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MULTI-OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEM BASED ON FUZZY GOAL PROGRAMMING

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ABSTRACT

This paper presents fuzzy goal programming approach for solving multi-objective linear fractional programming problem with multiple linear fractional objective functions. To construct the fractional membership functions, optimal solution of the objective functions are determined subject to the system constraints. The fractional membership functions are transformed into linear membership functions by first order Taylor series approximation. Then fuzzy goal programming approach is used to obtain highest degree of each of membership goals by minimizing negative deviational variables. A numerical example is solved to demonstrate the efficiency of the proposed approach.

Keywords: *Fractional programming; Fuzzy goal programming; Multi-objective linear fractional programming; Fuzzy mathematical programming; Taylor series.*

AMS subject Classification: *90C29; 90C32; 90C70.*

1. INTRODUCTION:

In this paper, we have considered multi-objective linear fractional programming problem (MOLFPP) consisting of multiple conflicting objectives. The objective functions are linear fractional in nature and the system constraints are linear functions.

MOLFPP is a special case of fractional programming (FP) [4]. Owing to computational difficulties, MOLFPP is transformed into equivalent single objective linear FP problem to solve the problem by using the variable transformation method due to Charnes and Cooper [3] or by adopting the updating objective function method by Bitran and Noveas [1]. Kornbluth and Steuer [7] studied goal programming approach to MOLFPP. To overcome the computational difficulties of using FP approaches to MOLFPPs, fuzzy set theory was incorporated. Luhandjula [8] proposed a linguistic variables approach to MOLFPP in 1984. Sakawa and Kato [13] presented interactive fuzzy programming approach to MOLFPPs with block angular structure consisting of fuzzy numbers. Dutta et al. [5] modified the linguistic approach of Luhandjula [8] and presented a fuzzy set theoretic approach to MOLFPP. Chakraborty and Gupta [2] studied fuzzy mathematical programming approach to MOLFPP by variable transformations. Fuzzy goal programming (FGP) procedure for MOLFPP was studied by Pal et al. [9]. They formulate fuzzy model at first involving fractional membership goals. Then, they transform fractional membership goals into linear membership goals by using the method of variable change as suggested by Kornbluth and Steuer [7]. So, computational burden is also inherently involved in the solution process. Guzel and Sivri [6] presented Taylor series solution procedure to deal with MOLFPP. Toksarı [14] developed Taylor series approach for solving MOLFPP in fuzzy environment.

In this study, we have transformed MOLFPP into equivalent multi-objective linear programming problem by using first order Taylor series. Then FGP approach is formulated for achieving highest degree of each of membership goals by minimizing negative deviational variables. To show the effectiveness of the proposed approach, we solve a MOLFPP and compare the results with the results obtained by Chakraborty and Gupta [2].

Our main results are as follows: (i) an alternative FGP approach for solving MOLFPP is presented. (ii) We formulate

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the fractional membership functions by finding individual best solution of the objective functions. Then we transform the fractional membership functions into linear membership functions by using first order Taylor series approximation at the best solution point. (iii) FGP procedure due to Pramanik and Dey [10] and Pramanik et al. [11] are used to solve to the transformed MOLFPP.

Rest of the paper is organized in the following way. Section 2 presents the formulation of MOLFPP. Section 3 discusses fuzzy programming formulation including formulation of membership functions of MOLFPP and linearization of membership functions by first order Taylor series. In subsection 3.1, FGP formulation to MOLFPP is presented. Section 4 presents the use of Euclidean distance function for comparison. Section 5 is devoted to present FGP algorithm for MOLFPP. In section 6, we provide a simple numerical example that helps to understand the proposed FGP approach. Finally, section 7 presents the concluding remarks.

2. FORMULATION OF MOLFPP:

The general formulation of MOLFPP can be written as:

$$\max F_t(\bar{x}) = \frac{\bar{c}_t^T \bar{x} + \alpha_t}{\bar{d}_t^T + \beta_t} \quad (t = 1, 2, \ldots, p) \tag{1}$$

subject to

$$\bar{x} \in S = \{\bar{x} \in \bar{R}^n \mid \bar{A}\bar{x}(\le, =, \ge)\bar{b}, \bar{x} \ge \bar{0}\} \tag{2}$$

Here, $\bar{c}_t^T, \bar{d}_t^T \in \bar{R}^n$ (t = 1, 2, …, p) and α_t, β_t are constants. $\bar{A} \in \bar{R}^{m\times n}$, $\bar{b} \in \bar{R}^m$. S is assumed to be non empty, convex and compact in $\bar{R}^n$ and $\{\bar{d}_t^T \bar{x} + \beta_t \mid \bar{x} \in S\} > \bar{0}$ (t = 1, 2… p). The symbol 'T' denotes transposition.

3. FUZZY PROGRAMMING FORMULATION OF MOLFPP:

To formulate the fuzzy programming model of a MOLFPP, the objective functions $F_t(\bar{x})$ (t = 1, 2… p) would be transformed into fuzzy goals by introducing an imprecise aspiration level to each of the objective.

The optimal solution of each objective function $F_t(\bar{x})$ (t = 1, 2… p) when calculated in isolation would be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal.

Let, $F_t^B = \max_{x\in S} F_t(\bar{x})$, $F_t^W = \min_{x\in S} F_t(\bar{x})$ (t = 1, 2, …, p). F_t^W and F_t^B are the best and the worst solutions of the t-th objective functions respectively. Then the fuzzy goal takes the form $F_t(\bar{x}) \gtrsim F_t^B$ (t = 1, 2, …, p)

The membership function associated with t-th objective goal can be written as:

$$\mu_t(\bar{x}) = \begin{cases} 1, & \text{if } F_t(\bar{x}) \ge F_t^B \\ \dfrac{F_t(\bar{x}) - F_t^W}{F_t^B - F_t^W}, & \text{if } F_t^W \le F_t(\bar{x}) \le F_t^B, \quad (t = 1, 2, \ldots, p) \\ 0, & \text{if } F_t(\bar{x}) \le F_t^W \end{cases} \tag{3}$$

Here, F_t^W and F_t^B are respectively the lower and the upper tolerance limits of the t-th fuzzy objective goal.

Then, the problem (1) reduces to the following problem:

$$\max \mu_t(\bar{x}) \quad (t = 1, 2, \ldots, p) \tag{4}$$

subject to

$$\bar{x} \in S = \{\bar{x} \in \bar{R}^n \mid \bar{A}\bar{x}(\le, =, \ge)\bar{b}, \bar{x} \ge \bar{0}\}.$$

Let, $\bar{x}_t^* = (x_{t1}^*, x_{t2}^*, ..., x_{tn}^*)$ be the individual best solution of the problem (4) subject to the system constraints, n = total number of variables of the system. Next, we transform the fractional membership function $\mu_t(\bar{x})$ (t = 1, 2… p) into equivalent linear membership function $\tilde{\mu}_t(\bar{x})$ (t = 1, 2... p) by first order Taylor series at the individual best solution point $\bar{x}_t^* = (x_{t1}^*, x_{t2}^*, ..., x_{tn}^*)$. The transformed linear membership function can be written as:

$$\mu_t(\bar{x}) \cong \mu_t(\bar{x}_t^*) + [(x_1 - x_{t1}^*)\frac{\partial}{\partial x_1}\mu_t(\bar{x}_t^*) + (x_2 - x_{t2}^*)\frac{\partial}{\partial x_2}\mu_t(\bar{x}_t^*) + ... + (x_n - x_{tn}^*)\frac{\partial}{\partial x_n}\mu_t(\bar{x}_t^*)]$$

$$= \tilde{\mu}_t(\bar{x}) \quad (5)$$

3.1 FORMULATION OF FGP MODEL OF MOLFPP:

The problem (4) reduces to the following problem:

$$\max \tilde{\mu}_t \ (t = 1, 2, \ldots, p) \quad (6)$$

subject to

$$\bar{x} \in S = \{\bar{x} \in \bar{R}^n \mid \bar{A}\bar{x}(\le,=,\ge)\bar{b}, \bar{x} \ge \bar{0}\}.$$

Since the maximum value of a membership function is one, the flexible membership goals having the aspired level one can be defined as:

$$\tilde{\mu}_t + d_t^- - d_t^+ = 1 (t = 1, 2, .., p) \quad (7)$$

Here, d_t^- (≥ 0) represents the negative deviational variable and d_t^+ (≥ 0) represents positive deviational variable. It may be noted that any over deviation from a fuzzy goal indicates full achievement of the membership goal [12]. Since the maximum value of a membership goal is one, positive deviation is not possible [10] and only the negative deviational variables are required to minimize. Then, following Pramanik and Dey [10] and Pramanik et al. [11], two FGP models are formulated for solving MOLFPP as follows:

model (I): minimize λ (8)

subject to

$$\mu_t(\bar{x}_t^*) + [(x_1 - x_{t1}^*)\frac{\partial}{\partial x_1}\mu_t(\bar{x}_t^*) + (x_2 - x_{t2}^*)\frac{\partial}{\partial x_2}\mu_t(\bar{x}_t^*) + ... + (x_n - x_{tn}^*)\frac{\partial}{\partial x_n}\mu_t(\bar{x}_t^*)] + d_t^- = 1, (t = 1, 2, \ldots, p)$$

$$\bar{x} \in S = \{\bar{x} \in \bar{R}^n \mid \bar{A}\bar{x}(\le,=,\ge)\bar{b}, \bar{x} \ge \bar{0}\},$$

$$\lambda \ge d_t^-, (t = 1, 2\ldots p)$$

$$d_t^- \ge 0 \ (t = 1, 2\ldots p).$$

model (II): minimize $\gamma = \sum_{t=1}^{p} w_t d_t^-$ (9)

$$\mu_t(\bar{x}_t^*) + [(x_1 - x_{t1}^*)\frac{\partial}{\partial x_1}\mu_t(\bar{x}_t^*) + (x_2 - x_{t2}^*)\frac{\partial}{\partial x_2}\mu_t(\bar{x}_t^*) + ... + (x_n - x_{tn}^*)\frac{\partial}{\partial x_n}\mu_t(\bar{x}_t^*)] + d_t^- = 1, (t = 1, 2\ldots p)$$

$$\bar{x} \in S = \{\bar{x} \in \bar{R}^n \mid \bar{A}\bar{x}(\le,=,\ge)\bar{b}, \bar{x} \ge \bar{0}\},$$

$$d_t^- \ge 0 \ (t = 1, 2\ldots p).$$

The weight w_t corresponding to the deviational variable d_t^- is determined by the decision maker (DM). If the normalized weight is considered by the DM, then $\sum_{t=1}^{p} w_t = 1$ with $w_t = 1/p$ (t = 1, 2… p).

4. USE OF DISTANCE FUNCTION TO COMPARE WITH OTHER APPROACH:

Yu [15] studied the concept of ideal point and the use of distance function for group decision analysis. In the FGP formulation, since the aspired level of each of the membership goals is one, the point comprising of highest membership value of each of the goals would represents the ideal point. The Euclidean distance function can be defined as:

$$D_2 = [\sum_{p=1}^{P} [1 - \mu_p(\bar{x})]^2]^{1/2} \tag{10}$$

Here, $\mu_p(\bar{x})$ represents the achieved membership value of the p -th objective goal. The solution for which D_2 is the minimum would be the optimal compromise solution.

5. THE FGP ALGORITHM FOR MOLFPP:

By the following steps, we present the proposed FGP algorithm for solving MOLFPP.

Step 1: Determine the best and the worst solutions of each objective function $F_t(\bar{x})$ (t = 1, 2… p) subject to the system constraints.

Step 2: Considering the best and the worst solutions as the upper and the lower tolerance limits of the fuzzy objective goals, formulate the membership function $\mu_t(\bar{x})$ (t = 1, 2… p) of t-th objective function as given by (3).

Step 3: Calculate the best solution of the membership function $\mu_t(\bar{x})$ (t = 1, 2… p) subject to the system constraints.

Step 4: Transform the fractional membership function $\mu_t(\bar{x})$ (t = 1, 2… p) into equivalent linear membership function $\tilde{\mu}_t$ (t = 1, 2… p) at the best solution point by first order Taylor polynomial series as given by (4).

Step 5: Formulate the FGP models (8) & (9).

Step 6: Solve the problems (8) & (9).

Step 7: Calculate Euclidean distances for the FGP models (8) & (9).

Step 8: Identify the optimal compromise solution for which the Euclidean distance D_2 is minimum.

Step 9: End.

6. NUMERICAL EXAMPLE:

Consider the following MOLFPP [2] with three objective functions studied by Chakraborty and Gupta:

$$\max \left(Z_1(\bar{x}) = \frac{(-3x_1 + 2x_2)}{(x_1 + x_2 + 3)}, Z_2(x) = \frac{(7x_1 + 2x_2)}{(5x_1 + 2x_2 + 1)}, Z_3(x) = \frac{(x_1 + 4x_2)}{(2x_1 + 3x_2 + 2)} \right) \tag{11}$$

subject to

$x_1 - x_2 \geq 1$,

$2x_1 + 3x_2 \leq 15$,

$x_1 + 9x_2 \geq 9$,

$x_1 \geq 3$,

$x_1, x_2 \geq 0$.

The best solutions subject to the system constraints are Z_1^B = -0.609 at (3.6, 2.6); Z_2^B = 1.358 at (7.2, 0.2); Z_3^B = 0.824 at (3.6, 2.6) and the worst solution Z_1^W = -2.038 at (7.2, 0.2); Z_2^W = 1.25 at (3, 2); Z_3^W = 0.471 at (7.2, 0.2).

Then, the fuzzy goals appear as $Z_1(\bar{x}) \gtrsim -0.609$, $Z_2(\bar{x}) \gtrsim 1.358$, $Z_3(\bar{x}) \gtrsim 0.824$.

We formulate the fractional membership functions as follows:

$$\mu_1(Z_1(\bar{x})) = \frac{Z_1(\bar{x}) + 2.038}{(-0.609 + 2.038)} = \frac{\frac{(-3x_1 + 2x_2)}{(x_1 + x_2 + 3)} + 2.038}{(-0.609 + 2.038)},$$

$$\mu_2(Z_2(\bar{x})) = \frac{Z_2(\bar{x}) - 1.25}{(1.358 - 1.25)} = \frac{\frac{(7x_1 + 2x_2)}{(5x_1 + 2x_2 + 1)} - 1.25}{(1.358 - 1.25)},$$

$$\mu_3(Z_3(\bar{x})) = \frac{Z_3(\bar{x}) - 0.471}{(0.824 - 0.471)} = \frac{\frac{(x_1 + 4x_2)}{(2x_1 + 3x_2 + 2)} - 0.471}{(0.824 - 0.471)}$$

The membership function $\mu_1(Z_1(\bar{x}))$ is maximal at the point (3.6, 2.6), the membership function $\mu_2(Z_2(\bar{x}))$ is maximal at the point (7.2, 0.2) and the membership function $\mu_3(Z_3(\bar{x}))$ is maximal at the point (3.6, 2.6).

Then, the fractional membership functions are transformed into linear at the best solution point by first order Taylor polynomial series as follows:

$$\tilde{\mu}_1(Z_1(\bar{x})) = \mu_1(Z_1(3.6, 2.6)) + (x_1 - 3.6)\frac{\partial}{\partial x_1}\mu_1(Z_1(3.6,2.6)) + (x_2 - 2.6)\frac{\partial}{\partial x_2}\mu_1(Z_1(3.6,2.6)),$$

$$\tilde{\mu}_1(Z_1(\bar{x})) = 1 + (x_1 - 3.6) \times (-0.182) + (x_2 - 2.6) \times (0.288),$$

$$\tilde{\mu}_2(Z_2(\bar{x})) = \mu_2(Z_2(7.2, 0.2)) + (x_1 - 7.2)\frac{\partial}{\partial x_1}\mu_2(Z_2(7.2,0.2)) + (x_2 - 0.2)\frac{\partial}{\partial x_2}\mu_2(Z_2(7.2,0.2)),$$

$$\tilde{\mu}_2(Z_2(\bar{x})) = 1 + (x_1 - 7.2) \times (0.052) + (x_2 - 0.2) \times (-0.177),$$

$$\tilde{\mu}_3(Z_3(\bar{x})) = \mu_3(Z_3(3.6, 2.6)) + (x_1 - 3.6)\frac{\partial}{\partial x_1}\mu_3(Z_3(3.6,2.6)) + (x_2 - 2.6)\frac{\partial}{\partial x_2}\mu_3(Z_1(3.6,2.6)),$$

$$\tilde{\mu}_3(Z_3(\bar{x})) = 1 + (x_1 - 3.6) \times (-0.107) + (x_2 - 2.6) \times (0.254).$$

The FGP model (I) can be written as:

minimize λ (12)

subject to

$1 + (x_1 - 3.6) \times (-0.182) + (x_2 - 2.6) \times (0.288) + d_1^- = 1,$

$1 + (x_1 - 7.2) \times (0.052) + (x_2 - 0.2) \times (-0.177) + d_2^- = 1,$

$1 + (x_1 - 3.6) \times (-0.107) + (x_2 - 2.6) \times (0.254) + d_3^- = 1,$

$x_1 - x_2 \geq 1,$

$2x_1 + 3x_2 \leq 15,$

$x_1 + 9x_2 \geq 9,$

$x_1 \geq 3,$

$\lambda \geq d_i^-$, (i = 1, 2, 3)

$d_i^- \geq 0$, (i = 1, 2, 3)

$x_1, x_2 \geq 0.$

Then, following the procedure, by solving FGP model (I), we obtain the optimal compromise solution as:

$Z_1^\bullet$ = -1.006, $Z_2^\bullet$ = 1.278, $Z_3^\bullet$ = 0.634 at (3, 0.985).

The resulting membership values are $\mu_1(Z_1) = 0.722$, $\mu_2(Z_2) = 0.261$, $\mu_3(Z_3) = 0.462$.

The FGP model (II) with equal weight can be formulated as:

minimize $\gamma = 1/3\,(d_1^- + d_2^- + d_3^-)$
(13)

subject to

$1 + (x_1 - 3.6) \times (-0.182) + (x_2 - 2.6) \times (0.288) + d_1^- = 1,$

$1 + (x_1 - 7.2) \times (0.052) + (x_2 - 0.2) \times (-0.177) + d_2^- = 1,$

$1 + (x_1 - 3.6) \times (-0.107) + (x_2 - 2.6) \times (0.254) + d_3^- = 1,$

$x_1 - x_2 \geq 1,$

$2x_1 + 3x_2 \leq 15,$

$x_1 + 9x_2 \geq 9,$

$x_1 \geq 3,$

$d_i^- \geq 0, \ (i = 1, 2, 3)$

$x_1, x_2 \geq 0.$

FGP model (II) offers the solution $Z_1^\bullet$ = -0.609, $Z_2^\bullet$ = 1.256, $Z_3^\bullet$ = 0.824 at (3.6, 2.6).

The obtained membership values are $\mu_1(Z_1) = 1$, $\mu_2(Z_2) = 0.057$, $\mu_3(Z_3) = 1$.

On comparing Euclidean distance, we observe that the proposed FGP model (I) and (II) offer better optimal solution than Chakraborty and Gupta [2].

Note 1: The solution set obtained by Chakraborty and Gupta [2] is given by $Z_1^\bullet$ = -0.625, $Z_2^\bullet$ = 1.250, $Z_3^\bullet$ = 0.786 at (3, 2). The corresponding membership values are $\mu_1(Z_1) = 0.989$, $\mu_2(Z_2) = 0$, $\mu_3(Z_3) = 0.893$.

Note 2: All solutions of the numerical example are obtained by Lingo software (version 6.0).

Table: 1 Comparison of optimal solutions of the numerical example based on different approaches.

Approach	$x_1,\ x_2$	$Z_1^\bullet, Z_2^\bullet, Z_1^\bullet$	$\mu_1(Z_1), \mu_2(Z_2), \mu_3(Z_3)$	Euclidean distance
Proposed FGP model (I)	3, 0.985	-1.006, 1.2782,0.6335	0.7218,0.2607,0.4617	0.9559
Proposed FGP model (II)	3.6, 2.6	-0.6087, 1.2562, 0.8235	1, 0.0572, 1	0.9428
Chakraborty and Gupta [2]	3, 2	-0.625, 1.25, 0.7857	0.9886, 0, 0.8929	1.0058

7. CONCLUSIONS:

An alternative fuzzy approach for solving MOLFPP is presented in this paper. In relation to the Pal et al. procedure [9], it is clear that computational burden of the proposed FGP approach is less than the Pal et al. approach [9] because the programming model to be computed has only negative deviational variables. In the Pal et al. approach [9] one has to solve FGP model involving negative and positive deviational variables. In the proposed approach, we transform MOLFPP into multi-objective linear programming problem by using first order Taylor polynomial series. Here we do not need any extra transformation variables. The proposed concept can be used to solve realistic MOLFPP such as inventory problems, production planning problems, agricultural planning problem, etc. It can also be applied to solve multi-objective decentralized bi-level as well as multilevel fractional programming problems.

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