International Journal of Mathematical Archive-11(3), 2020, 20-29
Available online through www.ijma.info ISSN 2229 – 5046

SOME STUDIES ON I-ROUGH TOPOLOGICAL SPACES

BOBY P. MATHEW*[1] & SUNIL JACOB JOHN[2]
[1]Department of Mathematics,
St. Thomas College, Pala, 686 574, Kerala, India.

[2]Department of Mathematics,
National Institute of Technology, Calicut, 673 601, Kerala, India.

(Received On: 21-01-20; Revised & Accepted On: 18-02-20)

ABSTRACT

This paper extends the concepts of interior and closure of a set in the point set topology to the I-rough topological spaces by introducing the concepts of I-rough interior and I-rough closure of an I-rough set. I-rough dense subset and I-rough boundary of an I-rough set are also introduced and their properties are discussed. A necessary and sufficient condition for an I-rough subset is an I-rough dense subset is studied. In order to swot up the new concepts in to the relative I-rough topology this paper also explores some properties of I-rough subspaces. The structure of I-rough open sets and I-rough closed sets in the relative I-rough topology are studied.

Keywords: *Topology, rough universe, I-rough sets, I-rough topology, I-rough topological spaces, I-rough open sets, I-rough closed sets, I-rough base, I-rough sub-base, I-rough subspace, I-rough interior, I-rough closure, I-rough dense subst.*

INTRODUCTION

Rough set theory proposed by Pawlak [8] is a mathematical tool to deal with incomplete and imprecise data. Rough set theory expresses vagueness by employing a boundary region of a set by a pair of lower and upper approximations. If the boundary region is empty then the set is crisp, and if it is non-empty the set is a rough set. The non-empty boundary region represents our knowledge about the set is not sufficient to define the set precisely. The successful applications of rough set models in a variety of problems have amply demonstrated their usefulness and versatility [9, 10].

Iwinski [1] presented the set oriented view of rough set in an algebraic method using a pair of definable sets. It is by the help of a complete sub-algebra of the Boolean algebra of power set of a non-empty set [1]. A beautiful and elaborated review can be seen in Yao [14]. Also a review that compares constructive and algebraic approaches in the study of rough sets can be seen in Yao [15].

A topology on a non-empty set is a collection of subsets of it, satisfying certain axioms. A detailed study and historical notes of topology can be seen in Willard [12]. Many theories and applications are presented in Munkres [7]. Some works are done regarding the combinations of topology and generalizations of rough set theory [3, 13]. Some work regarding topological structures of rough sets induced by an equivalence relation can be seen in Kondo & Dudek [2]. Also Thuan [11] studied the covering based rough sets from a topological point of view.

Mathew and John [4, 5, 6] introduced some general topological structure on an arbitrary rough universe and several topological properties of the resultant I-rough topological spaces are studied. They focused on the topological properties of the rough universe based on the set oriented view of rough set given by Iwinski [1].

This paper is an attempt to extend the concepts of the I-rough topological space introduced by Mathew & John [5]. The structure of I-rough open sets and I-rough closed sets in the relative I-rough topology are studied. I-rough interior and I-rough closure of an I-rough set and I-rough dense subsets are introduced and their properties are discussed. A necessary and sufficient condition for an I-rough subset is an I-rough dense subset is introduced. Also I-rough clopen sets and I-rough boundary of an I-rough set are defined and a characterization of the I-rough boundary of an I-rough clopen set is discussed. By the investigation of these ideas this paper is an attempt to strengthen the I-rough topological spaces, the topology of the rough universe.

Corresponding Author: Boby P. Mathew[1]*
[1]Department of Mathematics, St. Thomas College, Pala, 686 574, Kerala, India.

PRELIMINARIES

Some of the basic definitions for our further study need to be quoted before introducing the new concepts.

Let U be any non-empty set and let β be a complete sub-algebra of the Boolean algebra 2^U of subsets of U. Then the pair (U,β) is called a rough universe [1]. Let (U,β) be a given fixed rough universe. Let R be a relation on β defined by $A=(A_1,A_2)\in R$ iff $A_1,A_2\in\beta$ and $A_1\subseteq A_2$. The elements of R are called rough sets and the elements of β are called exact sets [1]. In order to distinguish this definition of rough sets from Pawlak's definition, this rough set is named as an I-rough set [14].

The element $(X,\,X)\in R$ is identified with the element $X\in\beta$ and hence an exact set is a rough set in the sense of the above definition. But a rough set need not be exact. For example, if U is non-empty, then (ϕ,U) is a rough set which is not exact [1].

Set theoretic operators on the rough sets are defined component wise using ordinary set operations as follows [1]. Let $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ be any two I-rough sets in the rough universe (U,β). Then,

$$X\cup Y=(X_1\cup Y_1,X_2\cup Y_2)$$
$$X\bigcap Y=(X_1\bigcap Y_1,X_2\bigcap Y_2)$$
$X\subseteq Y$ if $X\bigcap Y=X$ .That is $X\subseteq Y$ if $X_1\subseteq Y_1$ and $X_2\subseteq Y_2$
$$X-Y=(X_1-Y_2,X_2-Y_1).$$

Hence $X^C=(U,U)-(X_1,X_2)=(U-X_2,U-X_1)=(X_2{}^C,X_1{}^C)$.

These operations satisfy De Morgan's Laws, and $(R,\cup,\cap)$ is a complete distributive lattice with zero element (ϕ,ϕ) and unit element (U,U) [1].

Ordinary set operations are frequently needed and hence for avoiding further confusions, the above set operations on I-rough sets are necessary to be named as I-rough union, I-rough intersection, I-rough inclusion, I-rough difference and I-rough complement respectively [5].

A topology on a set X is a collection τ of subsets of X, called the open sets, satisfying the following:

1) Any union of elements of τ belongs to τ.
2) Any finite intersection of elements of τ belongs to τ.
3) ϕ and X belongs to τ. [12].

If τ is a topology on X then (X, τ) is a topological space. If (X, τ) is a topological space, then $E\subseteq$ X is closed iff X-E is open [12].

Let X be any non-empty set. Let $\tau=\{\phi,X\}$. Then τ is a topology on X, the indiscrete topology on X. Let τ be the collection of all subsets of X. Then τ is a topology on X, the discrete topology on X [12].

Let (U,β) be a fixed rough universe associated with the complete sub-algebra β of the Boolean algebra 2^U . Then a sub collection τ of R, the set of all I-rough sets in (U,β) is an I-rough topology on (U,β) if the following 1, 2 and 3 hold.

1. $(\phi,\phi)\in\tau$ and $(U,U)\in\tau$
2. τ is closed under finite I-Rough intersection
3. τ is closed under arbitrary I-Rough union [5].

If τ be an I-rough topology on the rough universe (U, β). Then the triple (U, β, τ) is an I-rough topological space. An I-rough set $A = (A_1, A_2)$ is an I-rough open set in an I-rough topological space (U, β, τ) if $A = (A_1, A_2) \in \tau$ and an I-rough set $A = (A_1, A_2)$ is an I-rough closed set if its I-rough complement $A^C = (U - A_2, U - A_1)$ is I-rough open [5].

Theorem: [5] Let F be the family of all I-rough closed sets of an I-rough topological space (U, β, τ). Then F has the following properties.

(i) $(\phi, \phi) \in F$ and $(U, U) \in F$
(ii) F is closed under finite I-Rough union
(iii) F is closed under arbitrary I-Rough intersection.

Every topological space (U, τ) can be considered as an I-rough topological space, since there is a space (U, β, τ) induced from (U, τ). The converse is not true. Hence topological spaces are properly contained inside the collection of all I-rough topological spaces [5].

A subfamily $\mathbf{B}$ of τ in an I-rough topological space (U, β, τ) is an I-rough base for τ if every member of τ can be expressed as the I-rough union of some sub collections of members of $\mathbf{B}$. That is τ can be recovered from $\mathbf{B}$ by taking all possible I-rough unions of sub collections from $\mathbf{B}$ [5].

A family S of I-rough sets of the rough universe (U, β) is an I-rough sub-base for the I-rough topological space (U, β, τ) if the family of all finite I-rough intersections of members of S is an I-rough base for τ [5].

Let τ be an I-rough topology defined on the rough universe (U, β). Let A is an exact subset of (U, β).Then $\tau / A = \{G \cap A = (G_1 \cap A, G_2 \cap A) / G = (G_1, G_2) \in \tau\}$ is an I-rough topology on the rough universe $(A, \beta / A)$ induced by τ, where $\beta / A = \{X \cap A / X \in \beta\}$ is the complete sub-algebra β of the Boolean algebra 2^U restricted to A. Then τ / A is known as the relative I-rough topology on A or the subspace I-rough topology on A and $(A, \beta / A, \tau / A)$ is known as an I-rough subspace of the I-rough topological space (U, β, τ) [5].

This paper uses the notations of crisp set operations on two different contests. While dealing with crisp sets, these notations represents ordinary crisp set operations and when dealing with I-rough sets they denotes I-rough set operations. For example the notation $C \cap D$ means C and D are crisp sets and $\cap$ is the ordinary crisp set intersection. But in the expression $(C_1, C_2) \cap (D_1, D_2)$, the notation $\cap$ is the I-rough set intersection and is given by $(C_1, C_2) \cap (D_1, D_2) = (C_1 \cap D_1, C_2 \cap D_2)$. Note that in $C_1 \cap D_1$ and $C_2 \cap D_2$, $\cap$ is the ordinary crisp set intersection.

I-ROUGH SUBSPACES

Even though the main objective of this paper is to introduce and investigate the properties of I-rough closure and I-rough interior of an I-rough set, it is important to explore some properties of I-rough subspaces. Then only we can swot up the new concepts in to the relative I-rough topology. Hence in this section the structure of I-rough open sets and I-rough closed sets in the relative I-rough topology are studied. Also the sufficient conditions for an I-rough open or I-rough closed sets with respect to the relative I-rough topology are respectively I-rough open or I-rough closed sets with respect to the I-rough topology are discussed.

Theorem 1: Let $(Y, \beta / Y, \tau / Y)$ be an I-rough subspace of the I-rough topological space (U, β, τ). Let (A_1, A_2) be an I-rough open subset with respect to the relative I-rough topology τ / Y and (Y, Y) is I-rough open with respect to τ. Then (A_1, A_2) is I-rough open with respect to τ.

Proof: Given that (A_1, A_2) is an I-rough open subset with respect to the relative I-rough topology τ/Y. Hence $(A_1, A_2) = (G_1, G_2) \cap (Y, Y)$, where $(G_1, G_2) \in \tau$. It is also given that $(Y, Y) \in \tau$. Hence $(G_1, G_2) \cap (Y, Y) = (A_1, A_2) \in \tau$, since an I-rough topology is closed under finite I-rough intersections. Thus (A_1, A_2) is I-rough open with respect to τ.

Theorem 2: Let $(Y, \beta/Y, \tau/Y)$ be an I-rough subspace of the I-rough topological space (U, β, τ). Then an I-rough set (D_1, D_2) is an I-rough closed subset with respect to the relative I-rough topology τ/Y iff $(D_1, D_2) = (C_1, C_2) \cap (Y, Y)$, where (C_1, C_2) is an I-rough closed subset of (U, β, τ).

Proof: First suppose that (D_1, D_2) is an I-rough closed subset with respect to the relative I-rough topology τ/Y. Then the I-rough complement $(Y - D_2, Y - D_1) \in \tau/Y$. Then by the definition of relative I-rough topology, $(Y - D_2, Y - D_1) = (G_1, G_2) \cap (Y, Y)$, where $(G_1, G_2) \in \tau$. That is $(Y - D_2, Y - D_1) = (G_1 \cap Y, G_2 \cap Y)$. This implies $Y - D_2 = G_1 \cap Y$ and $Y - D_1 = G_2 \cap Y$. That is $D_2 = (U - G_1) \cap Y$ and $D_1 = (U - G_2) \cap Y$. Thus $(D_1, D_2) = ((U - G_2) \cap Y, (U - G_1) \cap Y)$. That is $(D_1, D_2) = (U - G_2, U - G_1) \cap (Y, Y)$. Which implies $(D_1, D_2) = (G_1, G_2)^C \cap (Y, Y)$. Since $(G_1, G_2) \in \tau$, $(G_1, G_2)^C$ is I-rough closed with respect to τ. Now let $(G_1, G_2)^C = (C_1, C_2)$. Then $(D_1, D_2) = (C_1, C_2) \cap (Y, Y)$, where (C_1, C_2) is an I-rough closed subset of (U, β, τ).

Conversely assume $(D_1, D_2) = (C_1, C_2) \cap (Y, Y)$, where (C_1, C_2) is an I-rough closed subset of (U, β, τ). That is $(D_1, D_2) = (C_1 \cap Y, C_2 \cap Y)$. Now since (C_1, C_2) is an I-rough closed subset implies $(C_1, C_2)^C$ is an I-rough open set in (U, β, τ). That is $(U - C_2, U - C_1) \in \tau$. Then $(C_1, C_2)^C \cap (Y, Y)$ is I-rough open with respect to $(Y, \beta/Y, \tau/Y)$. That is $(C_1, C_2)^C \cap (Y, Y) = (U - C_2, U - C_1) \cap (Y, Y) = ((U - C_2) \cap Y, (U - C_1) \cap Y) = (Y - D_2, Y - D_1) \in \tau/Y$. That is $(Y, Y) - (D_1, D_2) \in \tau/Y$. Hence (D_1, D_2) is an I-rough closed subset with respect to the relative I-rough topology τ/Y.

Theorem 3: Let $(Y, \beta/Y, \tau/Y)$ be an I-rough subspace of the I-rough topological space (U, β, τ). Let (D_1, D_2) be an I-rough closed subset with respect to the relative I-rough topology τ/Y and (Y, Y) is I-rough closed with respect to τ. Then (D_1, D_2) is I-rough closed with respect to τ.

Proof: Given that (D_1, D_2) be an I-rough closed subset with respect to the relative I-rough topology τ/Y. Then $(D_1, D_2) = (C_1, C_2) \cap (Y, Y)$, where (C_1, C_2) is an I-rough closed set of the I-rough topological space (U, β, τ). Then $(C_1, C_2) \cap (Y, Y)$ is I-rough closed, since (Y, Y) is I-rough closed with respect to τ and I-rough intersection of I-rough closed sets are again I-rough closed. Hence $(D_1, D_2) = (C_1, C_2) \cap (Y, Y)$ is I-rough closed with respect to τ.

I-ROUGH CLOSURE

Any I-rough open set of the I-rough topological space (U, β, τ) generates an I-rough closed set; since an I-rough set is an I-rough open set iff its I-rough complement is an I-rough closed set. Besides the I-rough open sets, associated with every I-rough set there is a unique I-rough closed set associated with it, the I-rough closure of it. It is defined and some properties are studied in this section.

Definition 1: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then the I-rough closure of (A_1, A_2) is denoted by $\overline{(A_1, A_2)}$, and is defined by the I-rough intersection of all I-rough closed subsets of (U, β, τ), containing (A_1, A_2). That is the I-rough closure $\overline{(A_1, A_2)} = \cap \left\{ (C_1, C_2) / \left(C_2{}^C, C_1{}^C\right) \in \tau \;\&\; (A_1, A_2) \subseteq (C_1, C_2) \right\}$.

Theorem 4: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then $\overline{(A_1, A_2)}$ is an I-rough closed subset. Moreover $\overline{(A_1, A_2)}$ is the smallest I-rough closed set containing (A_1, A_2).

Proof: We have $\overline{(A_1, A_2)} = \bigcap\left\{(C_1, C_2)/\left(C_2^{\,C}, C_1^{\,C}\right) \in \tau \ \& \ (A_1, A_2) \subseteq (C_1, C_2)\right\}$. Since arbitrary I-rough intersections of I-rough closed sets are again I-rough closed, $\overline{(A_1, A_2)}$ is an I-rough closed subset. Also being the I-rough intersections of all I-rough closed sets containing (A_1, A_2), $\overline{(A_1, A_2)}$ is the smallest I-rough closed set containing (A_1, A_2).

Theorem 5: If (A_1, A_2) is an I-rough closed subset of the I-rough topological space (U, β, τ), iff $\overline{(A_1, A_2)} = (A_1, A_2)$.

Proof: If (A_1, A_2) is itself an I-rough closed subset of the I-rough topological space (U, β, τ), then (A_1, A_2) is the smallest I-rough closed set containing (A_1, A_2). Then by theorem 4, $\overline{(A_1, A_2)} = (A_1, A_2)$.
Conversely if $\overline{(A_1, A_2)} = (A_1, A_2)$, then clearly (A_1, A_2) is I-rough closed, since $\overline{(A_1, A_2)}$ is an I-rough closed subset by theorem 4.

Corollary 1: In an I-rough topological space (U, β, τ), $\overline{(\phi, \phi)} = (\phi, \phi)$ and $\overline{(U, U)} = (U, U)$.

Proof: Since $(\phi, \phi) = (U, U)^C$ and $(U, U) = (\phi, \phi)^C$, in any I-rough topological space (U, β, τ), (ϕ, ϕ) and (U, U) are I-rough closed subsets. Hence the proof follows directly by theorem 5.

Theorem 6: In an I-rough topological space (U, β, τ), if $(A_1, A_2) \subseteq (C_1, C_2)$,then $\overline{(A_1, A_2)} \subseteq \overline{(C_1, C_2)}$

Proof: From theorem 4, $\overline{(C_1, C_2)}$ is the smallest I-rough closed set containing (C_1, C_2). Then since $(A_1, A_2) \subseteq (C_1, C_2)$, $\overline{(C_1, C_2)}$ is an I-rough closed set containing (A_1, A_2). Then $\overline{(A_1, A_2)} \subseteq \overline{(C_1, C_2)}$, since $\overline{(A_1, A_2)}$ is the smallest I-rough closed set containing (A_1, A_2).

Theorem 7: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then $\overline{\overline{(A_1, A_2)}} = \overline{(A_1, A_2)}$.
Proof: Let (A_1, A_2)be any subset of the I-rough topological space (U, β, τ), then $\overline{(A_1, A_2)}$ is an I-rough closed set by theorem 4. Then by theorem 5, $\overline{\overline{(A_1, A_2)}} = \overline{(A_1, A_2)}$.

Theorem 8: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) and (D_1, D_2) be any two I-rough sets of the rough universe (U, β), then $\overline{(A_1, A_2) \cup (D_1, D_2)} = \overline{(A_1, A_2)} \cup \overline{(D_1, D_2)}$.

Proof: Clearly $(A_1, A_2) \subseteq \overline{(A_1, A_2)}$and $(D_1, D_2) \subseteq \overline{(D_1, D_2)}$, since the I-rough closure of an I-rough set is the smallest I-rough closed set containing it. Which implies $(A_1, A_2) \cup (D_1, D_2) \subseteq \overline{(A_1, A_2)} \cup \overline{(D_1, D_2)}$. But being the I-rough union of two I-rough closed set, $\overline{(A_1, A_2)} \cup \overline{(D_1, D_2)}$ is an I-rough closed set. Hence $\overline{(A_1, A_2)} \cup \overline{(D_1, D_2)}$ is an I-rough closed set containing $(A_1, A_2) \cup (D_1, D_2)$. Now $\overline{(A_1, A_2) \cup (D_1, D_2)}$ is the smallest I-rough closed set containing $(A_1, A_2) \cup (D_1, D_2)$. Therefore, $\overline{(A_1, A_2) \cup (D_1, D_2)} \subseteq \overline{(A_1, A_2)} \cup \overline{(D_1, D_2)}$.

Also $(A_1, A_2) \subseteq (A_1, A_2) \cup (D_1, D_2)$ and $(D_1, D_2) \subseteq (A_1, A_2) \cup (D_1, D_2)$. Then by theorem 6, $\overline{(A_1, A_2)} \subseteq \overline{(A_1, A_2) \cup (D_1, D_2)}$ and $\overline{(D_1, D_2)} \subseteq \overline{(A_1, A_2) \cup (D_1, D_2)}$. Then $\overline{(A_1, A_2)} \cup \overline{(D_1, D_2)} \subseteq \overline{(A_1, A_2) \cup (D_1, D_2)}$. From thee two implications $\overline{(A_1, A_2) \cup (D_1, D_2)} = \overline{(A_1, A_2)} \cup \overline{(D_1, D_2)}$.

I-ROUGH DENSE SUBSETS

I-rough dense subsets are introduced and their properties are discussed in this section. The necessary and sufficient condition for an I-rough set is an I-rough dense subset in an I-rough topological space is studied. The nature of I-rough closure of an I-rough set with respect to the relative I-rough topology is also investigated.

Definition 2: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then (A_1, A_2) is an I-rough dense subset if $\overline{(A_1, A_2)} = (U, U)$.

Theorem 9: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then (A_1, A_2) is an I-rough dense subset iff for every non-empty I-rough open set (G_1, G_2) of (U, β, τ), $(A_1 \cap G_1, A_2 \cap G_2) \neq (\phi, \phi)$.

Proof: Suppose (A_1, A_2) is an I-rough dense subset of an I-rough topological space (U, β, τ). If possible suppose (G_1, G_2) be a non-empty I-rough open set of (U, β, τ) such that $(A_1 \cap G_1, A_2 \cap G_2) = \phi$. Which implies $(A_1, A_2) \subseteq (U, U) - (G_1, G_2)$. That is $(A_1, A_2) \subseteq (U - G_2, U - G_1)$. Now (G_1, G_2) is a non-empty I-rough open set implies $(U - G_2, U - G_1)$ is an I-rough closed set and also is a proper I-rough set of (U, U). Since $\overline{(A_1, A_2)}$ is the smallest I-rough closed set containing (A_1, A_2), and (A_1, A_2) is included in the I-rough closed set $(U - G_2, U - G_1)$ implies $\overline{(A_1, A_2)} \subseteq (U - G_2, U - G_1)$.This is a contradiction, since $\overline{(A_1, A_2)} = (U, U)$. Hence our assumption is wrong and $(A_1 \cap G_1, A_2 \cap G_2) \neq (\phi, \phi)$.

Conversely suppose for every non-empty I-rough open set (G_1, G_2) of (U, β, τ), $(A_1 \cap G_1, A_2 \cap G_2) \neq (\phi, \phi)$. Let (C_1, C_2) be any proper I-rough closed set containing (A_1, A_2), then $(U - C_2, U - C_1)$ is a non-empty I-rough open set such that $(A_1, A_2) \cap (U - C_2, U - C_1) = \phi$. That is a contradiction to our assumption. Hence the only I-rough closed set containing (A_1, A_2) is (U, U). Hence $\overline{(A_1, A_2)} = (U, U)$. That is (A_1, A_2) is I-rough dense in (U, β, τ).

Remark: In the above theorem it is noticed that an I-rough set in an I-rough topological space is an I-rough dense set if it has non empty I-rough set intersection with every I-rough open sets. Actually it is enough to the I-rough set have non-empty I-rough intersection with every I-rough sets in an I-rough base. It is investigated in the following theorem.

Theorem 10: Let **B** be an I-rough base for an I-rough topological space (U, β, τ). Then an I-rough set (A_1, A_2) of the rough universe (U, β) is an I-rough dense subset iff $(A_1 \cap G_1, A_2 \cap G_2) \neq (\phi, \phi)$, for every (G_1, G_2) in **B**.

Proof: Suppose (A_1, A_2) is an I-rough dense subset of an I-rough topological space (U, β, τ). Let $(G_1, G_2) \in \mathbf{B}$ be arbitrary. Then clearly $(G_1, G_2) \in \tau$. Then by theorem 9, $(A_1 \cap G_1, A_2 \cap G_2) \neq (\phi, \phi)$.

Conversely suppose (A_1, A_2) is an arbitrary I-rough set such that for every (G_1, G_2) in **B**, the I-rough intersection $(A_1 \cap G_1, A_2 \cap G_2) \neq (\phi, \phi)$. Let (D_1, D_2) be an arbitrary I-rough open set of the I-rough topological space (U, β, τ). Since **B** is an I-rough base for (U, β, τ), (D_1, D_2) can be expressed as the I-rough union of some sub collection of members of **B**. Each of the I-rough sets in this sub collection are proper I-rough subset of (D_1, D_2), and by our assumption each of them has non-empty I-rough intersection with (A_1, A_2).
Hence $(A_1, A_2) \bigcap (D_1, D_2) \neq (\phi, \phi)$. That is $(A_1 \cap D_1, A_2 \cap D_2) \neq (\phi, \phi)$. Since (D_1, D_2) is arbitrary, by theorem 9, (A_1, A_2) is an I-rough dense subset of the I-rough topological space (U, β, τ).

Theorem 11: Let $(Y, \beta/Y, \tau/Y)$ be an I-rough subspace of the I-rough topological space (U, β, τ). Let (A_1, A_2) be an I-rough subset of (Y, Y). Let $\overline{(A_1, A_2)}$ be the I-rough closure of (A_1, A_2) in (U, β, τ). Then the I-rough closure of (A_1, A_2) in $(Y, \beta/Y, \tau/Y)$ is $\overline{(A_1, A_2)} \cap (Y, Y)$.

Proof: Let (C_1, C_2) be the I-rough closure of (A_1, A_2) in $(Y, \beta/Y, \tau/Y)$. Given that $\overline{(A_1, A_2)}$ is the I-rough closure of (A_1, A_2) in (U, β, τ). Then $\overline{(A_1, A_2)}$ is I-rough closed in (U, β, τ). Then $\overline{(A_1, A_2)} \cap (Y, Y)$ is I-rough closed in $(Y, \beta/Y, \tau/Y)$, by theorem 2. Now $(A_1, A_2) \subseteq \overline{(A_1, A_2)}$ and $(A_1, A_2) \subseteq (Y, Y)$, implies $(A_1, A_2) \subseteq \overline{(A_1, A_2)} \cap (Y, Y)$. That is $\overline{(A_1, A_2)} \cap (Y, Y)$ is an i-rough closed set containing (A_1, A_2). Then $(C_1, C_2) \subseteq \overline{(A_1, A_2)} \cap (Y, Y)$, since (C_1, C_2) is the smallest I-rough closed set in $(Y, \beta/Y, \tau/Y)$ containing (A_1, A_2).
Since (C_1, C_2) is the I-rough closure of (A_1, A_2) in $(Y, \beta/Y, \tau/Y)$, (C_1, C_2) is I-rough closed in $(Y, \beta/Y, \tau/Y)$. Hence $(C_1, C_2) = (D_1, D_2) \cap (Y, Y)$, where (D_1, D_2) is I-rough closed in (U, β, τ). Which implies $(C_1, C_2) \subseteq (D_1, D_2)$ and hence $(A_1, A_2) \subseteq (D_1, D_2)$. That is (D_1, D_2) is an I-rough closed set containing (A_1, A_2). Since $\overline{(A_1, A_2)}$ is the smallest I-rough closed set containing (A_1, A_2), it is clear that $\overline{(A_1, A_2)} \subseteq (D_1, D_2)$. Hence $\overline{(A_1, A_2)} \cap (Y, Y) \subseteq (D_1, D_2) \cap (Y, Y) = (C_1, C_2)$. That is $\overline{(A_1, A_2)} \cap (Y, Y) \subseteq (C_1, C_2)$. From these two implications $(C_1, C_2) = \overline{(A_1, A_2)} \cap (Y, Y)$.

Theorem 12: Let (U, β, τ), be an I-rough topological space and (A_1, A_2) is an I-rough dense subset of it. Let (Y, Y) be an I-rough open set of (U, β, τ). Then $(A_1, A_2) \cap (Y, Y)$ is an I-rough dense subset of the relative I-rough topology τ/Y.

Proof: Let (A_1, A_2) is an I-rough dense subset of (U, β, τ). Then to prove $(A_1, A_2) \cap (Y, Y)$ is an I-rough dense subset of the relative I-rough topology τ/Y, let (D_1, D_2) be any I-rough open set of the relative I-rough topology τ/Y. Then from the definition of the relative I-rough topology, $(D_1, D_2) = (G_1, G_2) \cap (Y, Y)$,
where $(G_1, G_2) \in \tau$ .Given that (Y, Y) is an I-rough open set of (U, β, τ). But $(G_1, G_2) \in \tau$ and $(Y, Y) \in \tau$ implies $(D_1, D_2) = (G_1, G_2) \cap (Y, Y) \in \tau$. Then by theorem 9, $(A_1, A_2) \bigcap (D_1, D_2) \neq (\phi, \phi)$. Since $(D_1, D_2) \cap (Y, Y) = (D_1, D_2)$, the expression can be rewritten as $(A_1, A_2) \bigcap [(D_1, D_2) \cap (Y, Y)] \neq (\phi, \phi)$. Since I-rough set intersection is associative, $[(A_1, A_2) \bigcap (Y, Y)] \cap (D_1, D_2) \neq (\phi, \phi)$. Since (D_1, D_2) is an arbitrary I-rough open set of the relative I-rough topology τ/Y, it is concluded that $(A_1, A_2) \cap (Y, Y)$ is an I-rough dense subset of the relative I-rough topology τ/Y.

Remark: The condition in which (Y, Y) is an I-rough open set of (U, β, τ) is necessary in the above theorem. If (Y, Y) is not an I-rough open set of (U, β, τ) then $(A_1, A_2) \cap (Y, Y)$ need not be an I-rough dense subset of the relative I-rough topology τ/Y. Consider the following example.

Example 1: Let $U = \{p, q, r, s, t\}$ and let $\beta = 2^U$. Let $\tau = \{ (\phi, \phi), (U, U), (\{p\}, \{p, q\}), (\{p, q\}, \{p, q\}), (\{p, q\}, \{p, q, r\}), (\{p, q, r\}, \{p, q, r\}) \}$. Then the family of I-rough closed sets are given by $F = \{ (\phi, \phi), (U, U), (\{r, s, t\}, \{q, r, s, t\}), (\{s, t\}, \{r, s, t\}), (\{s, t\}, \{s, t\}), (\{r, s, t\}, \{r, s, t\}) \}$. Now consider the I-rough set $(A_1, A_2) = (\{p, s, t\}, \{p, r, s, t\})$. Then the I-rough closure of it is (U, U), since there is no other I-rough closed set contains it. Thus $(\{p, s, t\}, \{p, r, s, t\})$ is I-rough dense subset of (U, β, τ). Now let $(Y, Y) = (\{q, s, t\}, \{q, s, t\})$. Then note that (Y, Y) is not an I-rough open subset of (U, β, τ). Now consider $(A_1, A_2) \cap (Y, Y) = (\{p, s, t\}, \{p, r, s, t\}) \cap (\{q, s, t\}, \{q, s, t\}) = (\{s, t\}, \{s, t\})$. Then $\overline{(A_1, A_2) \cap (Y, Y)} = \overline{(\{s, t\}, \{s, t\})} = (\{s, t\}, \{s, t\})$. That is $\overline{(A_1, A_2) \cap (Y, Y)} \neq (Y, Y)$. That is $(A_1, A_2) \cap (Y, Y)$ is not an I-rough dense subset with respect to the relative I-rough topology.

I-ROUGH INTERIOR

I-rough open sets are those I-rough sets which are members of the I-rough topology. There are I-rough sets which are neither I-rough open nor I-rough closed. Just like the I-rough closure, associated with every I-rough set there is a unique I-rough open set associated with it, the I-rough interior of it. It is defined and some properties are studied in this section.

Definition 3: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then the I-rough interior of (A_1, A_2) is denoted by $(A_1, A_2)^0$ and is defined by the I-rough union of all I-rough open subsets of (U, β, τ), contained in (A_1, A_2). That is the I-rough interior

$$(A_1, A_2)^0 = \cup\{(G_1, G_2)/(G_1, G_2)\subseteq(A_1, A_2)\,\&\,(G_1, G_2)\in\tau\}.$$

Theorem 13: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), Then the I-rough interior $(A_1, A_2)^0$ of (A_1, A_2) is I-rough open. Moreover $(A_1, A_2)^0$ is the largest I-rough open set contained in (A_1, A_2).

Proof: Since arbitrary I-rough union of I-rough sets are again I-rough open, the I-rough interior $(A_1, A_2)^0$ of (A_1, A_2) is always I-rough open set of the I-rough topological space (U, β, τ). Also being the I-rough union of all I-rough sets contained in (A_1, A_2), it is the largest I-rough set contained in (A_1, A_2).

Theorem 14: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then (A_1, A_2) is I-rough open iff $(A_1, A_2)^0 = (A_1, A_2)$.

Proof: Suppose (A_1, A_2) is an I-rough open subset of the I-rough topological space (U, β, τ). Then (A_1, A_2) is the largest I-rough open set contained in (A_1, A_2). Hence by theorem 13, $(A_1, A_2)^0 = (A_1, A_2)$.

Also if (A_1, A_2) is an I-rough subset such that $(A_1, A_2)^0 = (A_1, A_2)$, then by theorem 13, (A_1, A_2) is an I-rough open subset of the I-rough topological space (U, β, τ).

Corollary 2: Let (U, β, τ) be an I-rough topological space, then $(\phi, \phi)^0 = (\phi, \phi)$ and $(U, U)^0 = (U, U)$.

Proof: The proof follows directly from theorem 14, since (ϕ, ϕ) and (U, U) are I-rough open subsets of (U, β, τ).

Theorem 15: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then $\left((A_1, A_2)^0\right)^0 = (A_1, A_2)^0$.

Proof: For any I-rough set (A_1, A_2), the I-rough interior $(A_1, A_2)^0$ is an I-rough open set by theorem 13. Then by theorem 14, $\left((A_1, A_2)^0\right)^0 = (A_1, A_2)^0$.

Theorem 16: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) and (D_1, D_2) be any two I-rough subsets of the rough universe (U, β), such that $(A_1, A_2) \subseteq (D_1, D_2)$, then $(A_1, A_2)^0 \subseteq (D_1, D_2)^0$.

Proof: From the definition of I-rough interior, clearly $(A_1, A_2)^0 \subseteq (A_1, A_2)$. Thus if $(A_1, A_2) \subseteq (D_1, D_2)$, then $(A_1, A_2)^0 \subseteq (D_1, D_2)$. That is $(A_1, A_2)^0$ is an I-rough open set contained in (D_1, D_2). Then $(A_1, A_2)^0 \subseteq (D_1, D_2)^0$, since $(D_1, D_2)^0$ is the largest I-rough open set contained in (D_1, D_2).

Theorem 17: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) and (D_1, D_2) be any two I-rough subsets of the rough universe (U, β), then $[(A_1, A_2) \cap (D_1, D_2)]^0 = (A_1, A_2)^0 \cap (D_1, D_2)^0$.

Proof: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) and (D_1, D_2) be any two I-rough subsets of the rough universe (U, β). Then from the definition of I-rough interior, $(A_1, A_2)^0 \subseteq (A_1, A_2)$ and $(D_1, D_2)^0 \subseteq (D_1, D_2)$. Then $(A_1, A_2)^0 \cap (D_1, D_2)^0 \subseteq (A_1, A_2) \cap (D_1, D_2)$. Now since, $(A_1, A_2)^0$ and $(D_1, D_2)^0$ are I-rough open by theorem 13 and I-rough intersection of two I-rough open sets are again an I-rough open set implies $(A_1, A_2)^0 \cap (D_1, D_2)^0$ is an I-rough open set. That is $(A_1, A_2)^0 \cap (D_1, D_2)^0$ is an I-rough open set contained in $(A_1, A_2) \cap (D_1, D_2)$. Hence $(A_1, A_2)^0 \cap (D_1, D_2)^0 \subseteq [(A_1, A_2) \cap (D_1, D_2)]^0$.

Also $[(A_1, A_2) \cap (D_1, D_2)] \subseteq (A_1, A_2)$ and $[(A_1, A_2) \cap (D_1, D_2)] \subseteq (D_1, D_2)$. Then by theorem 16, $[(A_1, A_2) \cap (D_1, D_2)]^0 \subseteq (A_1, A_2)^0$ and $[(A_1, A_2) \cap (D_1, D_2)]^0 \subseteq (D_1, D_2)^0$. Which implies $[(A_1, A_2) \cap (D_1, D_2)]^0 \subseteq (A_1, A_2)^0 \cap (D_1, D_2)^0$. Hence $[(A_1, A_2) \cap (D_1, D_2)]^0 = (A_1, A_2)^0 \cap (D_1, D_2)^0$.

I-ROUGH BOUNDARY

I-rough boundary of an I-rough set is defined in this section. Also I-rough clopen sets are defined and a characterization of the I-rough boundary of an I-rough clopen set is discussed here.

Definition 4: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then (A_1, A_2) is I-rough clopen set if (A_1, A_2) is both I-rough open and I-rough closed. That is if $(A_1, A_2) \in \tau$ and $(U, U) - (A_1, A_2) = (U - A_2, U - A_1) \in \tau$.

Definition 5: Let (U, β, τ) be an I-rough topological space and (A_1, A_2) be any I-rough subset of the rough universe (U, β), then the I-rough boundary of (A_1, A_2) is the I-rough set defined by $\overline{(A_1, A_2)} \cap \overline{(U, U) - (A_1, A_2)} = \overline{(A_1, A_2)} \cap \overline{(U - A_2, U - A_1)}$ and is denoted by $\partial(A_1, A_2)$.

Remark: The I-rough boundary of an I-rough set is always an I-rough closed set, since it is the I-rough intersection of two I-rough closed sets. Also from the definition it is clear that the I-rough boundary of an I-rough set is same as the I-rough boundary of its I-rough complement.

Theorem 18: For any I-rough set (A_1, A_2), the I-rough boundary $\partial(A_1, A_2) = (\phi, \phi)$ iff (A_1, A_2), is an I-rough clopen set.

Proof: First suppose that (A_1, A_2), is an I-rough clopen subset of the I-rough topological space (U, β, τ). That is (A_1, A_2) is both I-rough open and I-rough closed. Since (A_1, A_2) is I-rough closed, $\overline{(A_1, A_2)} = (A_1, A_2)$. Also since (A_1, A_2) is I-rough open, $(U, U) - (A_1, A_2) = (U - A_2, U - A_1)$ is I-rough closed and hence $\overline{(U - A_2, U - A_1)} = (U - A_2, U - A_1)$. Hence $\partial(A_1, A_2) = \overline{(A_1, A_2)} \cap \overline{(U - A_2, U - A_1)} = (A_1, A_2) \cap (U - A_2, U - A_1) = (A_1, A_2) \cap (A_1, A_2)^C = (\phi, \phi)$.

Conversely suppose that for an I-rough set (A_1, A_2), the I-rough boundary $\partial(A_1, A_2) = (\phi, \phi)$. That is $\overline{(A_1, A_2)} \cap \overline{(U, U) - (A_1, A_2)} = (\phi, \phi)$. Since $((U, U) - (A_1, A_2)) \subseteq \overline{(U, U) - (A_1, A_2)}$, it is clear that $\overline{(A_1, A_2)} \cap ((U, U) - (A_1, A_2)) = (\phi, \phi)$. But their I-rough union is (U, U). This means that $\overline{(A_1, A_2)} = (A_1, A_2)$. Hence by theorem 5, (A_1, A_2) is an I-rough closed set. In the preceding proof, by reversing the role of (A_1, A_2) and $(U,U)-(A_1, A_2)$ it follows that $(U,U)-(A_1, A_2)$ is also I-rough closed set. Which implies (A_1, A_2) is I-rough open. That is (A_1, A_2) is both I-rough open and I-rough closed and hence it is an I-rough clopen set.

CONCLUSION

This paper studies some basic concepts of the I-rough topological spaces. I rough topological spaces are the generalization of topological spaces in to a rough universe. The paper mainly focuses on I-rough subspaces, I-rough closure, I-rough interior, I-rough dense subsets, I-rough clopen sets and I-rough boundary of an I-rough set and several properties related to them are discussed. The structure of I-rough open sets and I-rough closed sets in the relative I-rough topology are studied. By the investigation of these ideas this paper is an attempt to enrich the I-rough topological spaces. With the help of the ideas presented here there are a lot of research scopes in this area.

REFERENCES

1. Iwinski, T.B., 1987. 'Algebraic approach to rough sets'. *Bull. Polish Acad. Sci. Math, 35*(2), pp.673-683.
2. Kondo, M. and Dudek, W.A., 2006. 'Topological Structures of Rough Sets Induced by Equivalence Relations'. *JACIII*, *10*(5), pp.621-624.
3. Lashin, E.F., Kozae, A.M., Khadra, A.A. and Medhat, T., 2005. 'Rough set theory for topological spaces'. *International Journal of Approximate Reasoning, 40*(1), pp.35-43.
4. Mathew, B. P. and John, S. J., 2012. 'On rough Topological Spaces'. *International Journal of Mathematical Archive*, 3(9), pp.3413-3421.
5. Mathew, B. P. and John, S. J., 2016. 'I-Rough Topological Spaces'. *International Journal of Rough Sets and Data Analysis, 3*(1), pp.98-113.
6. Mathew, B. P., & John, S. J. (2016). Some special properties of I-rough topological spaces. *Annals of Pure and Applied Mathematics*, *12*(2), 111-122.
7. Munkres, J.R. (1975), Topology. New Delhi: Printice-Hall of India.
8. Pawlak, Z., 1982. 'Rough sets'. *International Journal of Computer & Information Sciences*, *11*(5), pp.341-356.
9. Pawlak, Z., 1984. 'Rough classification'. *International Journal of Man-Machine Studies*, *20*(5), pp.469-483.
10. Pawlak, Z. (1991) Rough sets-theoretical aspect of reasoning about data. Boston: Kluwer Academic.
11. Thuan, N.D., 2012. 'Covering rough sets from a topological point of view'. *arXiv preprint arXiv:1207.6560*.
12. Willard, S. (1970) *General topology*. New York: Dover Publications Inc.
13. Zhu, W., 2007. 'Topological approaches to covering rough sets'. *Information sciences*, *177*(6), pp.1499-1508.
14. Yao, Y.Y., 1996. 'Two views of the theory of rough sets in finite universes'. *International journal of approximate reasoning*, *15*(4), pp.291-317.
15. Yao, Y.Y., 1998. 'Constructive and algebraic methods of the theory of rough sets'. *Information sciences*, *109*(1), pp.21-47.

Source of support: Nil, Conflict of interest: None Declared.