International Journal of Mathematical Archive-11(4), 2020, 37-41
Available online through www.ijma.info ISSN 2229 – 5046

DISCRITIZATION OF WAVELET FUNCTION FOR COMPUTATION OF DERIVATIVE

RAJENDRA PANDEY*[1] AND PREETI NAGAR[2]

Department of Mathematics,
Govt. Model Science College, Jabalpur (M.P.), India.

Ph.D. Student,
Department of Mathematics and Com. Science,
R.D. University, Jabalpur (M.P.), India.

(Received On: 08-02-20; Revised & Accepted On: 12-03-20)

ABSTRACT

Maurice Hasson and Rajendra Pandey have constructed wavelet function using Gaussian function and Maxican Hat function. Richardson extrapolation technique has been used in their constructions. There wavelet functions are applicable for computation of derivatives. In this paper, the authors have derived the discrete analogue of the wavelet function which can be applied to determine the first derivative of function. The discrete version is use full for better approximation of derivatives and the estimate of error can be analysed properly.

Keywords: *Band pass filters, Error estimate, Dirac delta function, Richardson extrapolation.*

1. INTRODUCTION

Maurice Hasson in 2006[4] constructed the wavelet by using Gaussian function with the help of it is second derivative and also discussed about discrete analogue of the wavelet. Furthermore continue this work R. Pandey [5] in 2012, constructed new wavelet by using first derivative of Gaussian function.

In this paper we discuss discrete analogue by using first derivative of Gaussian function and prove some important result and also find more optimal value of step length by applying the idea of error estimate. This result is more optimal comparison of machine procedure.

2. PRELIMINARIES

In this paper we use the following notations for the Fourier transform f (ω) of a function f(x).

$$\hat{f}(\omega)= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx$$

and the inverse formula takes the form

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x)e^{i\omega x}\, d\omega$$

We begin with the function Ψ(x) which is the derivative of the Gaussian function.

$$\Psi(x) = -xe^{\frac{-x^2}{2}}$$

It's Fourier transform is

$$\hat{f}(\omega) = \sqrt{2\pi}\,(i\omega)e^{\frac{-\omega^2}{2}}$$

and

$$\hat{\Psi}1(x) = \frac{\Psi(\omega)}{\sqrt{2\pi}}$$

$$\hat{\Psi}1(\omega) = (i\omega)e^{-\frac{\omega^2}{2}}$$

$$\hat{\Psi}1(\omega) = (i\omega)(1-\frac{\omega^2}{2} + \frac{\omega^4}{8} - \frac{\omega^8}{48} \dots \dots)$$

$$\int_{-\infty}^{\infty} x\Psi_1(x)\, dx = -1$$

$$\hat{\Psi}_1\left(\frac{\omega}{2}\right) = \frac{i\omega}{2}(1 - \frac{\omega^2}{2} + \frac{\omega^4}{128} - \frac{\omega^6}{3072} \dots \dots \dots)$$

Corresponding Author: Rajendra Pandey*[1],
Department of Mathematics, Govt. Model Science College, Jabalpur (M.P.), India.

By using classical Richardson extrapolation

$$\widehat{\Psi}_2(\omega)=\frac{\widehat{\Psi}_1(\omega)-\widehat{\Psi}_1(\frac{\omega}{2})}{-3}$$

We have it Is inverse Fourier transform

$$\Psi_2(x)=\frac{\Psi_1(x)-16\Psi_1(2x)}{-3}$$

The wavelet $\Psi_2(x)$ satisfying the following property

$$\int_{-\infty}^{\infty}t^k\Psi_2(t)dt=0,k=0,2,3,4 \tag{1}$$

$$\int_{-\infty}^{\infty}t\Psi_2(t)dt=-1,k=1 \tag{2}$$

3. CONSTRUCTION OF THE DISCRETE ANALOGUE OF THE WAVELET $\boldsymbol{\Psi_2}(x)$.

This section is devoted in building the discrete analogue of the wavelet $\Psi_2(x)$. Our result deals with the discrete analogue which we call $T_2(x,h)$ of the wavelet $\frac{1}{h}\Psi_2(\frac{x}{h})$. The properties of distributions and their fourier transform will be used in this section. The aim of building $T_2(x,h)$ is twofold. First is to analysis the error $\int_{-\infty}^{\infty}f(x-t)\frac{1}{h}\Psi_2\left(\frac{t}{h}\right)dt-f'(x)$ for a given value of the machine epsilon of the computer and the second is to compare the filtering feature of $\frac{1}{h}\Psi_2(\frac{t}{h})$with those of difference quotient. The process of building $T_2(x,h)$ is performed as follows. Let

$$T_1(x,h)=\frac{\delta(x+h)}{2}-\frac{\delta(x-h)}{2}-\delta(x)$$

Here δ(x) is usual direct mass at 0. Then

$$\int_{-\infty}^{\infty}f(x-t)T_1(x,h)dt=\frac{f(x+h)-f(x-h)}{2}-f(x)$$

Hence

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)T_1(x,h)dt=f'(t)$$

We also have

$$\int_{-\infty}^{\infty}t^kT_1(t,1)dt=0,k=0,2$$

$$\int_{-\infty}^{\infty}tT_1(t,1)dt=-1,k=1$$

Here $T_1(x,h)$is discrete analogue of the wavelet $\frac{1}{h}\Psi_1(\frac{x}{h})$. Recall now that

$$\Psi_2(x)=\frac{\Psi_1(x)-16\Psi_1(2x)}{-3}$$

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

The distribution $T_2(x,h)$ is defind by

$$T_2(x)=\frac{1}{-3}\left[\frac{\delta(x+h)}{2}-\frac{\delta(x-h)}{2}-\delta(x)\right]+\frac{16}{3}\left[\frac{\delta\left(x+\frac{h}{2}\right)}{4}-\frac{\delta\left(x-\frac{h}{2}\right)}{4}-\frac{\delta(x)}{2}\right]$$

$$T_2(x)=\frac{-\delta(x+h)}{6}+\frac{\delta(x-h)}{6}+\frac{\delta(x)}{3})+\frac{16\delta(x+\frac{h}{2})}{12}-\frac{16\delta(x-\frac{h}{2})}{12}-\frac{16\delta(x)}{6}$$

By using the analogue result for $T_2(x,h)$ that we also have

$$\int_{-\infty}^{\infty}t^kT_2(t,1)dt=0,k=0,2,3,4$$

and

$$\int_{-\infty}^{\infty}tT_2(t,1)dt=-1,k=1$$

and

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)T_2(t,h)dt=f'(x)$$

Theorem1:

$$\lim_{h\to0}\int_{-\infty}^{\infty}\frac{1}{h}T_2(x,h)dx=\lim_{n\to\infty}\frac{1}{h}\left(\frac{1}{h}\Psi_2\left(\frac{x}{h}\right)\right)=\delta'(x) \tag{3}$$

In the sence of distribution. Here δ'(x) is the distribution.

$<\delta'(x),\rho(x)>=\rho'(0)$ for ρ belongs to S.

$$\int_{-\infty}^{\infty}\frac{1}{h}f(x-t)T_2(x,t)dt-f'(x)=C_1f^5(x)h^4+C_2f^7(x)h^6+O(h^8) \tag{4}$$

$$C_1=\frac{1}{5!}\int_{-\infty}^{\infty}t^5T_2(t,1)dt \tag{5}$$

$$C_2=\frac{1}{7!}\int_{-\infty}^{\infty}t^7T_2(t,1)dt \tag{6}$$

$f^{'}(x)$ is approximated with $f_{pp}(x,h)$ where

$$f_{pp}(x,h) = \frac{w_1 f_1(x,h) + w_2 f_1(x,\frac{h}{2})}{h} \quad (7)$$

Here

$$f_1(x,h) = \frac{f(x+h) + f(x-h)}{2} - f(x)$$

$w_1 = \frac{-1}{3}$ and $w_2 = \frac{16}{3}$ recall that the machine epsilon $\varepsilon, \varepsilon_m$ of a computer is the smallest positive floating point number. Such that $1 + \varepsilon_m > 1$ when a number x entered as of the order $x\varepsilon_m$ is absolute error. The error occurring in $f(x)$ is $|e| \approx |xf^{'}(x)|$ by the mean value theorem. This is also a bound for the error in $f(x+h)$ because $h$is small.An error occurs also when we enter$h$. However, $\frac{1}{h(1+\varepsilon_m)} = \frac{1}{h}(1-\varepsilon_m)^{-1} = \frac{1}{h}(1 - \varepsilon_m + O(\varepsilon_m{}^2) \approx \frac{1}{h}$ Hence the absolute value of the round off error, E_{round}, in calculating the difference quotient $\frac{f(x-h)-f(x)}{h}$ is bounded by $|E_{round}| \leq \frac{2|e|}{h} \approx \frac{2|xf^{'}(x)|\varepsilon_m}{h}$ the truncation error E_{round} is $E_{trunc} = \frac{hf^{''}(\xi)}{2}$ where $x < \xi < x + h, (if\ h\ is\ small)$.

$$E_{tot} = E_{round} + E_{trunc}$$
$$|E_{tot}| \leq |E_{round}| + |E_{trunc}|$$

We conclude that the optimal step size h can be obtain by minimizing

$$|E_{round}| + |E_{trunc}| \leq \frac{2|xf^{'}(x)|\varepsilon_m}{|h|} + \frac{hf^{''}(x)}{2}$$

We consider the approximation of $f^{'}(x)$ by $\frac{f(x+h)-f(x-h)}{2h}$ is similar as above and $|E_{round}| \leq \frac{|xf^{'}(x)|\varepsilon_m}{h}$

Where the truncation error $|E_{trunc}| \leq \frac{|f^3(x)|h^2}{6}$ as before the truncation error

$$|E_{round}| + |E_{trunc}| \leq \frac{|xf^{'}(x)\varepsilon_m|}{|h|} + \frac{h^2 f^3(x)|}{6}$$

hence in the case of approximation $\frac{f(x-h)-f(x)}{h}$ the optimal h that we find is $\sqrt{c_1\varepsilon_m}$ and optimal error is $\sqrt{d_1\varepsilon_m}$, where as in the case $\frac{f(x+h)-f(x-h)}{2h}$ the optimal h is $(c_2\varepsilon_m)^{\frac{3}{2}}$ is $(d_2\varepsilon_m{}^2)^{\frac{3}{2}}$ where c_1 and c_2 ,d_1 and d_2 are constant.

Proof: As in the case of difference quotient

$$\frac{f(x+h)-f(x)}{h} \text{ and } \frac{f(x+h)-f(x-h)}{2h}$$

We assume because h is small, that $|e| \approx |xf^{'}(x)|\varepsilon_m$ beside being a bound for the error in $f(x)$ is also a bound for the error in $f(x+h), f(x-h), f\left(x+\frac{h}{2}\right)$ and $f(x-\frac{h}{2})$. it followes that the absolute value of the round off error E_{round} , in calculating the difference quotient $f_{pp}(x,h)$ is given by (7) is bounded by

$$|E_{trunc}| \leq \frac{2|e|}{h^2} \approx \frac{2(1+16)}{3}\left(\frac{|xf^{'}(x)|\varepsilon_m}{h^2}\right)$$

The value of C_1 ginen in (5). It is easily to found to be

$$C_1 = \frac{1}{5!}\frac{1}{3}[1 - \frac{1}{2^2}]$$

Hence truncation error $|E_{trunc}| = \frac{1}{5!}\frac{1}{3}\frac{3}{4}h^2|f^5(x)|$

For the total error $|E_{tot}| \leq |E_{round}| + |E_{trunc}|$

The optimal step size h can be obtained by minimizing h.

$$|E_{round}| + |E_{trunc}| = \frac{2(1+16)}{3}\frac{|xf^{'}(x)|\varepsilon}{h^2} + \frac{1}{5!}\frac{h^4|f^5(x)|}{4}$$

Or

$$|E_{round}| + |E_{trunc}| = 11.333333\frac{|xf^{'}(x)|\varepsilon}{h^2} + 0.002083333h^2|f^5(x)|$$

$$|E_{round}| + |E_{trunc}| = 11.333333\frac{|xf^{'}(x)|\varepsilon}{h^2} + 2.083333h^2 10^{-3}|f^5(x)|$$

The optimal step size can be obtain by minimizing (for h)

$$11.333333|xf^{'}(x)|\varepsilon_m \frac{d}{dh}(h^2) + 2.083333 \times 10^{-3}|f^5(x)|\frac{d}{dh}h^4$$

$$h^6 = \frac{11.333333 \times 10^3|xf^{'}(x)|\varepsilon_m}{2\times 2.083333\,|f^5(x)|} \quad \text{and} \quad h_{opt} = O(\sqrt[6]{\varepsilon_m})$$

$$h = \sqrt[6]{\frac{5440.0007104\,|xf^{'}(x)|\varepsilon_m}{2|f^5(x)|}}$$

$$h = 4.193724\sqrt[6]{\frac{|xf^{'}(x)|\varepsilon_m}{2|f^5(x)|}}$$

Assuming that

$$\sqrt[6]{\frac{|xf^{'}(x)|\varepsilon_m}{2|f^5(x)|}} \approx 1$$

we see that, With $\varepsilon_m = 2^{-52}$

$$\sqrt[6]{\varepsilon_m} = 0.002461$$
$$h_{opt} = 0.002461 \times 4.193724$$
$$h_{0pt} = 0.010321$$

The optimal value of the error E_{opt} obtained by using $h = h_{opt}$ is then

$$E_{opt} = \frac{11.333333\,|xf^{'}(x)|\varepsilon_m}{4.193724\,x}$$
$$x = \sqrt[6]{\frac{x^2}{2}}$$

4. THE FILTERING CHARACTERISTICS OF $T_2(x, h)$ COMPARED TO THOSE OF $\frac{1}{h}\Psi_2\left(\frac{x}{h}\right)$.

The error in approximating $f^{'}(x)$ by $\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)T_2(x,h)dx$ and wavelet $\frac{1}{h}\Psi_2(\frac{x}{h})$ is gives us the same error estimate The $T_2(x, h)$ has no filtering characteristics. Indeed it is fourier transform

$$\hat{T}(\omega) = \sum_0^1 a_k\left(e^{i2^{-k}h\omega} + e^{-i2^{-k}h\omega}\right) + C$$

has no decay. The wavelet $\frac{1}{h}\Psi_2(\frac{x}{h})$ is very effective band pass filter.

5. THE ERROR ESTIMATE $\frac{1}{h}\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)\Psi_2(\frac{t}{h})\,dt$ IN THE PRESENCE OF ε_m.

Theorem 2: $\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)\Psi_2(\frac{t}{h})\,dt - f^{'}(x) = C_1 - f^{'}(x) = C_1f^5(x)h^4 + C_2f^7(x)h^6 + O(h^8)$

Lemma 1: let ε, δ > 0 Then there exist η that $|h| < \eta$

$$\int_{|t|>\delta}\frac{1}{h}\left|\Psi_2\left(\frac{t}{h}\right)\right|dt < \varepsilon$$

Proof of theorem: Let a function of t and for fixed x, we have

$$\left|(x-t)f^{'}(x-t)\right| \le M$$

If f belongs to the class S the round off error due to the presence of ε_m, in computing $f(x-t)$, is bounded by $|(x-t)f'(x-t)|\varepsilon_m$. Hence the round off error E_{round} , in computing $\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)\Psi_2(\frac{t}{h})\,dt$ is bounded by $\int_{-\infty}^{\infty}\frac{1}{h}\left|f^{'}(x-h)\right|(x-h)\varepsilon_m\Psi_2\left(\frac{t}{h}\right)|dt$

For $\varepsilon, \delta > 0$, let $\eta > 0$ as given by lemma1.

Hence w for $|h| < \eta$ it follows that for the error E_{round} we have

$$E_{round} \le \int_{|t|<\delta}\frac{1}{h}\left|f'(x-h)(x-h)\varepsilon_m\Psi_2\left(\frac{t}{h}\right)\right|dt + \varepsilon_m$$

Because ε can be chosen so small, we have

$$E_{round} \le |xf^{'}(x)|\varepsilon_m\int_{-\infty}^{\infty}|\Psi_2(t)|dt$$

In view of

$$\int_{|t|<\delta}\left|\Psi_2\left(\frac{t}{h}\right)\right|dt < \int_{-\infty}^{\infty}\frac{1}{h}\left|\Psi_2\left(\frac{t}{h}\right)\right|dt = \int_{-\infty}^{\infty}|\Psi_2(t)|dt$$

Hence the round off error E, due to the precence of ε_m, in computing

$$\frac{1}{h^2}\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)\Psi_2\left(\frac{t}{h}\right)dt$$

Is $E \le \frac{|xf^{'}(x)|\varepsilon_m}{h^2}\int_{-\infty}^{\infty}|\Psi_2(t)|dt$ (8)

The truncation error in

$$\frac{1}{h}\int_{-\infty}^{\infty}\frac{1}{h}f(x-h)\Psi_2\left(\frac{t}{h}\right)dt - f^{'}(x) \text{ is}$$

$$E_{trunc} = C_1 f^5(x)h^4 + C_2 f^7(x)h^6 + O(h^8) \tag{9}$$

Hence theorem 2 follows from (8) and (9)

6. CONCLUSION

By applying the idea of error estimation of discrete analogue, error is precisely minimized for $h = 0.010254$. This result is more optimal for error approximation of machine procedure.

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