International Journal of Mathematical Archive-11(7), 2020, 17-22
Available online through www.ijma.info ISSN 2229 – 5046

RIGHT ANGLE TRIANGLE WITH APPLICATION OF TWO CIRCLES ON THE BASE OF TRIANGLE

RAJBALI YADAV*
Ordnance Factory Khamaria, Jabalpur (M.P,) - 482005, India.

(Received On: 17-06-20; Revised & Accepted On: 27-06-20)

ABSTRACT

In this paper relation among diameter and radius of two circles drawn on the base of right angle triangle and height of the triangle is shown. By the application of two circles on the base of right angled triangle it is shown that this right angle triangle is extension of an obtuse triangle in which difference of two angle is, 90˚. Finally it will be proved that Square difference of diameter of larger circle and radius of smaller circle, drawn on base of right angle triangle is equal to height square of right angle triangle.

Keywords: *Circle, Maximum area, Obtuse triangle, Right angle triangle.*

1. INTRODUCTION

This theory is, a deduction from maximum or minimum area of Δ ADE (as shown in diagram1). For being maximum or minimum area of Δ ADE difference of two angle in Δ ABC must be 90˚. In this condition Δ ABC become an obtuse triangle. *For constructing such triangle, in two circles of different diameter (as shown in diagram1) if larger diameter circle pass through centre of smaller diameter circle, line drawn from intersection of circles and initial point of center line of first circle and a perpendicular line drawn from end point of second circle center line, which cut diagonal at some point, through this point line drawn to last point of first circle, an obtuse triangle (ΔABC) with difference of two angle 90 is obtain. In right angle Δ ADC, Square difference of, diameter of larger circle and radius of smaller circle is equal to height square of right angle triangle.*

Definition 1.1: Obtuse Triangle. An obtuse-angled triangle is a triangle in which one of the interior angles measures more than 90˚. In an obtuse triangle, if one angle measures more than 90˚ than the sum of remaining two angles is less than 90˚.

2. RIGHT ANGLED TRIANGLE WITH APPLICATION OF TWO CIRCLES ON THE BASE OF TRIANGLE

2.1. In *right angle triangle if base of right angle triangle is equal to sum of diameter and radius of two circles in which larger diameter circle pass through center of smaller radius circle and Hypotenuse passes from intersection of these two circles then*
'Square difference of diameter of larger circle and radius of smaller circle is equal to height square of right angle triangle'.
It can be mathematically written as: $\mathbf{d^2 - r^2 = h^2}$

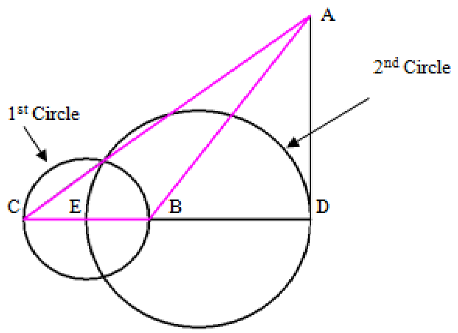

Diagram-1

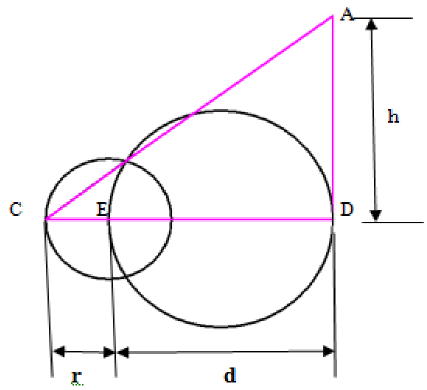

Diagram-2

Corresponding Author: Rajbali Yadav*
Ordnance Factory Khamaria, Jabalpur (M.P,) - 482005, India.

Note:
C, initial point of center line CD 1^{st} circle
D, End point of center line CD 2^{nd} circle
B, Last point of 1^{st} circle
E is mid of BC
AB = c, BC =a, AC = b
Angle ABC = β, Angle BAC = α, Angle ACB = C

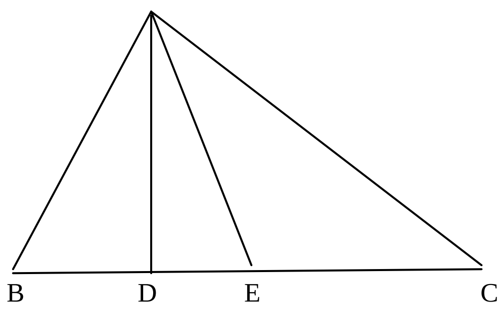

Diagram-3

Steps involved to prove above theorem:
For max. area of Δ ADE in ΔABC

1^{st} $a = \dfrac{b^2-c^2}{\sqrt{(b^2+c^2)}}$

2^{nd} $\alpha + 2\beta = 270^\circ$ Or $\beta - C = 90^\circ$

3^{rd} BC $\sin(\alpha+2\beta) + 2DE \sin\alpha = 0$

4^{th} $AB^2 + AC^2 = (2DE)^2$

And finally

$DE^2 - CE^2 = AD^2$ Or $d^2 - r^2 = h^2$

2.1 Proof.
in Δ ABC (Diagram 3)
E is mid of BC
D is perpendicular at at BC
BD = (a/2-DE), DC = (a/2+DE)

In Δ ADB
$AD^2 = c^2 - (a/2-DE)^2$ ……… 1

In Δ ADC
$AD^2 = b^2 - (a/2+DE)^2$………2

After solving 1 &2
$c^2 - (a/2-DE)^2 = b^2 - (a/2+DE)^2$
$c^2 - (a/2)^2 - DE^2 + DE\times a = b^2 - (a/2)^2 - DE^2 - DE\times a$
$c^2 + DE\times a = b^2 - DE\times a$

$$DE = \frac{b^2-c^2}{2a}$$

Put the value of DE in equation 2

$$AD^2 = b^2 - \left[\frac{a}{2} - \frac{(b^2-c^2)}{2a}\right]^2 \text{ OR } AD = \frac{\sqrt{(2ab)^2 - (a^2+b^2-c^2)^2}}{2a}$$

Area of Δ ADE (A1) = ½ AD× DE

$$= \frac{1}{2}\times\frac{\sqrt{(2ab)^2 - (a^2+b^2-c^2)^2}}{2a}\times\frac{(b^2-c^2)}{2a}$$

$$A1 = \frac{(b^2-c^2)}{8a^2}\times\sqrt{\left[(2ab)^2 - (a^2+b^2-c^2)^2\right]} \qquad (1)$$

$$A1^2 = \frac{(b^2-c^2)}{64a^4}\times\sqrt{\left[(2ab)^2 - (a^2+b^2-c^2)^2\right]}$$

For being area of Δ ADE Max. Or Mini.
dA1/da = dA1/db =dA1/dc = 0

Take dA1/da =0

$$2A1\frac{dA1}{da} = \frac{1}{64}\frac{d}{da}\frac{(b^2-c^2)^2}{a^4}\times\left[(2ab)^2 - (a^2+b^2-c^2)^2\right]$$

$$128A1\ \frac{dA1}{da} = (b^2-c^2)^2\ \frac{d}{da}\frac{1}{a^4}\times\left[(2ab)^2 - (a^2+b^2-c^2)^2\right]$$

$$= (b^2-c^2)^2\ \frac{d}{da}\ [4b^2/a^2 - (1 + b^4/a^4 + 2b^2/a^2 + c^4/a^4 - 2c^2/a^2 - 2b^2c^2/a^4]$$

$d/da\ [4b^2/a^2 - (1 + b^4/a^4 + 2b^2/a^2 + c^4/a^4 - 2c^2/a^2 - 2b^2c^2/a^4] = 0$ Put $dA1/da = 0$
$-8b^2/a^3 + 4b^4/a^5 + 4b^2/a^3 + 4c^4/a^5 - 4c^2/a^3 - 8b^2c^2/a^5 = 0$
$-2b^2 + b^4/a^2 + b^2 + c^4/a^2 - c^2 - 2b^2c^2/a^2 = 0$
$b^4/a^2 + c^4/a^2 - 2b^2c^2/a^2 = b^2 + c^2$
$[(b^2 - c^2)/a]^2 = b^2 + c^2$
$\mathbf{a = b^2 - c^2/\sqrt{(b^2 + c^2)}}$ (2)
For Max. area of Δ ADE

2.2 According to Cosine rule
$\text{Cos}\alpha = (b^2 + c^2 - a^2)/2bc$ (3)
Put the value of **a** in equation (3)

$$\text{Cos}\alpha = \frac{b^2 + c^2 - \dfrac{(b^2 - c^2)^2}{(b^2 + c^2)}}{2bc} = \frac{b^4 + c^4 + 2b^2c^2 - b^4 - c^4 + 2b^2c^2}{2bc(b^2 + c^2)} = \frac{4b^2c^2}{2bc(b^2 + c^2)}$$

$\text{Cos}\alpha = 2bc/(b^2 + c^2)$

Add 1 both side
$\text{Cos}\alpha + 1 = (2bc + b^2 + c^2)/(b^2 + c^2)$
$\text{Cos}\alpha + 1 = (b + c)^2/(b^2 + c^2)$

Equation (2) can also be written as
$[a/(b - c)]^2 = (b + c)^2/(b^2 + c^2)$

Hence
$\text{Cos}\alpha + 1 = [a/(b - c)]^2$
$a/(b - c) = \sqrt{(1 + \text{Cos}\alpha)}$
$= \sqrt{(1 + 2\,\text{Cos}^2\alpha/2 - 1)}$
$= \sqrt{2}\,\text{Cos}\alpha/2$ or $\dfrac{(b - c)}{a} = \dfrac{1}{\sqrt{2}\,\text{Cos}\,\alpha/2}$

Using Sine rule

$$\frac{(\text{Sin}\beta - \text{SinC})}{\text{Sin}\alpha} = \frac{1}{\sqrt{2}\,\text{Cos}\,\alpha/2} \qquad C = 180^\circ - (\alpha + \beta)$$

$$\text{Sin}\beta - \text{Sin}\left[180 - (\alpha + \beta)\right] = \frac{\text{Sin}\alpha}{\sqrt{2}\,\text{Cos}\,\alpha/2}$$

$$\text{Sin}\beta - \text{Sin}(\alpha + \beta) = \frac{2\text{Sin}(\alpha/2)\text{Cos}(\alpha/2)}{\sqrt{2}\,\text{Cos}\,\alpha/2}$$

$\text{Sin}\beta - \text{Sin}(\alpha + \beta) = \sqrt{2}\,\text{Sin}\,\alpha/2$
$2\text{Cos}[(\beta+\alpha+\beta)/2]\text{Sin}[(\beta-\alpha-\beta)/2] = \sqrt{2}\,\text{Sin}(\alpha/2)$

Using (SinC –SinD) formula

$$-2\text{Cos}\frac{(\alpha + 2\beta)}{2} = \sqrt{2}$$

$$\text{Cos}\frac{(\alpha + 2\beta)}{2} = -1/\sqrt{2} = \text{Cos}(180^\circ - 45^\circ)$$

$\alpha + 2\beta = 270^\circ$
$\alpha + \beta + \beta = 270^\circ$
$180 - C + \beta = 270^\circ$ or $\mathbf{\beta - C = 90^\circ}$

This shows that for maximum area of Δ ADE, difference of two angles in Δ ABC will be 90˚ and for constructing such triangle when between two circles of different diameters (as shown in diagram1) larger diameter circle pass through centre of smaller dia. circle, line drawn from intersection of circles and initial point of center line of first circle and a perpendicular line drawn from end point of second circle center line, which cut diagonal at some point, through this point line drawn to last point of first circle, an obtuse triangle with difference of two angle 90 is obtain. This means for maximum area of Δ ADE all relation found in Δ ABC (diagram3), will also be true for Obtuse Δ ABC with difference of two angle 90˚, constructed using two circles as shown in diagram1.

In diagram (3)

E is mid of BC line

$$\text{Area of } \Delta \text{ ADE } (A1) = \frac{(b^2 - c^2)}{8a^2} \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}$$

And

$$\text{Area of } \Delta \text{ ABC } (A) - \frac{1}{4} \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}$$

Dividing A1/A

$$A1/A = \frac{(b^2 - c^2)/8a^2 \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}}{1/4 \times \sqrt{\left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right]}}$$

$A1/A = (b^2 - c^2)/2a^2$ (4)

According to diagram no.3

$$A1 = \frac{1}{2} \times DE \times AD$$

$$A = \frac{1}{2} \times BC \times AD$$

Hence A1/A =DE/BC,

Put the value of A1/A in equation (4)

$DE/BC = (b^2 - c^2)/2a^2$ (5)

Using Sine rule

$$\frac{2DE}{BC} = \frac{\sin^2\beta - \sin^2 C}{\sin^2\alpha}$$

$$= \frac{(\sin\beta + \sin C)\ (\sin\beta - \sin C)}{\sin^2\alpha}$$

$$= \frac{[\sin\beta + \sin(180^\circ - \alpha + \beta)][\ (\sin\beta - \sin(180^\circ - \alpha + \beta)]}{\sin^2\alpha} \quad C = 180^\circ - (\alpha + \beta)$$

$$= \frac{[\sin\beta + \sin(\alpha + \beta)][\ \sin\beta - \sin(\alpha + \beta)]}{\sin^2\alpha}$$

$$= \frac{2\sin\left[(\alpha + 2\beta)/2\right]\cos(\alpha/2) \times\ - 2\cos\left[(\alpha + 2\beta)/2\right]\sin(\alpha/2)}{\sin^2\alpha} \text{ use}$$

$\sin C + \sin D$ and $\sin C - \sin D$ formula

$$= \frac{-2\sin\left[(\alpha + 2\beta)/2\right]\ \cos\left[(\alpha + 2\beta)/2\right] \times 2\sin(\alpha/2)\cos(\alpha/2)}{\sin^2\alpha}$$

$= -\sin[(\alpha + 2\beta) \times \sin\alpha \times /\sin^2\alpha$

$2DE/BC = -\sin[(\alpha + 2\beta)/\sin\alpha$

$2DE \sin\alpha = - BC \sin[(\alpha + 2\beta)$

$$\mathbf{BC\ \sin[(\alpha + 2\beta) + 2DE\ \sin\alpha = 0}$$

By previous relation it is proved that for maximum area of Δ ADE

$\alpha + 2\beta = 270^\circ$

Put this value in above equation

$BC \sin 270^\circ + 2DE \sin\alpha = 0$, $- BC + 2DE \sin\alpha = 0$

$\sin\alpha = BC/2DE$ as $BC = a$ and $DE = a1$ So, $\sin\alpha = a/2a1$ (6)

According to equation (5) $2DE/BC = (b^2 - c^2)/a^2$

$\sin\alpha = BC/2DE$

$= 1/(b^2 - c^2)/a^2$

$\sin\alpha = a^2/(b^2 - c^2)$ (7)

According to equation no. 2

$$a^2 = \frac{(b^2 - c^2)^2}{b^2 + c^2}$$

put the value of a^2 in equation no. 7

$$\operatorname{Sin}\alpha = \frac{\dfrac{\left(b^2-c^2\right)^2}{\left(b^2+c^2\right)}}{\dfrac{\left(b^2-c^2\right)}{1}}$$

$$\operatorname{Sin}\alpha = \frac{b^2-c^2}{b^2+c^2} \tag{8}$$

According to equation (6) $\operatorname{Sin}\alpha = a/2a1$

Put the value of a from equation no.2 and Sinα from equation (8) in equation (6)

$$\frac{\left(b^2-c^2\right)}{\left(b^2+c^2\right)} = \frac{\dfrac{b^2-c^2}{\sqrt{\left(b^2+c^2\right)}}}{2a1}$$

$$\frac{\left(b^2-c^2\right)}{\left(b^2+c^2\right)} = \frac{b^2-c^2}{2a1\times\sqrt{\left(b^2+c^2\right)}}$$

$$2a1 = \frac{\left(b^2+c^2\right)}{\sqrt{\left(b^2+c^2\right)}}$$

$(2a1)^2 = b^2 + c^2$ OR $(2DE)^2 = AC^2 + AB^2$

As , $AB = c$ $AC = b$ & $DE = a1$

$$\mathbf{AC^2 + AB^2 = (2DE)^2} \tag{9}$$

For maximum area of Δ ADE

Finally it will be proved that in diagram 1 & 2

$DE^2 - CE^2 = AD^2$

Or $d^2 - r^2 = h^2$

In Δ ABD

$AB^2 = AD^2 + BD^2$

$= AD^2 + (DE+BE)^2$

$= AD^2 + DE^2 + BE^2 + 2DE\times BE$

$$AB^2 = AD^2 + DE^2 + BE^2 + DE\times BC \tag{10}$$

In diagram 1 & 2 $BE = BC/2$

In Δ ADC

$AC^2 = AD^2 + DC^2$

$= AD^2 + (CE-DE)^2$

$= AD^2 + CE^2 + DE^2 - 2CE\times DE$

$$= AD^2 + CE^2 + DE^2 - BC\times DE \tag{11}$$

In diagram 1 & 2 $CE = BC/2$

ADD eqn. (10) & (11)

$$AB^2 + AC^2 = 2AD^2 + 2DE^2 + BC^2/2 \tag{12}$$

According to eqn. (9) $AC^2 + AB^2 = (2DE)^2$

Put the value of $AC^2 + AB^2$ in eqn (12)

$4DE^2 = 2AD^2 + 2DE^2 + BC^2/2$

$2DE^2 = 2AD^{2\,2} + BC^2/2$

$4DE^2 = 4AD^2 + BC^2$ $\qquad$ $BC = 2CE$

$4DE^2 = 4AD^2 + 4CE^2$

$DE^2 = AD^2 + CE^2$

Or

$DE^2 - CE^2 = AD^2$

In diagram 1 & 2

$DE = d$

$CE = r$

$AD = h$,

hence

$d^2 - r^2 = h^2$

3. CONCLUSION

This theory can help to understand right angled triangle after knowing one set of constraint i.e diameter of circles, in which one pass through centre of other. Height and Hypotenuse of right triangle can be find easily.

4. REFERENCE

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2. https://www.splashlearn.com

Source of support: Nil, Conflict of interest: None Declared.