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COMMON FIXED POINT THEOREMS IN G-METRIC SPACES

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ABSTRACT

The intent of this paper is to establish the common fixed point theorems through semi-compatibility in G-metric spaces for six self maps. In our theorems the completeness of the space X and the continuity of maps is replaced with a set of four alternative conditions for functions satisfying implicit relations.

1. INTRODUCTION AND PRELIMINARIES

Mustafa and Sims [9] introduced the concept of G-metric spaces in the year 2004 as a generalization of the metric spaces. In this type of spaces a non-negative real number is assigned to every triplet of elements. In [11] Banach contraction mapping principle was established and a fixed point results have been proved. After that several fixed point results have been proved in these spaces. Some of these works may be noted in [2–4, 10–13] and [14]. Several other studies relevant to metric spaces are being extended to G-metric spaces. For instances we may note that a best approximation result in these type of spaces established by Nezhad and Mazaheri in [15], the concept of w-distance, which is relevant to minimization problem in metric spaces [8], has been extended to G-metric spaces by Saadati et al. [23]. Also one can note that fixed point results in G-metric spaces have been applied to proving the existence of solutions for a class of integral equations [25].

Now we give some preliminaries and basic definitions which are used throughout the paper.

Definition 1.1: G-metric Space

Let X be a non empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following:

1. $G(x, y, z) = 0 \ if \ x = y = z$
2. $0 < G(x, x, y), for \ all \ x, y \in X \ with \ x \neq y$
3. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \ (symmetry \ in \ all \ three \ variable)$
4. $G(x, x, y) \leq G(x, y, z), for \ all \ x, y, z \in X \ with \ z \neq y$
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \ for \ all \ x, y, z, a \in X \ (rectangle \ inequality)$ then the function is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is a G-metric space

Definition1.2: [10] Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence of points in X We say that $\{x_n\}$ is G-convergent to x if $G(x, x_n, x_m) = 0$, that is, for each $\in > 0$ there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$. We call that x is the limit of the sequence and we write $x_{n \to} x$ or $\lim_{n\to\infty} x_n = x$

It has been shown in [10] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

Proposition 1.1: [10] Let (X, G) be a G-metric space then the following are equivalent:

1. $\{x_n\}$ is convergent to x,
2. $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
3. $G(x_n, x, x) \to 0$ as $n \to \infty$,
4. $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

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Definition 1.3: [10] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is said to be a G-Cauchy sequence for each $\varepsilon > 0$ there exists a positive integer N such that $G(x_{n,}, x_m, x_l) < \varepsilon$ for all $l,m,n \geq N$.

Proposition 1.2: [10] Let (X,G) be a G-metric space then the following are equivalent:

1. the sequence $\{x_n\}$ is G-Cauchy,
2. for each $\varepsilon > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) >\in$ for all $l,m,n \geq N.$

Proposition 1.3: [10] Let (X,G) be a G-metric space then the function $G(x,y,z)$ is jointly continuous in all three variable

Definition 1.4: [10] A G-metric space (X,G) is called a symmetric G-metric space if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$

Proposition 1.4: [10] Every G-metric (X,G) defines a metric space (X,d_G) by

1. $d_G(x,y) = G(x,y,y) + G(y,x,x)$ for all $x,y \in X$

   If (X,G) is a symmetric G-metric space, then
2. $d_G(x,y) = 2G(x,y,y)$ for all $x,y \in X$

However, if (X,G) is not a symmetric G-metric space, then it follows from the G-metric properties that

3. $\frac{3}{2}G(x,y,y) \leq d_G(x,y) = 3G(x,y,y)$ for all $x,y \in X$.

Proposition 1.5: [10] A G-metric space (X,G)) is G-complete if and only if (X,d_G) is a complete metric space.

Proposition 1.6: [10] Let (X,G) be a G-metric space. Then, for any $x,y,z,a \in X$ it follows that

1. if $G(x,y,z) = 0$ then $x = y = z$
2. $G(x,y,z) \leq G(x,x,y) + (x,x,z)$,
3. $G(x,y,y) \leq 2G(y,x,x)$,
4. $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,
5. $G(x,y,z) \leq \frac{2}{3}(G(x,a,a) + G(y,a,a) + G(z,a,a))$.

Next we give two examples of non-symmetric G-metric spaces.

Example 1.1: [10] Let $X = (a,b)$ let $G(a,a,a) = G(b,b,b) = 0,\ G(a,a,b) = 1, G(a,b,b) = 2$ and extend G to all of $X \times X \times X$ by symmetry in the variables. Then G is a G-metric. It is non-symmetric since $G(a,b,b) \neq G(a,a,b)$

Definition 1.5: [6] Let f and g be two self mappings on a metric space (X,d). The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$

In particular, now we look in the context of common fixed point theorem in G-metric spaces. Start with the following contraction condition:

Definition 1.6: Let (X,G) be a G-metric space and $T : X \to X$ be a self mapping on (X,G). Now T is said to be a contraction if

$$G(Tx,Ty,Tz) \leq \alpha G(x,y,z) \tag{1.1}$$

$x,y,z \in X$ For all where $0 \leq \alpha \leq 1$

It is clear that every self mapping $T : X \to X$ satisfying condition (1.1) is continuous. Now we focus to generalize the condition (1.1) for a pair of self mappings S and T on X in the following way:

$$G(Sx, Sy, Sz) \le \alpha G(Tx, Ty, Tz) \tag{1.2}$$

$x, y, z \in X$ For all where $0 \le \alpha \le 1$

Let $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to} gx_n = z$ for some $z \in X$. To prove the existence of common fixed points for mappings satisfying inequality (1.2), it is necessary to add additional assumptions of the following type:

1. construction of the sequence $\{x_n\}$
2. some mechanism to obtain common fixed point and this problem was overcomed by imposing additional hypothesis of commutative pair {S,T}

Most of the theorems followed a similar pattern of mappings:

1. contraction,
2. continuity of functions (either one or both) and
3. commuting pair of mappings were given.

2. MAIN RESULTS

Now we come to our main result for a pair of compatible maps.

Theorem 2.1: Let (X, G) be a complete G-metric space and f, g be two self mappings on (X, G) satisfies the following conditions:

1. $f(X) \subseteq g(X)$, (2.1)
2. f or g is continuous, (2.2)
3. $G(fx, fy, fz) \le \alpha G(fx, gy, gz) + \beta G(gx, fy, gz) + \gamma G(gx, gy, fy)$, (2.3)

For every $x, y, z \in X$ and $\alpha, \beta, \gamma \ge 0$ with $0 \le \alpha + 3\beta + 3\gamma \le 1$. Then f and g have a unique common fixed point in X provided f and g are compatible maps.

Proof: Let x_0 be an arbitrary point in X. By (2.1), one can choose a point $x_1 \in X$ such that $fx_0 = gx_1$. In general one can choose x_{n+1} such that $y_n = fx_n = gx_{n+1}, n = 0,1,2.........$

From (2.3), we have

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) &\le \alpha G(fx_n, gx_{n+1}, gx_{n+1}) + \beta G(gx_n, fx_{n+1}, gx_{n+1}) + \gamma G(gx_n, gx_{n+1}, fx_{n+1}) \\ &= \alpha G(fx_n, fx_n, fx_n) + \beta G(fx_{n-1}, fx_{n+1}, fx_n) + \gamma G(fx_{n-1}, fx_n, fx_{n+1}) \\ &= (\beta + \gamma) G(fx_{n-1}, fx_n, fx_{n+1}). \end{aligned} \tag{2.4}$$

By the rectangular inequality of G-Metric space, we have

$$\begin{aligned} G(fx_{n-1}, fx_n, fx_{n+1}) &\le G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1}) \\ &\le G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_n, fx_{n+1}) \end{aligned}$$

By using proposition (1.6)

From (2.3), we have

$$(1 - 2\beta - 2\gamma) G(fx_n, fx_{n+1}, fx_{n+1}) \le (\beta + \gamma) G(fx_{n-1}, fx_n, fx_n),$$

That is, $G(fx_n, fx_{n+}, fx_{n+1}) \le \dfrac{(\beta + \gamma)}{(1 - 2\beta - 2\gamma)} G(fx_{n+1}, fx_n, fx_n)$

That is, $G(fx_n, fx_{n+1}, fx_{n+1}) \le qG(fx_{n-1}, fx_n, fx_n)$ where q = $\dfrac{(\beta + \gamma)}{(1 - 2\beta - 2\gamma)} < 1$.

Continuing in the same way, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q^n G(fx_0, fx_1, fx_1).$$

Therefore, for all $n, m \in N, n < m$, we have by rectangle inequality that

$$\begin{aligned} G(y_{n,} y_{m,} y_m) &\le G(y_{n,} y_{n+1,} y_{n+1}) + G(y_{n+1,} y_{n+2,} y_{n+2}) + ... + G(y_{m-1,} y_{m,} y_m) \\ &\le (q^n + q^{n+1} + ... + q^{m-1}) G(y_0, y_1, y_1) \\ &\le \frac{q^n}{1-q} G(y_0, y_1, y_1). \end{aligned}$$

Letting as $n, m \to \infty$, we have $\lim_{n\to\infty} G(y_{n,} y_{m,} y_m) = 0.$. Thus $\{y_n\}$ is a G-Cauchy sequence in X . Since (X,G) is complete G-metric space, therefore, there exists a point $z \in Z$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$ Since the mapping f or g is continuous, for definiteness one can assume that g is continuous, therefore $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gz.$ Further f and g are compatible ,therefore $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, implies $\lim_{n\to\infty} fgx_n = gz.$

Form (2.3), we have

$$G(fgx_n, fx_n, fx_n) \le \alpha G(fgx_n, gx_n, gx_n) + \beta G(ggx_n, fx_n, gx_n) + \gamma G(ggx_n, gx_n, fx_n) .$$

Proceeding limt as $n \to \infty$, we have $gz = z$.

Again from (2.3), we have

$$G(fx_n, fz, fz) \le \alpha G(fx_n, gz, gz) + \beta G(fx_n, fz, gz) + \gamma G(gx_n, gz, fz) .$$

Taking limit $n \to \infty$, we have $z = fz$. Therefore, we have $gz = fz = z$. Thus z is a common fixed point of f and g .

For uniqueness, we assume that $z_1 (\neq z)$ be another common fixed point of f and g . Then $G(z, z_1, z_1) > 0$ and

$$\begin{aligned} G(z, z_1, z_1) &= G(fz, fz_1, fz_1) \\ &\le \alpha G(fz, gz_1, gz_1) + \beta G(gz, fz_1, gz_1) + \gamma G(gz, gz_1, fz_1) \\ &= (\alpha + \beta + \gamma) G(z, z_1, z_1) \\ &< G(z, z_1, z_1), \text{ a contradiction,} \end{aligned}$$

Which demands that $z = z_1$

This completes the proof of the theorem.

Corollary 2.1: Let (X,G) be a complete G-metric space and f, g be two compatible self mappings on (X,G) satisfies(2.1), (2.2) and the following condition:

$G(fx, fy, fz) \le qG(x, y, z)$ for every $x, y, z \in X$ and $0 < q < 1$ Then f and g have a unique common fixed point in X .

Proof: Proof follows easily from above theorem.

Theorem 2.2: Let f and g be weakly compatible self maps of a G-metric space (X,G) satisfying conditions (2.1) and (2.3) and any one of the subspace $f(x)$ or $g(x)$ is complete. Then f and g have a unique common fixed point in X .

Proof: From Theorem 2.1, we conclude that $\{y_n\}$ is a G-Cauchy sequence in X . Since either $f(x)$ or $g(x)$ is complete, for definiteness assume that $g(x)$ is complete subspace of X then the subsequence of $\{y_n\}$ must get a limit in $g(x)$. Call it be z . Let $u \in g^{-1}z$ then $gu = z$ as $\{y_n\}$ is a G-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also convergent implying there by the convergence of subsequence of the convergent sequence. Now we show that $fu = z$.

On setting $x = u, y = x_n$ and $z = x_n$, in (2.3), we have

$$G(fu, fx_n, fx_n) \le \alpha G(fu, gx_n, gx_n) + \beta G(gu, fx_n, gx_n) + \gamma G(gu, gx_n, fx_n).$$

Letting as $n \to \infty$ in the above inequality, we have

$$G(fu, z, z) \le \alpha G(fu, z, z),$$

Which implies that, $fu = z$.

Therefore, $fu = gu = z,$ i.e., u is a coincident point of f and g . Since f and g are weakly compatible, it follows that $fgu = gfu,$ i.e. $fz = gz$

We now show that $fz = z$. Suppose that $fz \neq z$, therefore $G(fz, z, z) > 0$. From (2.3), on setting $x = z, y = u, z = u,$ we have

$$
\begin{aligned}
G(fz, z, z) &= G(fz, fu, fu) \\
&\leq \alpha G(fz, gu, gu) + \beta G(gz, fu, gu) + \gamma G(gz, gu, fu) \\
&= (\alpha + \beta + \gamma)(fz, z, z) \\
&< G(fz, z, z)
\end{aligned}
$$

Which implies that $fu = z$.

Therefore, $fu = gu = z$ i.e. z is common fixed point of f and g Uniqueness follows easily.

3. PROPERTY $(E.A.)$ IN G-METRIC SPACES

Recently, Amari and Moutawakil [1] introduced a generalization of non compatible maps as property $(E.A.)$ in metric spaces as follows:

Definition 3.1: Let A and S be two self-maps of a metric space (X, d) The pair (A, S) is said to satisfy property $(E.A.)$ if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

In [22] property $(E.A.)$ in metric spaces has been used to prove a common fixed point result. In similar mode we use property $(E.A.)$ in G-metric spaces. Now we prove a common fixed point theorem for a pair of weakly compatible maps along with property $(E.A.)$

Theorem 3.1: Let f f and g be two self maps on a G-metric space (X, d) satisfying condition (2.3) and the following conditions:

1. f and g satisfy property $(E.A.)$, (3.1)
2. $g(X)$ is a closed subspace of X. (3.1)

Then f and g have a unique common fixed point in X provided f and g are weakly compatible self maps.

Proof: Since f and g satisfy property $(E.A.)$, therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. Since . $g(X)$ is a closed subspace of X, therefore every convergent sequence of points of $g(X)$ has a limit point in $g(X)$. Therefore, $\lim_{n\to\infty} fx_n = uuga = \lim_{n\to\infty} gx_n$ for some $a \in X$. This implies that $u = ga \in g(X)$

Now from (2.3), we have

$$G(fa, fx_n, fx_n) \leq \alpha G(fa, gx_n, gx_n) + \beta G(ga, fx_n, ga) + \gamma G(ga, gx_n, fx_n).$$

Letting $n \to \infty$ and using 0 · $0 \leq \alpha + 3\beta + 3\gamma \leq 1$. we have $u = fa$. This implies $u = ga = fa$. Thus a is the coincidence point of f and g. Since f and g are weakly compatible, therefore, $fu = fga = gfa = gu$.

Again from (2.3), we have

$$G(fu, fa, fa) \leq \alpha G(fu, ga, ga) + \beta G(gu, fa, ga) + \gamma G(gu, ga, fa).$$

since $0 \leq \alpha + 3\beta + 3\gamma \leq 1$, above inequality implies that $u = fa$. Hence u is common fixed point of f and g. Uniqueness follows easily.

Corollary 3.1: Let (X, G) be a complete G-metric space and f, g be two self mappings on (X, G) satisfying (3.1),(3.2) and the following condition:

$G(fx, fy, fz) \leq qG(gx, gy, gz)$ for every $x, y, z \in X$ and $0 < q < 1$. Then f and g have a unique common fixed point in X provided f and g are weakly compatible self maps.

Proof: Proof follows easily from above Theorem 3.1.

CONCLUSIONS

Our results involve the followings:

1. to relax the continuity requirement of maps completely,
2. to minimize the commutativity requirement of the maps to the point of coincidence,
3. to weaken the completeness requirement of the space,
4. Property $(E.A.)$ buys containment of ranges without any continuity requirement to the points of coincidence.

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