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REFINEMENT OF S. BERNSTEIN INEQUALITY

JAHANGEER HABIBULLAH GANAI*[1] AND DR. ANJNA SINGH[2]

[1,2]Department of Mathematical Sciences,
A.P.S. University, Rewa (M.P.) 486003 India,

Govt. Girls P.G. College Rewa, (M.P), India.

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ABSTRACT

In the present paper we will discuss refinements of Bernstein's Inequality for the polynomials and will prove some results which will among other things also generalize.

Keywords: *Polynomial, Derivative, Inequality.*

INTRODUCTION

Suppose $F(x)$ be a polynomial of degree m and $F'(x)$ be its derivative. Concerning the estimate modulus of $F'(x)$ on the unit circle $|x|=1$ we know the inequality called as Bernstein's inequality

$$\max_{|x|=1}|F'(x)| \le m \max_{[x[=1}|F(x)| \tag{1.1}$$

Concerning the estimate modulus of $F(x)$ on a large circle $|x| = R > 1$, we get

$$\max_{|x|=R>1}|F(x)| \le R^m \max_{|x|=1}|F(x)| \tag{1.2}$$

Inequality (1.1) is an consequence of S.Bernstein's theorem on the derivative of a trigonometric polynomials. Inequality (1.2) is a simple deduction consequence of maximum modulus principle.

For the both (1.1) and (1.2) holds for the polynomial $F(x) = \beta x^m, |\beta| \ne 0,$ that is, if and only if $F(x)$ has all its zeros at the origin. It has been proved by Frappier, Ruscheweyh and Rahman that if $F(x)$ is a polynomial of degree m, then

$$\max_{|x|=1}|F'(x)| \le m \max_{1\le k\le 2m}\left|F(e^{\frac{ik\pi}{m}})\right| \tag{1.3}$$

Equation (1.3) clears represents a refinement of (1.1). since the maximum of $|F(x)|$ on the $|x|=1$ may be large than the maximum of $F(x)$ taken over the $2n^{th}$ roots of unity. take an example $F(x) = x^m + ib, b > 0.$ As it has been proved by the A.Aziz interesting refinement of (1.3) and hence Bernstein's Inequality (1.1) as well.

Corresponding Author: Jahangeer Habibullah Ganai[1],*
[1]Department of Mathematical Sciences, A.P.S. University, Rewa (M.P.) – 486003, India,

Theorem 1.1: If $F(x)$ is a polynomial of degree m, then for every given real β

$$\max_{|x|=1}|F'(x)| \leq \frac{m}{2}\left[M_{\beta} + M_{\beta+\pi}\right] \tag{1.4}$$

Where $M_{\beta} = \max_{1\leq k\leq m}\left|F(e^{\frac{i(\beta+2k\pi}{m}})\right|$ (1.5)

$M_{\beta+\pi}$ is obtained from (1.5) by replacing β by $\beta+\pi$. The result is best possible and equality in (1.4) holds for $F(x) = x^m + re^{i\beta},\ 1\leq r\leq 1$.

Theorem 1.2: If $F(x)$ is a polynomial of degree m, then for all real β and R >1.

$$\max_{|x|=1}|F(Rx) - F(x)| \leq \left(\frac{R^n-1}{2}\right)\left(M_{\beta} + M_{\beta+\pi}\right) \tag{1.6}$$

The result is best possible and equaliy in (1.6) holds for the polynomial $F(x) = x^m + re^{i\beta},\ -1\leq r\leq 1$. If we restrict ourselves to the class of polynomials having no zero in $|x|<1$, inequality (1.1) is sharpened. In fact P.Erdos conjectured and later P.D.Lax [5] verified that

$$\max_{|x|=1}|F'(x)| \leq \frac{m}{2}\max|F(x)| \tag{1.7}$$

Theorem 1.3: If $F(x)$ is a polynomial of degree having no zero in $|x|<1$, then for every real β

$$\max_{|x=1|}|F'(x)| \leq \frac{m}{2}\left[M_{\beta}^2 + M_{\beta+\pi}^2\right]^{1/2} \tag{1.8}$$

The result is best possible and equality in (1.8) holds for $F(x) = x^m + e^{i\beta}$.

Theorem 1.4: If $F(x)$ is a polynomial of degree m having no zero in $|x|<1$, then for every real β and R>1

$$\max_{|x|=1}|F(R(x) - F(x)| \leq \left(\frac{R^n-1}{2}\right)\left(M_{\beta}^2 + M_{\beta+\pi}^2\right)^{1/2} \tag{1.9}$$

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The result is sharp and equality in (1.9) holds for $F(x) = x^m + e^{i\beta}$

Now we will prove the theorem one by one.

Theorem A: If $F(x)$ is a polynomial of degree m having all its zeros in $|x|\geq k\geq 1$, then

$$\max_{|x|=1}|F'(x)|^2 \leq \frac{n^2}{2(1+k^2)}\left[M_{\beta}^2 + M_{\beta+2}^2\right] \tag{1.10}$$

Where M_{β} is defined by (1.5)

Taking k=1, Theorem A reduces to Theorem 1.3.

Theorem B: If $F(x)$ is a polynomial of degree n having all its zeros in $|x|\geq k\geq 1$, then for all real α and R>1,

$$|F(Rx) - F(x)| \leq \frac{R^n-1}{\sqrt{2(1+k^2)}}\left[M_{\beta}^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}} \tag{1.11}$$

Where $M_{\beta} + M_{\beta+\pi}$ are defined as in Theorem 1.1

Corollary 1: If $F(x)$ is a polynomial of degree m, then for all real β and $r \leq 1$,

$$\max_{|x|=1}\left|F(rz) - r^n F(x)\right| \leq \frac{1-r^n}{\sqrt{2(1+k^2)}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}} \tag{1.12}$$

Theorem C: If $F(x)$ is a polynomial of degree m having all its zeros in $|x| < k, k \leq 1$ then

$$\max_{|x|=1}\left|F'(x)\right| \leq \frac{n}{\sqrt{2(1+k^2)}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}} \tag{1.13}$$

Theorem D: If $F(x)$ is a polynomial of degree m having all its zeros on $|x| \leq k, k \leq 1$, then for all real α and R>1,

$$\left|F(Rx) - F(x)\right| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2n})}}\left[M_\beta + M_{\beta+\pi}^2\right]^{\frac{1}{2}} \tag{1.14}$$

Theorem E: If $F(x)$ is self inverse polynomial of degree m, then

$$\max_{|x|=1}\left|F'(x)\right| \leq \frac{n}{2}\sqrt{M_\beta^2 + M_{\beta+\pi}^2}, \tag{1.15}$$

where M_β is defined by (1.5)

Proof of theorem: for the proof of these theorems we need the following leemas

Leema 1: If $F(x)$ is a polynomial of degree m, then for $|x| = 1$ ad for every real β,

$$\left|F'(x)\right|^2 + \left|mF(x) - xF'(x)\right|^2 \leq \frac{m^2}{2}\left[M_\beta^2 + M_{\beta+\pi}^2\right] \tag{1.16}$$

Where $\left[M_\beta^2 + M_{\beta+\pi}^2\right]$ are defined as in Theorem 1.

Leema 2: If $F(x)$ is a polynomial of degree m having all its zeros in $|x| \geq k \geq 1$, then

$$k^s\left|F(e^{i\theta})\right| \leq \left|Q^s(e^{i\theta})\right|, 0 \leq \theta \leq 2\pi, \tag{1.17}$$

Where $Q(x) = zn\overline{F(\frac{1}{x})}$

Leema3: If $F(x)$ is a polynomial of degree m having all its zeros in $|x| < k, k \leq 1$, then

$$km\max_{|x|=1}\left|F'(x)\right| \leq \max_{|x|=1}\left|Q'(x)\right| \tag{1.18}$$

Where Q(x) is as mentioned in Leema 2.

Proof of theorem A: Let $Q(x) = xn\overline{F(\frac{1}{x})}$. Then

$$\left|Q'(x)\right| = \left|mF(x) - xF'(x)\right|, \ for |x| = 1$$

By using (1.10), we will get

$$\left|F'(x)\right| + \left|Q'(x)\right| \leq \frac{m^2}{2}\left[M_\beta^2 + M_{\beta+\pi}^2\right] \tag{1.19}$$

From equation (1.17) with s=1, we have

$$k\left|F'(x)\right| \leq \left|Q'(x)\right|, \ for |x| = 1$$

Hence

$$\begin{aligned}(1+k^2)\left|F'(x)\right|^2 &= \left|F'(x)\right|^2 + k^2\left|F'(x)\right|^2 \\ &\le \left|F'(x)\right|^2 + \left|Q'(x)\right|^2 \\ &\le \frac{m^2}{2}\left[M_\beta^2 + M_{\beta+\pi}^2\right]\end{aligned}$$

This gives

$$\left|F'(x)\right|^2 \le \frac{m^2}{2(1+k^2)}\left[M_\beta^2 + M_{\beta+\pi}^2\right] \tag{1.20}$$

Hence proved Theorem A.

Proof of Theorem B: We have $\forall t \ge 1$ and $0 \le \theta \le 2\pi$

$$\left|F'(te^{i\theta})\right| \le t^{n-1}\max_{|x|=1}\left|F'(x)\right| \tag{1.21}$$

Now applying Theorem 1 to the polynomial F(x) which is of degree m-1, we will get

$$\left|F'(tei\theta)\right| \le t^{m-1}\frac{m}{\sqrt{2(1+k^2)}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}$$

Hence for each $\theta, 0 \le \theta \le 2\pi$ and R>1, we have

$$\begin{aligned}\left|F(\mathrm{Re}^{i\theta}) - F(e^{i\theta})\right| &= \left|\int_1^R e^{i\theta}F'(te^{i\theta})dt\right| \\ &\le \int_1^R \left|F'(te^{i\theta})dt\right| \\ &\le \frac{m}{\sqrt{2(1+k^2)}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}\int_1^R t^{m-1}dt \\ &= \frac{R^m-1}{\sqrt{2(1+k)^2}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]\end{aligned}$$

This gives

$$\left|F(Rx) - F(x)\right| \le \frac{R^m-1}{\sqrt{2(1+k)^2}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}$$

Hence we get the required result.

Proof of Theorem C: We have from Leema 1

$$\left|F'(x)\right|^2 + \left|mF(x) - xF'(x)\right| \le \frac{m^2}{2}\left[M_\beta^2 + M_{\beta+\pi}^2\right] \tag{1.22}$$

$$\begin{aligned}(1+k^{2m})\max\left|F'(x)\right|^2 &= \left|F'(x)\right|^2 + k^{2m}\left|F'(x)\right|^2 \\ &= \left|F'(x)\right|^2 + \left|k^mF'(x)\right|^2\end{aligned}$$

$$\begin{aligned}(1+k^{2m})\max_{|x|=1}\left|F'(x)\right|^2 &\le \max_{|x|=1}\left[\left|F'(x)\right|^2 + \left|Q'(x)\right|^2\right] \\ &= \max_{|x|=1}\left[\left|F'(x)\right|^2 + \left|mF(x) - xF'(x)\right|^2\right] \\ &\le \frac{m^2}{2}\left[M_\beta^2 + M_{\beta+\pi}^2\right]\end{aligned}$$

$$\max_{|x|=1}\left|F'(x)\right| \le \frac{m}{\sqrt{2(1+k^{2m})}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}$$

Hence completes the Proof of the theorem.

Proof of Theorem D:

We have $\forall t \geq 1$ and $0 \leq \theta \leq 2\pi$

$$\left|F'(te^{i\theta})\right| \leq t^{n-1} \max_{|x|=1}\left|F'(x)\right|$$

Now applying Theorem 1 to the polynomial F(x) which is of degree m-1, we will get

$$\left|F'(tei\theta)\right| \leq t^{m-1}\frac{m}{\sqrt{2(1+k^2)}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}$$

Hence for each $\theta, 0 \leq \theta \leq 2\pi$ and R>1, we have

$$\begin{aligned}\left|F(\mathrm{R}e^{i\theta}) - F(e^{i\theta})\right| &= \left|\int_1^R e^{i\theta}F'(te^{i\theta})dt\right| \\ &\leq \int_1^R \left|F'(te^{i\theta})dt\right| \\ &\leq \frac{m}{\sqrt{2(1+k^2)}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}\int_1^R t^{m-1}dt \\ &= \frac{R^m - 1}{\sqrt{2(1+k)^2}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]\end{aligned}$$

This gives

$$\left|F(Rx) - F(x)\right| \leq \frac{R^m - 1}{\sqrt{2(1+k)^2}}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}}$$

Hence we get the required result.

Proof of Theorem E: Now $F(x) = x m\overline{F(\frac{1}{x})}$

We have

$$F'(x) = mx^{m-1}F(\frac{1}{x}) - x^{m-2}F(\frac{1}{x})$$

$$xF'(x) = mx^{m}F(\frac{1}{x}) - x^{m-1}F(\frac{1}{x})$$

$$xF'(x) = mF(x) - x^{m-1}F(\frac{1}{x})$$

$$\left|mF(x) - xF'(x)\right| = \left|F'(x)\right|, \ for \left|x\right| = 1$$

By using lemma 1 we have

$$2\left|F'(x)\right|^2 = \left|F'(x)\right|^2 + \left|mF(x) - xF'(x)\right|^2$$

$$\left|F'(x)\right|^2 \leq \frac{m^2}{4}\left[M_\beta^2 + M_{\beta+\pi}^2\right]$$

$$\left|F'(x)\right| \leq \frac{m}{2}\left[M_\beta^2 + M_{\beta+\pi}^2\right]^{\frac{1}{2}},$$

Hence proved.

REFERENCES

1. Abdul Aziz and Q.G.Mohammad, Simple Proof of a theorem of Erdos and Lax, Proc. Amer. Math. Soc.80 (1980) 119-122.
2. Abdul Aziz, Inequalities for Polynomials with a Prescribed Zero, J. Approx. Theory, 41(1984), 15-20.
3. Abdul Aziz, A Refinement of an Inequality of S. Bernstein, Journal of Mathematical Analysis and Applications,Vol.144 No.1 November 1989, 226-235
4. A. C. Schaeffer, Inequalities of A. Markoff and S. Bernstein for Polynomials and related functions, Bull. Amer. Math. Soc.47 (1941), 565-579.
5. C.Frappier, Q. I. Rahman and St. Ruscheweyh, New Inequalities for Polynomials, Trans. Amer. Math. Soc. 288(1985) 69-99.
6. G. Polya and G. Szego, Aufgaben und Lehrsatze aus der Analysis, Springer-Verlag, Berlin 1925.
7. M. Riesz, Uber einen Satz des Herrn Serge Bernstein, Acta Math. 40(1916), 337-347.
8. N. K. Govil and Q. I. Rahman, Functions of exponential type not vanishing in a half plane and Related Polynomials,Tran.Amer.Math.Soc.137(1969), 501-517.
9. N. K. Govil, Some Inequalities for Derivatives of Polynomials, J. Approx. Theory, 66(1991), 29-35.
10. P. D. Lax, Proof of a Conjecture of P.Erdos on the derivative of a Polynomial, Bull. Amer. Math. Soc. 50 (1944) 509 -513.

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