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ON $f_{\lambda,\mu}(2\nu+3,\pm 1)$-STRUCTURE MANIFOLDS

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ABSTRACT

The idea of f-structure on a differentiable manifold was initiated and developed by Yano [6], Ishihara [1], Andreaou [2] and others. In the present paper, we have defined a structure $f_{\lambda,\mu}(2\nu+3,\pm 1)$ *of rank r on a manifold with a tensor field f of type (1, 1) satisfying* $\left(f^{2\nu+3}-\lambda^2 f\right)\left(f^{2\nu+3}+\mu^2 f\right)=0$ *Some results on this structure have been proved in this paper.*

(1.1) PRELIMINARIES:

Let M^n be an n-dimensional differentiable manifold of class C^∞ and of rank r and $\mathrm{f}(\neq 0)$ be a tensor field of type (1, 1), such that

(1.1.1) $$\left(f^{2\nu+3}-\lambda^2 f\right)\left(f^{2\nu+3}+\mu^2 f\right)=0$$

where $f^{2\nu+2}\neq\lambda^2$, $f^{2\nu+2}\neq-\mu^2$; $\lambda,\ \mu\in R^+$, $\lambda\neq\mu$ and rank of

$$f=\frac{1}{2}\left(\text{rank}\, f^{2\nu+2}+\dim M^n\right)=r=\text{constant}.$$

Let us define tensor fields $'l'$ and $'m'$ of type (1, 1) on M^n, by

(1.1.2) $$l=-\frac{f^{2\nu+2}-\lambda^2}{\mu^2+\lambda^2}\ ,\quad m=\frac{f^{2\nu+2}+\mu^2}{\lambda^2+\mu^2}$$

We have the following theorem:

Theorem 1.1.1: Let M^n be an $f_{\lambda,\mu}(2\nu+3,\pm 1)$-structure manifold, then

(1.1.3) $$l+m=I\ ,\ l^2=l\ ,\ m^2=m\ ,\ lm=ml=0$$

Proof: In view of the equations (1.1.1) and (1.1.2), the proof of the theorem follows in obvious manner.

Thus, the operators l and m when applied to tangent space of M^n at a point are complementary projection operators. Thus there exist complementary distributions L and M corresponding to projection operators l and m respectively. Let us call such a structure as then -structure of rank *r*.

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For the manifold M^n equipped with $f_{\lambda,\mu}(2\nu+3,\pm 1)$ -structure of rank r, we have the following theorem:

Theorem 1.1.2: On an $f_{\lambda,\mu}(2\nu+3,\pm 1)$-structure manifold, we have

(1.1.4) $$fl = -\frac{f^{2\nu+3} - f\lambda^2}{\lambda^2+\mu^2} \quad , \quad fm = \frac{f^{2\nu+3} + f\mu^2}{\lambda^2+\mu^2}$$

(1.1.5) $$f^{2\nu+2} l = -\mu^2 l \quad , \quad f^{2\nu+2} m = \lambda^2 m$$

(1.1.6) $$m - l = \frac{2f^{2\nu+2} - \lambda^2 - \mu^2}{\lambda^2+\mu^2}$$

Proof: In view of the equations (1.1.1), (1.1.2) and (1.1.3), the proof of the theorem follows immediately.

(1.2) $f_{\lambda,\mu}(2\nu+3,\pm 1)$-STRUCTURE IN LOCAL COORDINATES:

We now introduce a local coordinate system in the manifold M^n represented by $\mathrm{f}^i_j, l^i_j, m^i_j$ in local components of tensors f, l and m respectively. We also introduce in M^n a positive definite Riemannian metric by taking r mutually orthogonal unit vectors $\mathrm{v}^j_a\,(a,b,c,.....=1,2,3,.....,r)$ in L and (n-r) mutually orthogonal unit vectors $\mathrm{v}^j_A\,(A,B,C,.....=r+1,r+2,.......,n)$ in M, and then we have

(1.2.1) $$l^i_j \mathrm{v}^j_b = \mathrm{v}^i_b, \; l^i_j \mathrm{v}^j_B = 0$$
$$m^i_j \mathrm{v}^j_b = 0, \; m^i_j \mathrm{v}^j_B = \mathrm{v}^i_B$$

Let $\left(s^a_j, s^A_j\right)$ represents the matrix inverse of $\left(\mathrm{v}^j_b, \mathrm{v}^j_B\right)$, then s^a_j and s^A_j are both components of linearly independent covariant vectors and satisfy the relations

(1.2.2) $$s^a_j \mathrm{v}^j_b = \delta^a_b, \; s^a_j \mathrm{v}^j_B = 0$$
$$s^A_j \mathrm{v}^j_b = 0, \; s^A_j \mathrm{v}^j_B = \delta^A_B$$

and

(1.2.3) $$s^a_j \mathrm{v}^i_a + s^A_j \mathrm{v}^i_A = \delta^i_j$$

δ^i_j being the Kronecker delta.

If we put

(1.2.4) $$g_{kj} = s^a_k s^a_j + s^A_k s^A_j$$

then g_{kj} is globally well defined positive Riemannian metric such that

$$s^a_k = g_{kj}\mathrm{v}^j_a \quad , \quad s^A_k = g_{kj}\mathrm{v}^j_A$$

In view of the equations (1.2.1) and (1.2.2), we get

$$\left(l^i_j \mathrm{s}^a_i\right)\mathrm{v}^j_b = \delta^a_b, \; \left(l^i_j \mathrm{s}^a_i\right)\mathrm{v}^j_B = 0$$

(1.2.5) $$\left(m^i_j \mathrm{s}^A_i\right)\mathrm{v}^j_b = 0, \left(m^i_j \mathrm{s}^A_i\right)\mathrm{v}^j_B = \delta^A_B$$

Thus we have,

$$l_j^i s_i^a = s_j^a \;,\; m_j^i s_i^A = s_j^A$$

(1.2.6)

$$l_j^i s_i^A = 0, m_j^i s_i^a = 0$$

Using $l_j^i v_a^j = v_a^i$ in equation (1.2.6), we have

$$l_k^i s_j^a v_a^k = s_j^a v_a^i \;,\; l_k^i \left(\delta_j^k - s_j^A v_A^k\right) = s_j^a v_a^i$$

Hence

(1.2.7) $$l_j^i = s_j^a v_a^i$$

Similarly, we get

(1.2.8) $$m_j^i = s_j^B v_B^i$$

Let us now put

(1.2.9) $$l_{kj} = l_k^r g_{rj} \quad \text{and} \quad m_{kj} = m_k^r g_{rj}$$

In view of the equations (1.2.4), (1.2.7) and (1.2.8), we have

(1.2.10) $$l_{kj} = s_k^a s_j^a \;,\; m_{kj} = s_k^A s_j^A$$

and

(1.2.11) $$l_{kj} = l_{jk} \;,\; m_{kj} = m_{jk}$$

Consequently, we have

(1.2.12) $$l_{jk} + m_{jk} = g_{jk}$$

The following equations can be proved easily:

(i) $l_k^r l_j^p g_{rp} = l_{kj}$

(ii) $l_k^r m_j^p g_{rp} = 0$

(1.2.13) and

(iii) $m_k^r m_j^p g_{rp} = m_{kj}$

For any two vectors X and Y with components X^i and Y^i, let us put

$$m(X,Y) = m_{rp} X^r Y^p$$

(1.2.14) and

$$g(X,Y) = g_{rp} X^r X^p$$

(1.2.15)

$$\tilde{g}(X,Y) = \frac{1}{2}\frac{1}{(\nu+1)}\left[g(X,Y) + g(fX,fY) + g\left(f^2X,f^2Y\right) + \ldots\ldots + g\left(f^{2\nu+1}X,f^{2\nu+1}Y\right) + m(X,Y)\right]$$

Thus we have

$$m\left(v_A,v_A\right) = g\left(v_A,v_A\right) = g\left(fv_A,fv_A\right) = g\left(f^2v_A,f^2v_A\right) = \ldots\ldots = g\left(f^{2\nu+1}v_A,f^{2\nu+1}v_A\right) = 0$$

and

$$\tilde{g}\left(\mathrm{v}_A,\mathrm{v}_A\right)=\frac{1}{2}\frac{1}{(\nu+1)}\left[g\left(\mathrm{v}_A,\mathrm{v}_A\right)+g\left(\mathrm{fv}_A,\mathrm{fv}_A\right)+g\left(\mathrm{f}^2\mathrm{v}_A,\mathrm{f}^2\mathrm{v}_A\right)+......+g\left(\mathrm{f}^{2\nu+1}\mathrm{v}_A,\mathrm{f}^{2\nu+1}\mathrm{v}_A\right)+m\left(\mathrm{v}_A,\mathrm{v}_A\right)\right]$$
$$=0$$

By virtue of the fact that the distributions L and M are orthogonal with respect to the Riemannian metric g , the distributions L and M are orthogonal with respect to $\tilde{g}$ also. Hence, we have the following theorem:

Theorem 1.2.1: Let M^n be an n-dimensional differentiable manifold equipped with $f_{\lambda,\mu}(2\nu+3,\pm 1)$- structure of rank r. Then there exist complementary distributions L and M and a positive definite Riemannian metric $\tilde{g}$ with respect to which distributions are orthogonal.

Further, by virtue of equations (1.2.9), (1.2.11) and (1.2.12) it is easy to verify that

$$g(\mathrm{fv}_a,\mathrm{fv}_b)=l_{rp}\mathrm{f}_h^r\mathrm{f}_j^p\mathrm{v}_a^h\mathrm{v}_b^j$$
$$g(\mathrm{fv}_a,\mathrm{fv}_b)+m(\mathrm{fv}_a,\mathrm{fv}_b)=g_{rp}\mathrm{f}_h^r\mathrm{f}_j^p\mathrm{v}_a^h\mathrm{v}_b^j$$
$$g(\mathrm{f}^2\mathrm{v}_a,\mathrm{f}^2\mathrm{v}_b)=g_{rp}\mathrm{v}_a^r\mathrm{v}_b^p$$

These relations lead to the following:

(1.2.16) $$\tilde{g}(\mathrm{fX},\mathrm{fY})=g(\mathrm{X},\mathrm{Y}) \quad ; \quad \text{for all X, Y in L.}$$

Let M_1 be a space such that $\mathrm{X}\in\mathrm{M}_1$, $\mathrm{f(X)}=\lambda\mathrm{X}$ and let M_2 be the distribution orthogonal to M_1 in M with respect to $\tilde{g}$. We choose an orthogonal basis $\mathrm{u}_{\mathrm{n-r+1}},..............,\mathrm{u}_{2(\mathrm{n-r})}$ with respect to $\tilde{g}$ for M_2. Further, let $\mathrm{e}_1,\mathrm{e}_2,.........,\mathrm{e}_{2\mathrm{r-n}}$ be an orthogonal basis for L with respect to $\tilde{g}$. Using $\tilde{g}$, we can define a Riemannian metric g on M^n by

$$g(\mathrm{e}_i,\mathrm{e}_k)=\tilde{g}(\mathrm{e}_i,\mathrm{e}_k)$$
$$g(\mathrm{e}_i,\mathrm{u}_\alpha)=\tilde{g}(\mathrm{e}_i,\mathrm{u}_\alpha)$$
$$g(\mathrm{u}_\alpha,\mathrm{u}_\beta)=\tilde{g}(\mathrm{u}_\alpha,\mathrm{u}_\beta)$$
$$g(\mathrm{e}_i,\mathrm{f}(\mathrm{u}_\alpha))=\tilde{g}(\mathrm{e}_i,\mathrm{f}(\mathrm{u}_\alpha))$$
$$g(\mathrm{f}(\mathrm{u}_\alpha),\mathrm{u}_\beta)=0$$
$$g(\mathrm{f}(\mathrm{u}_\alpha),\mathrm{f}(\mathrm{u}_\beta))=\delta_{\alpha\beta}$$

where,

$$1\le i,k\le 2\mathrm{r-n} \quad , \quad \mathrm{n-r}+1\le\alpha,\beta\le 2n-r$$

then g is well defined because if $\tilde{\mathrm{u}}_{\mathrm{n-r+1}},............,\tilde{\mathrm{u}}_{2(\mathrm{n-r})}$ is another orthonormal basis for M_2, then for $\tilde{\mathrm{u}}_\alpha=z_\alpha^\beta\mathrm{u}_\beta$, we have

$$\tilde{g}(\tilde{\mathrm{u}}_\alpha,\tilde{\mathrm{u}}_\gamma)=\tilde{g}(\mathrm{z}_\alpha^\beta\mathrm{u}_\beta,\mathrm{z}_\gamma^\varepsilon\mathrm{u}_\varepsilon)$$
$$=\mathrm{z}_\alpha^\beta\mathrm{z}_\gamma^\varepsilon\delta_{\beta\varepsilon}$$
$$=\mathrm{z}_\alpha^\beta\mathrm{z}_\gamma^\beta$$
$$=\delta_{\alpha\gamma}$$

and, $$g(\mathrm{f}(\tilde{\mathrm{u}}_\alpha),\mathrm{f}(\tilde{\mathrm{u}}_\gamma))=g(z_\alpha^\beta\mathrm{f}(\mathrm{u}_\beta),z_\gamma^\varepsilon\mathrm{f}(\mathrm{u}_\varepsilon))$$

$$= z_\alpha^\beta z_\gamma^\varepsilon g\left(f\left(u_\beta\right), f\left(u_\varepsilon\right)\right)$$
$$= z_\alpha^\beta z_\gamma^\beta$$
$$= \delta_{\alpha\gamma}$$

This signifies that there is a Riemannian metric g with respect to which L, M_1, M_2 are mutually orthogonal and

$$g(fX, fY) = -\mu^2 f \quad ; \quad \text{for all X, Y in L}$$
$$g(fX, fY) = \lambda^2 f \quad ; \quad \text{for all X, Y in M}$$

Thus, we have:

Theorem 1.2.2: Let M^n be an n-dimensional differentiable manifold with $f_{\lambda,\mu}(2\nu+3,\pm 1)$-structure of rank r. Then there exists complementary distribution L of dimension (2r-n) and distribution M of dimension 2(n-r) and a positive definite Riemannian metric g with respect to which L and M are orthogonal and furthermore

$$g(fX, fY) = -\mu^2 f \quad ; \quad \forall\ X, Y \in L$$
$$g(fX, fY) = \lambda^2 f \quad ; \quad \forall\ X, Y \in L.$$

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