

NEW SUB CLASSES OF THE CLASS OF CLOSE TO CONVEX FUNCTIONS  
WITH RESPECT TO SYMMETRIC POINTS

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ABSTRACT

We introduce some subclasses of the class of close to convex functions w.r.t. symmetric points, derive inclusion relations, establish integral representation formulas and determine coefficient estimates for the functions of such classes. The results are sharp.

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1. INTRODUCTION AND DEFINITIONS

Principle of subordination ([5], [7])

Let  $f(z)$  and  $F(z)$  be two analytic functions in the unit disc  $E = \{z; |z| < 1\}$ . Then  $f(z)$  is called subordinate to  $F(z)$  if there exists a function  $w(z)$  analytic in  $E$  and satisfying the conditions  $w(0) = 0, |w(z)| < 1$  such that  $f(z) = F(w(z))$  and we write as  $f(z) \prec F(z)$ . If  $F(z)$  is univalent in  $E$ , the above definition is equivalent to  $f(0) = F(0)$  and  $f(E) \subset F(E)$ .

By  $\mathcal{U}$ , we denote the class of analytic bounded functions of the form

$$(1.1) \quad w(z) = \sum_{k=1}^{\infty} d_k z^k, \quad z \in E$$

with the conditions  $w(0) = 0$  and  $|w(z)| < 1$ .

Let  $\mathcal{A}$  be the class of functions of the form

$$(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc  $E$ .

Sakaguchi [8] introduced the concept of starlike univalent functions w.r.t. symmetric points. A function  $f(z) \in \mathcal{A}$  is called univalent star-like w.r.t. symmetric points if and only if

$$(1.3) \quad \operatorname{Re} \left\{ \frac{z f'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in E,$$

and the class of functions satisfying (1.3) may be denoted by  $S_s^*$ . Das and Singh [2] extended the idea of symmetric points to convex and close to convex functions. A function  $f(z) \in \mathcal{A}$  is said to be univalent convex w. r. t. symmetric points if and only if

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$$(1.4) \quad \operatorname{Re} \left\{ \frac{(zf'(z))'}{(f(z)-f(-z))'} \right\} > 0, \quad z \in E,$$

and the class of such functions is denoted by  $K_S$ . A function  $f(z) \in \mathcal{A}$  is said to belong to the class  $C_S$  of close to convex functions w.r.t. symmetric points if there exists a function

$$(1.5) \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_S^*$$

for which

$$(1.6) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{g(z)-g(-z)} \right\} > 0, \quad z \in E.$$

If

$$(1.7) \quad h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in K_S$$

for which

$$(1.8) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{h(z)-h(-z)} \right\} > 0, \quad z \in E.$$

the class of functions satisfying (1.8) is denoted by  $C_{1(S)}$ ,  $C_{1(S)} \subset C_S$ .

Let  $C'_S$  represent the class of functions  $f(z) \in \mathcal{A}$  which satisfy the condition

$$(1.9) \quad \operatorname{Re} \left\{ \frac{(zf'(z))'}{(g(z)-g(-z))'} \right\} > 0, \quad g \in S_S^*, \quad z \in E.$$

If

$$(1.10) \quad \operatorname{Re} \left\{ \frac{(zf'(z))'}{(h(z)-h(-z))'} \right\} > 0, \quad h \in K_S, \quad z \in E,$$

the corresponding class is denoted by  $C'_{1(S)}$ . Goel and the first author [4] introduced and studied the subclass  $S_S^*(A, B)$  of  $S_S^*$ .  $f(z) \in S_S^*(A, B)$  if

$$(1.11) \quad \left[ \frac{2zf'(z)}{f(z)-f(-z)} \right] < \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E.$$

If

$$(1.12) \quad \left[ \frac{2(zf'(z))'}{(f(z)-f(-z))'} \right] < \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E,$$

the class of functions satisfying (1.12) is denoted by  $K_S(A, B)$  which is a subclass of  $K_S$ . It is obvious that

$$f(z) \in S_S^*(A, B) \text{ implies that } zf'(z) \in K_S(A, B).$$

The author et al. [6] introduced the subclass  $C_S(A, B; C, D)$  of  $C_S$  and obtained its coefficient estimates.

A function  $f(z)$  in  $\mathcal{A}$  belongs to  $C_S(A, B; C, D)$  if

$$(1.13) \quad \left[ \frac{2zf'(z)}{g(z)-g(-z)} \right] < \frac{1+Cz}{1+Dz}, \quad g \in S_S^*(A, B), \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E.$$

If

$$(1.14) \quad \left[ \frac{2zf'(z)}{h(z)-h(-z)} \right] < \frac{1+Cz}{1+Dz}, \quad h \in K_S(A, B), \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E.$$

the corresponding class of functions satisfying (1.14) is denoted by  $C_{1(S)}(A, B; C, D)$ ,  $C'_S(A, B; C, D)$  is the class of functions  $f(z)$  in  $\mathcal{A}$  for which

$$(1.15) \quad \left[ \frac{2(zf'(z))'}{(g(z)-g(-z))'} \right] < \frac{1+Cz}{1+Dz}, \quad g \in S_S^*(A, B), \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E.$$

Moreover  $C'_{1(s)}(A, B; C, D)$  is the class of functions  $f(z)$  in  $\mathcal{A}$  such that

$$(1.16) \quad \left[ \frac{2(zf'(z))'}{(h(z)-h(-z))'} \right] < \frac{1+Cz}{1+Dz}, \quad h \in K_s(A, B), \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E.$$

Let  $\alpha \geq 0$  and  $\frac{f(z)f'(z)}{z} \neq 0$ . Then  $C_s(\alpha)$  is the class of  $\alpha$ -close to convex functions  $f(z)$  in  $\mathcal{A}$  w.r.t. symmetric points if there exists  $g(z) \in S_s^*$  such that

$$(1.17) \quad \operatorname{Re} \left\{ \frac{2(1-\alpha)zf'(z)}{g(z)-g(-z)} + \frac{2\alpha(zf'(z))'}{(g(z)-g(-z))'} \right\} > 0, \quad z \in E.$$

If

$$(1.18) \quad \operatorname{Re} \left\{ \frac{2(1-\alpha)zf'(z)}{h(z)-h(-z)} + \frac{2\alpha(zf'(z))'}{(h(z)-h(-z))'} \right\} > 0, \quad h \in K_s(A, B),$$

the class of functions satisfying (1.18) and is denoted by  $C_{1(s)}(\alpha)$ .

Also  $C_s(\alpha; A, B; C, D)$  and  $C_{1(s)}(\alpha; A, B; C, D)$  represent the classes of functions  $f(z)$  in  $\mathcal{A}$  which satisfy, respectively, the conditions

$$(1.19) \quad \left[ \frac{2(1-\alpha)zf'(z)}{g(z)-g(-z)} + \frac{2\alpha(zf'(z))'}{(g(z)-g(-z))'} \right] < \frac{1+Cz}{1+Dz}, \quad g \in S_s^*(A, B), \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E.$$

and

$$(1.20) \quad \left[ \frac{2(1-\alpha)zf'(z)}{h(z)-h(-z)} + \frac{2\alpha(zf'(z))'}{(h(z)-h(-z))'} \right] < \frac{1+Cz}{1+Dz}, \quad h \in K_s(A, B), \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E.$$

Throughout this paper we assume that

$$(1.21) \quad \left\{ \begin{array}{l} \alpha \geq 0, g \in S_s^*(A, B), h \in K_s(A, B), G(z) = \frac{g(z)-g(-z)}{2}, H(z) = \frac{h(z)-h(-z)}{2}, \\ \wp(z) = \frac{1+Cw(z)}{1+Dw(z)}, w \in u, \quad -1 \leq D \leq B < A \leq C \leq 1, \quad z \in E. \end{array} \right\}$$

## 2. PRELIMINARY LEMMAS

**Lemma: 2.1[1]** Let  $\alpha \geq 0$  and  $D(z)$  be starlike in  $E$ . Let  $\mathcal{N}(z)$  be analytic in  $E$  such that  $\mathcal{N}(0) = D(0) = 0 = \mathcal{N}'(0) - 1 = D'(0) - 1$ , then  $\operatorname{Re} \left\{ \frac{\mathcal{N}(z)}{D(z)} \right\} > 0$  whenever

$$\operatorname{Re} \left\{ (1-\alpha) \frac{\mathcal{N}(z)}{D(z)} + \alpha \frac{\mathcal{N}'(z)}{D'(z)} \right\} > 0.$$

**Lemma: 2.2** Under the same conditions of lemma 2.1,  $\frac{\mathcal{N}(z)}{D(z)} < \frac{1+Cz}{1+Dz}$  whenever

$$\left\{ (1-\alpha) \frac{\mathcal{N}(z)}{D(z)} + \alpha \frac{\mathcal{N}'(z)}{D'(z)} \right\} < \frac{1+Cz}{1+Dz}.$$

**Proof:** By definition of subordinates,  $\left\{ (1-\alpha) \frac{\mathcal{N}(z)}{D(z)} + \alpha \frac{\mathcal{N}'(z)}{D'(z)} \right\} = \frac{1+Cw(z)}{1+Dw(z)}$ .

This implies that

$$\operatorname{Re} \left\{ (1-\alpha) \frac{\mathcal{N}(z)}{D(z)} + \alpha \frac{\mathcal{N}'(z)}{D'(z)} \right\} = \operatorname{Re} \left\{ \frac{1+Cw(z)}{1+Dw(z)} \right\} \geq \frac{1-Cr}{1-Dr} > \frac{1-C}{1-D} = \beta \quad (0 \leq \beta < 1)$$

which can be put into the form

$$(2.1) \quad \frac{1}{(1-\beta)} \operatorname{Re} \left\{ (1-\alpha) \left( \frac{\mathcal{N}(z)}{D(z)} - \beta \right) + \alpha \left( \frac{\mathcal{N}'(z)}{D'(z)} - \beta \right) \right\} > 0.$$

Setting

$$(2.2) \quad \mathcal{M}(z) = \frac{\mathcal{N}(z) - \beta D(z)}{(1-\beta)},$$

$$(2.1) \text{ takes the form } \operatorname{Re} \left\{ (1-\alpha) \frac{\mathcal{M}(z)}{D(z)} + \alpha \frac{\mathcal{M}'(z)}{D'(z)} \right\} > 0.$$

By lemma 2.1,  $\operatorname{Re} \left\{ \frac{\mathcal{M}(z)}{D(z)} \right\} > 0$  whenever  $\operatorname{Re} \left\{ (1-\alpha) \frac{\mathcal{M}(z)}{D(z)} + \alpha \frac{\mathcal{M}'(z)}{D'(z)} \right\} > 0$ .

This means that  $\operatorname{Re} \left\{ \frac{\mathcal{N}(z)}{D(z)} \right\} > \beta$  whenever  $\operatorname{Re} \left\{ (1-\alpha) \frac{\mathcal{N}(z)}{D(z)} + \alpha \frac{\mathcal{N}'(z)}{D'(z)} \right\} > \beta$ .

That is  $\frac{\mathcal{N}(z)}{D(z)} < \frac{1+Cz}{1+Dz}$  whenever  $\left\{ (1-\alpha) \frac{\mathcal{N}(z)}{D(z)} + \alpha \frac{\mathcal{N}'(z)}{D'(z)} \right\} < \frac{1+Cz}{1+Dz}$ .

**Lemma: 2.3[3]** Let  $\wp(z) = \frac{1+Cw(z)}{1+Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$ . Then

$$|p_n| \leq (C - D).$$

The estimates are sharp for the functions  $\wp_n(z) = \frac{1+C\delta z^n}{1+D\delta z^n}$ ,  $|\delta| = 1$  and  $n \geq 1$ .

**Lemma: 2.4 [4]** Let  $g \in S_s^*(A, B)$ . Then for  $n \geq 1$ ,

$$|b_{2n}| \leq \frac{(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j),$$

$$|b_{2n+1}| \leq \frac{(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j).$$

The bounds are sharp being attained for the function  $g_0(z)$  defined by

$$(2.3) \quad g_0(z) = \begin{cases} \log \left( \frac{(1+z)^{(1-A)/2(1-B)}(1+Bz)^{(A-B)/(1-B^2)}}{(1-z)^{(1+A)/2(1+B)}} \right), & B \neq -1; \\ \log \left( \frac{1+z}{1-z} \right)^{(1-A)/4} + \left( \frac{1+A}{2} \right) \left( \frac{z}{1-z} \right), & B = -1; \\ \log \left( \frac{(1+z)^{(1-A)/2}}{(1-z)^{(1+A)/2}} \right), & B = 0. \end{cases}$$

Since  $f(z) \in S_s^*(A, B)$  implies that  $zf'(z) \in K_s(A, B)$ , we have the following

**Lemma: 2.5** Let  $f \in K_s(A, B)$ , then

$$|c_{2n}| \leq \frac{1}{2n} \left[ \frac{(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j) \right],$$

$$|c_{2n+1}| \leq \frac{1}{2n+1} \left[ \frac{(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j) \right].$$

The extremal function is given by

$$(2.4) \quad h_0(z) = \int_0^z \frac{g_0(t)}{t} dt, \quad g_0(z) \text{ is defined by (2.3).}$$

### 3. INCLUSION RELATIONS AND INTEGRAL REPRESENTATION FORMULAS

**Theorem: 3.1**  $C_s(\alpha; A, B; C, D) \subset C_s(A, B; C, D) \subset C_s$ .

**Proof:** Putting  $\mathcal{N}(z) = zf'(z)$  and  $\mathcal{D}(z) = G(z)$  (which is odd star-like) in lemma 2.2 so that

$$\frac{zf'(z)}{G(z)} < \frac{1+Cz}{1+Dz}, \text{ whenever } \left\{ (1-\alpha) \frac{zf'(z)}{G(z)} + \alpha \frac{(zf'(z))'}{G'(z)} \right\} < \frac{1+Cz}{1+Dz}$$

and the desired result follows.

From the above theorem it follows that all  $\alpha$ -close to convex functions w. r. t. symmetric points are close to convex w. r. t. symmetric points. Similarly we can prove

**Theorem: 3.2**  $C_{1(s)}(\alpha; A, B; C, D) \subset C_{1(s)}(A, B; C, D) \subset C_{1(s)}$ .

**Theorem: 3.3** Let  $f \in C_s(\alpha; A, B; C, D)$ , then

(i) for  $\alpha = 0$ ,  $f(z) = \int_0^z \frac{G(t)\wp(t)}{t} dt$ .

(ii) for  $\alpha > 0$ ,  $f(z) = (1+c) \left[ \int_0^z \frac{1}{t(G(t))^c} \left\{ \int_0^t (G(u))^c G'(u)\wp(u) du \right\} dt \right], \left( c = \frac{1}{\alpha} - 1 \right)$ .

**Proof:** We have

$$(3.1) \quad (1-\alpha) \frac{zf'(z)}{G(z)} + \alpha \frac{(zf'(z))'}{G'(z)} = \wp(z)$$

For  $\alpha = 0$ , there is nothing to prove.

Consider the case when  $\alpha > 0$ .

Dividing (3.1) by  $\alpha$  and putting  $c = \frac{1}{\alpha} - 1$ , (3.1) takes the form

$$(3.2) \quad \frac{czf'(z)}{G(z)} + \frac{(zf'(z))'}{G'(z)} = (1+c)\wp(z)$$

Multiplying (3.2) by  $(G(z))^c G'(z)$ , we get

$$czf'(z)(G(z))^{c-1}G'(z) + (zf'(z))'(G(z))^c = (1+c)[(G(z))^c G'(z)\wp(z)]$$

which reduces to

$$(3.3) \quad \frac{d}{dz} [zf'(z)(G(z))^c] = (1+c)[(G(z))^c G'(z)\wp(z)]$$

Integrating (3.3) from 0 to z,

$$f'(z) = \frac{(1+c)}{z(G(z))^c} \left[ \int_0^z (G(u))^c G'(u)\wp(u) du \right]$$

which on integration gives the desired result.

On the same lines we can prove

**Theorem: 3.4** Let  $f \in C_{1(s)}(\alpha; A, B; C, D)$ , then

(i) for  $\alpha = 0$ ,  $f(z) = \int_0^z \frac{H(t)\wp(t)}{t} dt$ .

(ii) for  $\alpha > 0$ ,  $f(z) = (1+c) \left[ \int_0^z \frac{1}{t(H(t))^c} \left\{ \int_0^t (H(u))^c H'(u)\wp(u) du \right\} dt \right], \left( c = \frac{1}{\alpha} - 1 \right)$ .

4. COEFFICIENT ESTIMATES FOR THE CLASSES  $C'_s(A, B; C, D)$  and  $C_s(\alpha; A, B; C, D)$

**Theorem: 4.1** Let  $f \in C'_s(A, B; C, D)$ , then

$$(4.1) \quad |a_{2n}| \leq \frac{(C-D)}{(2n)^2} \left\{ 1 + (A-B) \left( \sum_{k=2}^n \frac{(2k-1)}{2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right) \right\},$$

$$(4.2) \quad |a_{2n+1}| \frac{1}{(2n+1)^2} \left[ \frac{(2n+1)(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j) + (C - D) \left\{ (A-B) \sum_{k=2}^n \frac{(2k-1)}{2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right\} \right].$$

**Proof:** We have

$$(zf'(z))' = G'(z)\wp(z) \text{ which on expansion yields}$$

$$(4.3) \quad 1 + 2^2 a_2 z + 3^2 a_3 z^2 + \dots + (2n)^2 a_{2n} z^{2n-1} + (2n+1)^2 a_{2n+1} z^{2n} + \dots \\ = [1 + p_1 z + p_2 z^2 + \dots + p_{2n-1} z^{2n-1} + p_{2n} z^{2n} + \dots] \\ [1 + 3b_3 z^2 + 5b_5 z^4 + \dots + (2n-1)b_{2n-1} z^{2n-2} + (2n+1)b_{2n+1} z^{2n} + \dots]$$

Equating coefficients of  $z^{2n-1}$  and  $z^{2n}$  in (4.3), we get

$$(4.4) \quad (2n)^2 a_{2n} = (2n-1)p_1 b_{2n-1} + (2n-3)p_3 b_{2n-3} + \dots + 3p_{2n-3} b_3 + p_{2n-1}.$$

$$(4.5) \quad (2n+1)^2 a_{2n+1} = (2n+1)b_{2n+1} + (2n-1)p_2 b_{2n-1} + \dots + 3p_{2n-2} b_3 + p_{2n}.$$

Using lemmas 2.3 and 2.4 in (4.4), we obtain

$$(2n)^2 |a_{2n}| \leq (C-D) \left( 1 + \sum_{k=2}^n (2k-1) |b_{2k-1}| \right) \leq (C-D) \left[ 1 + (A-B) \left\{ \sum_{k=2}^n \frac{(2k-1)}{2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right\} \right]$$

from which (4.1) follows.

Again applying lemmas 2.3 and 2.4 in (4.5), we have

$$(2n+1)^2 |a_{2n+1}| \leq (2n+1) |b_{2n+1}| + (C-D) \left( 1 + \sum_{k=2}^n (2k-1) |b_{2k-1}| \right) \\ \leq \frac{(2n+1)(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j) + (C-D) \left[ 1 + (A-B) \left\{ \sum_{k=2}^n \frac{(2k-1)}{2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right\} \right]$$

which gives the desired result.

The extremal function is obtained by choosing  $G(z) = \frac{g_0(z) - g_0(-z)}{2}$ ,  $g_0(z)$  is defined by (2.3), and  $\wp(z) = \frac{1+Cz}{1+Dz}$  in the integral representation formula

$$f(z) = \left[ \int_0^z \frac{1}{t} \left\{ \int_0^t G'(u) \wp(u) du \right\} dt \right].$$

Taking  $A = C = 1$  and  $B = D = -1$  in the theorem, we have the following:

**Corollary: 4.1** If  $f \in C_s$ , then  $|a_{2n}| \leq \frac{1}{2}$  and  $|a_{2n+1}| \leq \frac{1+(2n+1)^2}{2(2n+1)^2}$ .

**Theorem 4.2** Let  $f \in C_s(\alpha; A, B; C, D)$ . Then

$$(4.6) \quad |a_2| \leq \frac{(C-D)}{2(1+\alpha)},$$

$$(4.7) \quad |a_3| \leq \frac{(1+2\alpha)(A-B)+2(C-D)}{6(1+2\alpha)},$$

$$(4.8) \quad |a_4| \leq \frac{(C-D)}{8(1+\alpha)(1+3\alpha)} [(1+5\alpha)(A-B) + 2(1+\alpha)],$$

$$(4.9) \quad |a_5| \leq \frac{1}{5(1+2\alpha)(1+4\alpha)} \left[ \frac{1}{8}(1+2\alpha)(1+4\alpha)(A-B)(A-B+2) \right. \\ \left. + 4(C-D)\{(1+8\alpha)(A-B) + 2(1+2\alpha)\} \right],$$

$$(4.10) \quad |a_6| \leq \frac{(C-D)}{48(1+\alpha)(1+3\alpha)(1+5\alpha)} \left[ 4\alpha|\alpha-1| \left( \frac{A-B}{4} \right)^2 + (1+3\alpha)(1+9\alpha)(A-B)(A-B+2) \right. \\ \left. + 4(1+\alpha)(1+11\alpha)(A-B) + 8(1+\alpha)(1+3\alpha) \right]$$

The bounds are sharp.

**Proof:** Since  $f \in C_s(\alpha; A, B; C, D)$ , therefore

$$(1-\alpha)zf'(z)G'(z) + \alpha(zf'(z))'G(z) = \wp(z)G(z)G'(z).$$

Expanding the series, we get

$$(1-\alpha)[z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + 6a_6z^6 + \dots][1 + 3b_3z^2 + 5b_5z^4 + \dots] \\ + \alpha[1 + 4a_2z + 9a_3z^2 + 16a_4z^3 + 25a_5z^4 + 36a_6z^5 + \dots][z + b_3z^3 + b_5z^5 + \dots] \\ = [1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + p_5z^5 + \dots][z + b_3z^3 + b_5z^5 + \dots][1 + 3b_3z^2 + 5b_5z^4 + \dots]$$

which on simplification yields

$$(4.11) \quad 1 + 2(1+\alpha)a_2z + \{3(1+2\alpha)a_3 + (3-2\alpha)b_3\}z^2 + \{4(1+3\alpha)a_4 + (6-2\alpha)a_2b_3\}z^3 + \{5(1+4\alpha)a_5 + \\ 5-4\alpha b_5+9a_3b_3z^4+61+5\alpha a_6+43+\alpha a_4b_3+21+\alpha a_2b_5z^5+\dots \\ = 1 + p_1z + (4b_3 + p_2)z^2 + (4p_1b_3 + p_3)z^3 + \{(6b_5 + 3b_3^2) + 4p_2b_3 + p_4\}z^4 \\ + \{p_1(6b_5 + 3b_3^2) + 4p_3b_3 + p_5\}z^5 + \dots.$$

Identifying terms in (4.11), we get

$$(4.12) \quad 2(1+\alpha)a_2 = p_1,$$

$$(4.13) \quad 3(1+2\alpha)a_3 = (1+2\alpha)b_3 + p_2,$$

$$(4.14) \quad 4(1+3\alpha)a_4 = (2\alpha-6)a_2b_3 + 4p_1b_3 + p_3,$$

$$(4.15) \quad 5(1+4\alpha)a_5 = (1+4\alpha)b_5 - 9a_3b_3 + 3b_3^2 + 4p_2b_3 + p_4,$$

$$(4.16) \quad 6(1+5\alpha)a_6 = 4(3+\alpha)a_4b_3 - 2(5-3\alpha)a_2b_5 + p_1(6b_5 + 3b_3^2) + 4p_3b_3 + p_5.$$

Applying lemma 2.3 to (4.12), we get (4.6). Using lemma 2.3 and 2.4 in (4.13), (4.7) follows. (4.14) in conjunction with (4.12) leads us to

$$(4.17) \quad 4(1+\alpha)(1+3\alpha)a_4 = (1+5\alpha)p_1b_3 + (1+\alpha)p_3.$$

With the application of lemmas 2.3 and 2.4 in (4.17), (4.8) follows.

Eliminating  $a_3$  from (4.13) and (4.15), we arrive at

$$(4.18) \quad 5(1+2\alpha)(1+4\alpha)a_5 = (1+2\alpha)(1+4\alpha)b_5 + (1+8\alpha)p_2b_3 + (1+2\alpha)p_4.$$

Using lemmas 2.3 and 2.4 in (2.18), (4.9) follows.

From (4.12), (4.14) and (4.16), we obtain

$$(4.19) \quad \begin{aligned} &6(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)a_6 \\ &= 4\alpha(\alpha - 1)p_1b_3^2 + (1 + 3\alpha)(1 + 9\alpha)p_1b_5 + (1 + \alpha)(1 + 11\alpha)p_3b_3 + (1 + \alpha)(1 + 3\alpha)p_5. \end{aligned}$$

With the application of lemma 2.3 and 2.4 in (4.19), (4.10) follows.

The extremal function is obtained by choosing  $G(z) = \frac{g_0(z) - g_0(-z)}{2}$ ,  $g_0(z)$  is defined by (2.3), and  $\wp(z) = \frac{1+Cz}{1+Dz}$  in the integral representation formula proved in theorem 3.3.

Letting  $A = C = 1$  and  $B = D = -1$  in the theorem, we have the following:

**Corollary: 4.2** If  $f \in C_s(\alpha)$ , then

$$\begin{aligned} |a_2| &\leq \frac{1}{(1+\alpha)}, \\ |a_3| &\leq \frac{3+2\alpha}{3(1+2\alpha)}, \\ |a_4| &\leq \frac{1}{(1+\alpha)}, \\ |a_5| &\leq \frac{5+26\alpha+8\alpha^2}{5(1+2\alpha)(1+4\alpha)}, \\ |a_6| &\leq \begin{cases} \frac{3+32\alpha+37\alpha^2}{3(1+\alpha)(1+3\alpha)(1+5\alpha)}, & 0 \leq \alpha \leq 1, \\ \frac{1}{(1+\alpha)}, & \alpha \geq 1. \end{cases} \end{aligned}$$

## 5. COEFFICIENT ESTIMATES FOR THE CLASSES $SC \langle \rangle_{1(s)}(A, B; C, D)$ , $C'_{1(s)}(A, B; C, D)$ and $C_{1(s)}(\alpha; A, B; C, D)$

**Theorem: 5.1** Let  $f \in C_{1(s)}(A, B; C, D)$ , then

$$(5.1) \quad |a_{2n}| \leq \frac{(C-D)}{2n} \left\{ 1 + \left( \sum_{k=2}^n \frac{(A-B)}{(2k-1)2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right) \right\},$$

$$(5.2) \quad |a_{2n+1}| \leq \frac{1}{2n+1} \left[ \frac{(A-B)}{(2n+1)2^n n!} \prod_{j=1}^{n-1} (A-B+2j) + (C - D) \left\{ 1 + \sum_{k=2}^n \frac{(A-B)}{(2k-1)2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right\} \right].$$

**Proof:** We have  $zf'(z) = H(z)\wp(z)$  which on expansion gives

$$(5.3) \quad \begin{aligned} &z + 2a_2z^2 + 3a_3z^3 + \dots + 2na_{2n}z^{2n} + (2n+1)a_{2n+1}z^{2n+1} + \dots \\ &= [1 + p_1z + p_2z^2 + \dots + p_{2n-1}z^{2n-1} + p_{2n}z^{2n} + \dots][z + c_3z^3 + \dots + c_{2n-1}z^{2n-1} + c_{2n+1}z^{2n+1} + \dots] \end{aligned}$$

Equating coefficients of  $z^{2n}$  and  $z^{2n+1}$  in (5.3), we have

$$(5.4) \quad 2na_{2n} = p_1c_{2n-1} + p_3c_{2n-3} + \dots + p_{2n-3}c_3 + p_{2n-1}.$$

$$(5.5) \quad (2n+1)a_{2n+1} = c_{2n+1} + p_2c_{2n-1} + \dots + p_{2n-2}c_3 + p_{2n}.$$



Using lemmas 2.3 and 2.5 in (5.4), we have

$$2n|a_{2n}| \leq (C - D) \left( 1 + \sum_{k=2}^n |c_{2k-1}| \right) \leq (C - D) \left[ 1 + \left\{ \sum_{k=2}^n \frac{(A - B)}{(2k - 1)2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right]$$

from which (5.1) follows.

By the application of lemmas 2.3 and 2.5 in (5.5), we get

$$\begin{aligned} (2n + 1)|a_{2n+1}| &\leq |c_{2n+1}| + (C - D) \left( 1 + \sum_{k=2}^n |c_{2k-1}| \right) \\ &\leq \frac{(A-B)}{(2n+1)2^n n!} \prod_{j=1}^{n-1} (A - B + 2j) + (C - D) \left[ 1 + \left\{ \sum_{k=2}^n \frac{(A-B)}{(2k-1)2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right] \end{aligned}$$

which gives (5.2).

Choosing  $H(z) = \frac{h_0(z) - h_0(-z)}{2}$ ,  $h_0(z)$  is defined by (2.4), and  $\wp(z) = \frac{1+Cz}{1+Dz}$ , the extremal function is

$$(z) = \int_0^z \frac{H(t)\wp(t)}{t} dt.$$

If  $A = C = 1$  and  $B = D = -1$ , we have following

**Corollary: 5.1** If  $f \in C_{1(s)}$ , then

$$\begin{aligned} |a_{2n}| &\leq \frac{1}{n} \left[ 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n+1} + \dots \right], \\ |a_{2n+1}| &\leq \frac{1}{2n+1} \left[ \frac{1}{2n+1} + 2 \left( 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \right) \right]. \end{aligned}$$

**Theorem: 5.2** Let  $f \in C'_{1(s)}(A, B; C, D)$ , then

$$(5.6) \quad |a_{2n}| \leq \frac{(C - D)}{(2n)^2} \left\{ 1 + \left( \sum_{k=2}^n \frac{(A - B)}{2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right) \right\},$$

$$(5.7) \quad |a_{2n+1}| \leq \frac{1}{(2n + 1)^2} \left[ \frac{(A - B)}{2^n n!} \prod_{j=1}^{n-1} (A - B + 2j) + (C - D) \left\{ 1 + \sum_{k=2}^n \frac{(A - B)}{2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right].$$

The results are sharp.

**Proof:** We have

$$(5.8) \quad (zf'(z))' = \wp(z)H'(z)$$

Equating coefficients of  $z^{2n-1}$  and  $z^{2n}$  in (5.8), we have

$$(5.9) \quad (2n)^2 a_{2n} = (2n - 1)p_1 c_{2n-1} + (2n - 3)p_3 c_{2n-3} + \dots + 3p_{2n-3} c_3 + p_{2n-1},$$

$$(5.10) \quad (2n + 1)^2 a_{2n+1} = (2n + 1)c_{2n+1} + (2n - 1)p_2 c_{2n-1} + \dots + 3p_{2n-2} c_3 + p_{2n}.$$

Using lemmas 2.3 and 2.5 in (5.9), we get

$$(2n)^2 |a_{2n}| \leq (C - D) \left( 1 + \sum_{k=2}^n (2k - 1) |c_{2k-1}| \right) \leq (C - D) \left( 1 + \left\{ \sum_{k=2}^n \frac{(A - B)}{2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right)$$

which is the result (5.6).

From (5.10) by lemmas 2.3 and 2.5, we obtain

$$(2n+1)^2 |a_{2n+1}| \leq (2n+1) |c_{2n+1}| + (C-D) \left( 1 + \sum_{k=2}^n (2k-1) |c_{2k-1}| \right) \\ \leq \frac{(A-B)}{2^n n!} \prod_{j=1}^{n-1} (A-B+2j) + (C-D) \left( 1 + \left\{ \sum_{k=2}^n \frac{(A-B)}{2^{k-1} (k-1)!} \prod_{j=1}^{k-2} (A-B+2j) \right\} \right)$$

from which (5.7) follows.

Choosing  $H(z) = \frac{h_0(z)-h_0(-z)}{2}$ ,  $h_0(z)$  is defined by (2.4), and  $\wp(z) = \frac{1+Cz}{1+Dz}$ , the extremal function is

$$f(z) = \left[ \int_0^z \frac{1}{t} \left\{ \int_0^t H'(u) \wp(u) du \right\} dt \right].$$

Taking  $A = C = 1$  and  $B = D = -1$  in the theorem, we get

**Corollary 5.2** If  $f \in C_{1(s)}$ , then  $|a_{2n}| \leq \frac{1}{2n}$  and  $|a_{2n+1}| \leq \frac{1}{2n+1}$ . That is  $|a_n| \leq \frac{1}{n}$ .

Proceeding as in theorem 4.2 and using lemmas 2.3 and 2.5, we can prove

**Theorem 5.3** Let  $f \in C_{1(s)}(\alpha; A, B; C, D)$ . Then

$$(5.11) \quad |a_2| \leq \frac{(C-D)}{2(1+\alpha)},$$

$$(5.12) \quad |a_3| \leq \frac{(1+2\alpha)(A-B)+6(C-D)}{18(1+2\alpha)},$$

$$(5.13) \quad |a_4| \leq \frac{(C-D)}{24(1+\alpha)(1+3\alpha)} [(1+5\alpha)(A-B) + 6(1+\alpha)],$$

$$(5.14) \quad |a_5| \leq \frac{1}{600(1+2\alpha)(1+4\alpha)} \left[ 3(1+2\alpha)(1+4\alpha)(A-B)(A-B+2) \right. \\ \left. + 20(C-D)\{(1+8\alpha)(A-B) + 6(1+2\alpha)\} \right],$$

$$(5.15) \quad |a_6| \leq \frac{(C-D)}{2160(1+\alpha)(1+3\alpha)(1+5\alpha)} \left[ 40\alpha|\alpha-1|(A-B)^2 + 9(1+3\alpha)(1+9\alpha)(A-B)(A-B+2) \right. \\ \left. + 60(1+\alpha)(1+11\alpha)(A-B) + 360(1+\alpha)(1+3\alpha) \right].$$

The extremal function is obtained by choosing  $H(z) = \frac{h_0(z)-h_0(-z)}{2}$  and  $\wp(z) = \frac{1+Cz}{1+Dz}$  in the integral representation formula

$$(i) \quad \text{for } \alpha = 0, f(z) = \int_0^z \frac{H(t)\wp(t)}{t} dt,$$

$$(ii) \text{ for } \alpha > 0, f(z) = (1+c) \left[ \int_0^z \frac{1}{t(H(t))^c} \left\{ \int_0^t (H(u))^c H'(u) \wp(u) du \right\} dt \right], \left( c = \frac{1}{\alpha} - 1 \right).$$

Letting  $A = C = 1$  and  $B = D = -1$ , we have

**Corollary 5.3** If  $f \in C_{1(s)}(\alpha)$ , then

$$|a_2| \leq \frac{1}{(1+\alpha)},$$

$$|a_3| \leq \frac{7+2\alpha}{9(1+2\alpha)},$$

$$|a_4| \leq \frac{2(1+2\alpha)}{3(1+\alpha)(1+3\alpha)},$$

$$|a_5| \leq \frac{43+158\alpha+24\alpha^2}{75(1+2\alpha)(1+4\alpha)},$$

$$|a_6| \leq \begin{cases} \frac{69+488\alpha+523\alpha^2}{135(1+\alpha)(1+3\alpha)(1+5\alpha)}, & 0 \leq \alpha \leq 1, \\ \frac{69+448\alpha+563\alpha^2}{135(1+\alpha)(1+3\alpha)(1+5\alpha)}, & \alpha \geq 1. \end{cases}$$

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**ABBREVIATED TITLE:** *New sub-classes of the class of close to convex functions with respect to symmetric points.*

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