 and b + 1 = 1 for every a, b in S.
ab + 1 = ab + a + 1 = a(b + 1) + 1 = a.1 + 1 = a + 1 = 1.
Hence, ab+ 1 = 1.

Theorem 1.23: If S is a Simple Semiring then a_1 a_2 a_3 a_4 …..a_n+ 1 = 1 for every a_i in S.

Theorem 1.24: Let S be a simple semiring and (S, +) be commutative. Then (S, .) is commutative if (S, +) is not a rectangularband.

Proof: Suppose (S, +) is a rectangular band

Consider ab + a = a, for all a, b in S$\Rightarrow$ ab + a + ab = a + ab$\Rightarrow$ a (b + 1 + b) = ab + a (Since (S, +) is commutative)
$\Rightarrow$ ab = ab+ a (Since (S,+) is a rectangularband) $\Rightarrow$ ab = a

Now ab + a = a (Put a = 1) then $\Rightarrow$ 1. b + 1=1$\Rightarrow$ b + 1=1, for all b in S

Also ba + b =b, for all a, b in S $\Rightarrow$ ba + b + ba = b + ba$\Rightarrow$ b (a + 1 + a) = ba + b (Since (S,+) is commutative)
$\Rightarrow$ba = ba+b (Since (S,+) is a rectangularband)$\Rightarrow$ba = b$\Rightarrow$ ab $\neq$ba, which proves the result. Also ab = a
$\Rightarrow$ab + b = a + b$\Rightarrow$ (a + 1) b = a + b$\Rightarrow$ 1. b = a+b (from b + 1 =1)$\Rightarrow$ b = a + b = b + a

This is evident from the following example

Example 1.25:

+	1	A	b
1	1	1	1
A	1	A	b
B	1	B	b

.	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

Theorem 1.26: Let S be a simple semiring. Let (S, +) be commutative and (S, .) is rectangular band then ab = a and ba =b

Proof: Consider ab + a = a for all a, b in S and ba + b = b for all b, a inS
$\Rightarrow$ ab = a (ba + b) $\Rightarrow$ ab = aba + ab $\Rightarrow$ ab = a+ ab (Since (S, .) is a rectangularband)
$\Rightarrow$ ab = ab+ a (Since (S,+) is commutative) $\Rightarrow$ ab = a
Also ba = b (ab + a) $\Rightarrow$ba = bab + ba$\Rightarrow$ba = b+ba (Since (S, .) is a rectangular band)$\Rightarrow$ba = ba+b (Since (S,+) is commutative) $\Rightarrow$ ba = b. Therefore, ab = a andba = b for all a, b in S.

Theorem 1.27: Let S be a simple semiring and (S,·) be a left singular, then (S, +) is a right singularsemigroup.

Proof: By hypothesis ab = a, for all a, b in S ($\because$(S,·) is left singular) $\Rightarrow$ ab + b = a + b $\Rightarrow$ (a + 1) b = a + b $\Rightarrow$ 1. b = a+b
($\because$S is simple semiring) $\Rightarrow$ b = a + b Also ba = b $\Rightarrow$ba + a = b + a $\Rightarrow$ (b + 1) a = b + a $\Rightarrow$ 1. a = b+a
($\because$S is simple semiring) $\Rightarrow$ a = b + a$\Rightarrow$
a + b = b and b + a = a, for all a, b in S. Hence (S, +) is a right singular semigroup.

Theorem 1.28: Let S be a simple semiring. If (S, +) is a right singular semigroup, then (S, +) is a rectangular band.

Proof: By hypothesis a + b = b, for all a, b in S ($\because$ (S, +) is rightsingular) $\Rightarrow$ a + b + a = b + a$\Rightarrow$ a + b + a = a, for all a, b in S, which proves the theorem. ($\because$(S, +) is a right singular semigroup) i.e., (S, +) is a rectangular band.

Theorem 1.29: Let S be a totally ordered simple semiring. If (S, +) is p.t.o (n.t.o.) and (S, ·) is commutative, then (S, ·) is n.t.o.(p.t.o.).

Proof: Since S istotally ordered simple semiring ab + a = a, for all a, b in S$\Rightarrow$ a = ab + a$\geq$ab ($\because$(S, +) is p.t.o.) $\Rightarrow$ a$\geq$ ab

Suppose ab > b$\Rightarrow$ ab + a $\geq$ b + a$\Rightarrow$a $\geq$ b+ a ($\because$ab + a =a)$\Rightarrow$ b + a $\leq$ a

Which contradicts the hypothesis that (S, +) is p.t.o.$\Rightarrow$ ab $\leq$ b
$\therefore$ab $\leq$ a & ab $\leq$ b Hence (S,·) is n.t.o.

Similarly we can prove that (S,·) is p.t.o if (S, +) is n.t.o.

Theorem 1.30: If S be a simple semiring then (S, +) is weakly seperative semigroup.

Proof: If S be a simple semiring then (S, +) is a band.

Consider a + a = a + b = b + b $\Rightarrow$ a = a + b = b $\Rightarrow$ a = b $\Rightarrow$ (S, +) is weakly seperative semigroup.

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