International Journal of Mathematical Archive-2(10), 2011, Page: 1865-1869
JMA Available online through www.ijma.info ISSN 2229 – 5046

TWO POINT BOUNDARY VALUE PROBLEMS ASSOCIATED WITH n th ORDER MATRIX LYAPUNOV DIFFERENTIAL SYSTEMS

[1]B. V. Appa Rao* and [2]K. A. S. N. Prasad

*[1]Sophomore Engineering Department, K. L. University,
Vaddeswaram-522502, Guntur Dt.; Andhra Pradesh; INDIA
E-mail: bvardr2010@gmail.com*

*[2]Academic consultant, Acharya Nagarjuna University,
P. G. center M. R. Appa Rao campus, Nuzvid; INDIA
E-mail: Prasad.krosuri@gmail.com*

(Received on: 24-09-11; Accepted on: 10-10-11)

ABSTRACT

This paper introduces the criteria for the existence and uniqueness of solutions of two point boundary value problems associated with the nth order matrix Lyapunov differential systems by applying the technique of Kronecker product of matrices and with the help of a Green's matrix.

Key words: *Kronecker product, Lyapunov systems, Matrix differential equations, Variation of parameters.*

AMS subject classification: *34D10, 49K 15.*

1. INTRODUCTION:

The importance of boundary value problems in the theory of differential equations and their applications to different areas of science and technology are well known. The existence and uniqueness of solutions of matrix Lyapunov systems is an important area of current research and many concepts of existence and uniqueness solutions have been recently developed for first order Lyapunov matrix differential systems.

In this paper is concerned with the problem of obtaining explicit solutions of two point boundary value problems with a system of matrix differential equation of the type

$$Z^{(n)} = A^n Z(t) + n_{c_1} A^{n-1} Z(t) B + n_{c_2} A^{n-2} Z(t) B^2 + \ldots + n_{c_r} A^{n-r} Z(t) B^r + \ldots + Z(t) B^n + F(t)$$

$$a \le t \le b \qquad (1.1)$$

Satisfying

$$M\,Z(a) + N\,Z(b) = P \qquad (1.2)$$

Where A, B and F are $n \times n$ real or complex valued matrices(members of $R^{n\times n}$ ($C^{n\times n}$) and 't' in the interval [a, b]. M, N and P are constant square matrices of order n. Also Z(t)$\in C[R_+ R^{n\times n}]$ is a variable matrix.

The main tool used in this paper is the technique of Kronecker product of matrices, which has been successfully applied in various fields of matrix theory, group theory and particle physics. The corresponding homogeneous equation [F=0] has wide range of applications in mathematical physics.

In section 2 we study various concepts of Kronecker products of matrices and develop primary results relating to corresponding boundary value problems. In section 3 we obtain existence and uniqueness solutions of two point boundary value problems with the help of a Green's matrix. The properties of a Green's Matrix are also studied in this section.

2 PRELIMINARIES:

In This section we study some properties and rules for kronecker products and basic results on related boundary value problems.

***Corresponding author: [1]B. V. Appa Rao*,*E-mail: bvardr2010@gmail.com**

Definition: [2.1] [1]: Let $R \in C^{m\times n}$ and $T \in C^{p\times q}$ then the kronecker product of R and T written $R \otimes T$ is defined to be partition matrix

$$R \otimes T = \begin{bmatrix} r_{11}T & r_{12}T & \cdots & r_{1n}T \\ r_{21}T & r_{22}T & \cdots & r_{2n}T \\ \vdots & \vdots & \cdots & \vdots \\ r_{m1}T & r_{m2}T & \cdots & r_{mn}T \end{bmatrix}$$

Is an $mp \times nq$ matrix and is in $C^{mp\times nq}$.

Definition: [2.2] [1] Let $R = \{r_{ij}\} \in C^{m\times n}$ we denote

$$\hat{R} = \text{Vec}R = \begin{bmatrix} R_{.1} \\ R_{.2} \\ \vdots \\ \vdots \\ R_{.n} \end{bmatrix} \text{ where } R_{.j} = \begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ \vdots \\ r_{nj} \end{bmatrix} \quad 1 \le j \le n.$$

The Kronecker product has the following properties and rules

(I) $(R \otimes T)^* = R^* \otimes T^*$ (R^* denotes the transpose of complex conjugate of R)
(II) $(R \otimes T)^{-1} = R^{-1} \otimes T^{-1}$
(III) The mixed product rule $(R \otimes T)(S \otimes Q) = (RS \otimes TQ)$

This rule holds good provided the dimensions of the matrices are such that the various expressions exit.
(IV) $\text{Vec}(RTS) = (S^* \otimes R)\,\text{Vec}\,T$
(V) $\text{Vec}(RQI) = (I^* \otimes R)\,\text{Vec}\,Q$
(VI) If R and T matrices both of order $n\times n$ then

i) $\text{Vec}(RT) = (I \otimes R)\,\text{Vec}\,T$
ii) $\text{Vec}(RT) = (T^* \otimes R)\,\text{Vec}\,I$

Theorem: 2.1 [3] If X is a fundamental matrix solution of $Z' = AZ$ then $X^{(n)}$ is a fundamental matrix solution of $Z^{(n)} = A^n Z$ where 'n' is a positive integer.

Proof: If X is a fundamental matrix of $Z' = AZ$ if, and only if $X' = AX$. This implies that $X'' = AX'(t) = A(AX(t)) = A^2X(t)$. This further implies that $X'''(t) = A^3X$ and so on.
Hence $X^{(n)}(t) = A^n X(t)$. Thus $X^{(n)}$ is also a fundamental matrix solution of $Z^{(n)} = A^n Z$.

Theorem: 2.2 Let C be a constant square matrix of order $n\times n$. Then any solution of

$$Z^{(n)}(t) = \sum_{r=0}^{n} n_{c_r} A^{n-r} Z(t) B^r \tag{2.1}$$

With $A^0 = B^0 = I$, is of the form $Z(t) = X(t)\,C\,Y^*(t)$, where X(t) is a fundamental matrix solution of $Z'(t) = AZ(t)$ and Y(t) is a fundamental matrix solution of $Z'(t) = B^* Z(t)$.

Theorem: 2.3[3] Any solution Z(t) of

$$Z^{(n)}(t) = \sum_{r=0}^{n} n_{c_r} A^{n-r} Z(t) B^r + F(t) \tag{2.2}$$

is of the form $Z(t) = X(t)CY^*(t) + Z_p(t)$. Where $Z_p(t)$ is a particular solution of (2.2) and

$$Z_p(t) = X(t)[\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} X^{-1}(s)F(s)Y^{-1}(s)ds]Y^*(t).$$

Theorem 2.4: Every solution Z (t) of the non homogeneous Lyapunov system (2.2) is of the form

$$Z(t)=X(t)C\,Y^*(t) + Z_p(t).$$

Proof: for n=1, the homogeneous system (2.1) reduces to Z'=AZ+ZB. The general solution in terms of the variation parameters formula for Z'=AZ+ZB+ F has been established in [2].

Now by applying the Vec operator to the homogeneous boundary value problem

$$Z^{(n)}(t) = \sum_{r=0}^{n} n_{c_r} A^{n-r} Z(t) B^r$$

Satisfying

$$M\,Z(a) + N\,Z(b)=0 \tag{2.3}$$

and by using Kronecker product properties we have

$$\hat{Z}^{(n)}(t) = \{I_n \otimes A^n + n_{c_1} B^* \otimes A^{n-1} + n_{c_2} B^{*2} \otimes A^{n-2} + \cdots + n_{c_r} B^{*r} \otimes A^{n-r} + \cdots + B^{*n} \otimes I_n\}\hat{Z}(t)$$

$$\hat{Z}^{(n)}(t) = (\sum_{r=0}^{n} n_{c_r} B^{*n} \otimes A^{n-r})\hat{Z}(t) \tag{2.4}$$

Satisfying

$$(I \otimes M)\hat{Z}(a) + (I \otimes N)\hat{Z}(b) = 0 \tag{2.5}$$

Let us denote $G= (\sum_{r=0}^{n} n_{c_r} B^{*n} \otimes A^{n-r})$

and $\psi(t) = \hat{Z}(t)$ then the boundary value problem(2.4) satisfying (2.5) can be written in the form

$$\psi^{(n)}(t) = G\psi(t) \tag{2.6}$$

Satisfying

$$(I \otimes M)\psi(a) + (I \otimes N)\psi(b) = 0 \tag{2.7}$$

Now we confine our attention to the homogeneous boundary value problem (2.6)satisfying(2.7). The corresponding non homogeneous boundary value problem (1.1) satisfying (1.2) is given by

$$\psi^{(n)} = G\psi + \hat{F} \tag{2.8}$$

$$(I \otimes M)\psi(a) + (I \otimes N)\psi(b) = \hat{P} \tag{2.9}$$

Where $\hat{F} = \text{vec}F$ and $\hat{P} = \text{vec}P$

Definition: 2.3 The dimension of the solution space of a boundary value problem is called the index of compatibility of the problem. If the index of the compatibility is zero then we say that the boundary value problem is incompatible.

The terms Fundamental matrix, Characteristic matrix are used in the usual sense [3].

Lemma: 2.1 Let X (t) and Y(t) stands for the fundamental matrix solutions of $Z'(t) = AZ(t)$ and $Z'(t) = B*Z(t)$ respectively. Then any solution $\Phi(t)=(Y^*(t) \otimes X(t))$ is a solution of(2.6) for any constant C of order n^2 and every solution is of this form.

Proof: Let $\Phi(t)=(Y^*(t) \otimes X(t))C$

Consider

$$\Phi'(t) = [B*\otimes I_n + I_n \otimes A](Y*(t) \otimes X(t))C$$

implies

$$\Phi^{(n)} = (\sum_{r=0}^{n} n_{c_r} B*^n \otimes A^{n-r})\ (Y*(t) \otimes X(t))C$$

Hence $\Phi(t)$ is a solution of (2.6) and it can be easily seen that every solution is of this form.

Theorem: 2.5 Let $Y^*(t) \otimes X(t))$ be a fundamental matrix of the homogeneous system (2.6). Then any solution then any solution of the non-homogeneous system (2.8) is of this form

$$\psi(t) = (Y*(t) \otimes X(t))C + (Y*(t) \otimes X(t))[\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}((Y*^{-1}(s) \otimes X^{-1}(s))(s)\hat{F}(s)ds].$$

where C is any constant vector of order n^2.

Proof: The proof is similar to that of theorems 3.31 &3.32 0f [3].

Theorem: 2.6 [4] If the homogeneous boundary value problem (2.6) satisfying (2.7) has a characteristic matrix D= $(I \otimes M)\ (Y*(a) \otimes X(a)) + (I \otimes N)\ (Y*(b) \otimes X(b))$ of rank 'r' then its index of compatibility is n^2-r.

3. MAIN RESULT

In this section, we shall be concerned with the existence and uniqueness of solutions to the boundary value problems (2.8) satisfying (2.9).

Theorem: 3.1 Let $(Y*(t) \otimes X(t))$ be a fundamental matrix of (2.6) and suppose that the homogeneous boundary value problem is incompatible and the characteristic matrix D is non-singular. Then there exists a unique solution to the non-homogeneous boundary value problem (2.8) satisfying (2.9) and is of the form

$$\psi(t) = (Y*(t) \otimes X(t)D^{-1}\hat{P} + \int_a^b H(t,s)\hat{F}(s)ds$$

where H(t, s) is the green's matrix for the homogeneous boundary value problem.

Proof: From Theorem 2.5 any solution of (2.8) is of the form

$$\psi(t) = (Y*(t) \otimes X(t))C + (Y*(t) \otimes X(t))[\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}((Y*^{-1}(s) \otimes X^{-1}(s))(s)\hat{F}(s)ds].$$

where $(Y*(t) \otimes X(t))$ be a fundamental matrix for the homogeneous system and C is any constant vector of order n^2 and will be determined uniquely from the fact that the solution $\Psi(t)$ must satisfy the boundary condition (2.9). Substituting the general form of $\Psi(t)$ in boundary condition (2.9) we have

$[(I \otimes M)\ (Y*(a) \otimes X(a)) + (I \otimes N)\ (Y*(b) \otimes X(b))]$C+

$$C= D^{-1}\hat{P} - D^{-1}(I \otimes N)(Y*(b) \otimes X(b))\int_a^b \frac{(b-s)^{n-1}}{(n-1)!}(Y*^{-1}(s) \otimes X^{-1}(s))\hat{F}(s)ds$$

$$\psi(t) = (Y^*(t) \otimes X(t))D^{-1}\hat{P} -$$

$$((Y^*(t) \otimes X(t))D^{-1}(I \otimes N)(Y^*(b) \otimes X(b))[\frac{1}{(n-1)!}\int_a^b (b-s)^{n-1}((Y^{*-1}(s) \otimes X^{-1}(s))(s)\hat{F}(s)ds].$$

$$+((Y^*(t) \otimes X(t))[\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}((Y^{*-1}(s) \otimes X^{-1}(s))(s)\hat{F}(s)ds].$$

$$\psi(t) = (Y^*(t) \otimes X(t)D^{-1}\hat{P} + \int_a^b H(t,s)\hat{F}(s)ds$$

where

$$H(t,s) = \begin{cases} (Y^*(t) \otimes X(t))[I_{n^2} - D^{-1}(I \otimes N)(Y^*(b) \otimes X(b))(\dfrac{(t-s)^{n-1} + (b-s)^{n-1}}{(n-1)!}](Y^{*-1}(s) \otimes X^{-1}(s)) & a \le s < t \le b \\ -(Y^*(t) \otimes X(t))D^{-1}(I \otimes N)(Y^*(b) \otimes X(b))(\dfrac{(b-s)^{n-1}}{(n-1)!})(Y^{*-1}(s) \otimes X^{-1}(s)) & a \le t < s \le b \end{cases}$$

Corollary: 3.1 When F=0 the boundary value problem(2.6) satisfying (2.9) has a unique solution is given by

$$\psi(t) = (Y^*(t) \otimes X(t)D^{-1}\hat{P}.$$

Theorem 3.2: The Greens matrix H(t,s) has the following properties

(i) H(t, s) has a function of 't' with fixed 's' having continuous derivatives everywhere except at t=s, H(t, s) has a jump discontinuity of unit magnitude and is given by

$$H(s^+, s) - H(s^-, s) = I_n^2.$$

(ii) H (t, s) is a formal solution of homogeneous boundary value problem(2.6)satisfying (2.7). H (t, s) fails to be a true solution because of the discontinuity at t=s.

(iii) H (t, s) satisfying the properties (i) and (ii) is unique.

REFERENCES:

[1] Alexander Graham, Kronecker products and Matrix calculus with applications, Ellis Horwood Ltd England (1981).

[2] M. S. N. Murty and B. V. Appa Rao, Two point boundary value problems for matrix differential equations, Journal of the Indian Math. Soc. Vol 73 Nos.1-2(1006), 1-7.

[3] R. H. Cole, The theory of ordinary differential equations, Applenton-Century Crofts, 1968.

[4] S. Barnet, Introduction to mathematical control theory, Clarenton press, Oxford, 1975.

****