International Journal of Mathematical Archive-11(3), 2020, 17-19
Available online through www.ijma.info ISSN 2229 – 5046

A SHORT NOTE ON LOGARITHMIC CONVEXITY OF STOLARSKY'S MEAN TYPE FUNCTION

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(Received On: 11-01-20; Revised & Accepted On: 06-02-20)

ABSTRACT

In this short note the condition for logarithmic convexity of two parameter Stolarsky's mean type homogeneous function by using Taylor's series expansions is established.

Keywords: *Stolarsky's mean, Gini means, Homogeneous function, logarithmic convexity.*

2010 Mathematics Subject Classification: *Primary 26D10, secondary 26D15.*

1. INTRODUCTION

In1971K.B.Stolarsky's introduced one parameter mean by using logarithmic mean and later in 1975, the family of bivariate means with two-parameterare introduced and named as the Stolarsky's means in mathematical literature [9] which are given by;

$$G_{(p,q)}(a,b)=\begin{cases}\left(\dfrac{p(a^q-b^q}{q(a^p-b^p)}\right)^{1/(p-q)} & pq(p-q)\neq 0\\ exp\left(\dfrac{-1}{q}+\dfrac{a^q loga-b^q logb}{a^q-b^q}\right) & p=q\neq 0\\ \left(\dfrac{a^q-b^q}{q(loga-logb)}\right)^{\frac{1}{q}} & q\neq 0, p=0\\ \sqrt{ab} & p=q=0\\ a & b=a>0\end{cases}$$

The following well known means are the particular cases of Stolarsky's mean $G_{p,q}(a,b)$. Forexample, $G_{1,2}=\left(\frac{a+b}{2}\right)=A-$ arithmetic mean, $G_{0,0}=G_{1,1}=\sqrt{ab}=G-$ geometrical mean, $G_{0,1}=\frac{a-b}{loga-logb}=L-$ logarithmic mean, $G_{1,1}=\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}=I-$ identric mean, $G_{p,2p}=\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}=P-r^{th}$ power mean etc. Several authors studied in depth about this mean and published good number of articles in two or more variables. The comparative study of this mean value with Gini mean. Established some inequalities and discussed few convexity related results in ([1]-[10]).

The notations for the generalized weighted arithmetic and contra harmonic mean respectively denoted by $A(r,s;a,b)$ and $C(r,s;a,b)$. In [1], Janardhan *et al.*, established an inequality chain involving harmonic mean, arithmetic means, contra harmonic mean and other well-known means in two and *n* variables refer [7]. The weighted form of Stolarsky's mean and its extended generalization forms are introduced and discussed the Schur convexity and logarithmic convexity results by good number of authors ([6], [8, 10]).

The weighted forms of arithmetic mean and contra harmonic mean defined as;

$A=ra+sb$; and $C=\frac{ra^2+sb^2}{ra+sb}$, where $r,s,a,b\in R_+$; $r+s=1$.

Note that $C>A$ for $a\neq b$.

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The weighted two parameter homogeneous function was introduced in [4], which is defined as extended Stolarsky's mean class *G* as a particular case. Namely, for $r, s, a, b \in R_+, r + s = 1,\ pq(p-q)(a-b) \neq 0$, is denoted by $N = N_{p,q}(r,s;a,b)$ and defined as;

$$N = \left[\frac{p^2 A(r,s;a^q,b^q) - C(r,s;a^q,b^q)}{q^2 A(r,s;a^p,b^p) - C(r,s;a^p,b^p)}\right]^{\frac{1}{(q-p)}} \quad (1.1)$$

$$= \left[\frac{p^2}{q^2}\left(\frac{ra^p + sb^p}{ra^q + sb^q}\right)\left(\frac{a^q - b^q}{a^p - b^p}\right)\right]^{\frac{1}{q-p}} \quad (1.2)$$

Authors established the following identities;

$$N_{p,q}(r,s;a,b) = N_{q,p}(r,s;a,b)$$
$$N_{p,q}(r,s;a,b) = N_{p,q}(r,s;b,a)$$
$$N_{p,q}(r,s;b,a) = abN_{p,q}(r,s,a^{-1},b^{-1})$$
$$N_{xp,xq}(r,s;a,b) = \left[N_{q,p}(r,s;a^x,b^x)\right]^{\frac{1}{x}}, \quad x \neq 0$$

Note that

$$N_{2p,2q}(1/2,1/2,a,b) = \left[\frac{p^2}{q^2}\left(\frac{a^{2p} + b^{2p}}{a^{2q} + b^{2q}}\right)\left(\frac{a^{2q} - b^{2q}}{a^{2q} - b^{2q}}\right)^2\right]^{\frac{1}{2(q-p)}}$$

The two-parameter homogeneous function $N = N_{p,q}(r,s;a,b)$ is given by;

$$N_{p,q}(r,s;a,b) = \begin{cases} \left[\frac{p^2}{q^2}\left(\frac{ra^p + sb^p}{ra^q + sb^q}\right)\left(\frac{a^q - b^q}{a^p - b^p}\right)^2\right]^{\frac{1}{p-q}} & pq(p-q)(a-b) \neq 0 \\ \left[\frac{2}{\log^2\left(\frac{a}{b}\right)}\left(\frac{1}{ra^q + sb^q}\right)\left(\frac{a^q - b^q}{q}\right)^2\right]^{\frac{1}{q-p}} & q(a-b) \neq 0, p = 0 \\ exp\left[\frac{-2}{q} - \frac{ra^q loga + sb^q logb}{ra^q + sb^q} + 2\frac{a^q loga - b^q logb}{a^q - b^q}\right] & q(a-b) \neq 0, p = q \\ a^{(r+1)/3} b^{(r+1)/3} & a \neq b, p = q = 0 \\ a \quad b = a & \end{cases}$$

In articles [10] extension and application of Gini mean and Stolarsky's mean for two or more variable has been discussed.

Based on above literature survey, the following results are established.

Result 1: [10] The function $N_{p,q}(r,s;a,b)$ are monotone increasing w.r.t p or q and a or b.

Result 2: The function $N_{p,q}(r,s;a,b)$ is
(i) $N_{p,q}(r,s;a,b) \equiv N_{p,q}(r,s;b,a)$ is symmetric.
(ii) $N_{p,q}(r,s;\lambda a,\lambda b) = \lambda N_{p,q}(r,s;a,b)$ is homogeneous.

Based on the propositions, the property of Stolarsky's mean and lemmas from the articles [9, 10], the theorem below is stated.

Theorem 1.1: The homogeneous function $N_{p,q}(r,s;a,b)$ is logarithmic-convex for all $p, q \in I$

Proof: If the function $Np, q(r,s;a,b)$ is logarithmic-convex for all $p, q \in I$, then the expression;

$$T: N_{p,p-x}(r,s;a,b)N_{p,p+x}(r,s;a,b) - \left[N_{p,p}(r,s;a,b)\right]^2$$

For $a, b \in R_+, T$ as definite sign for all values.

Putting $b = 1, a^x = e^t, p/x = y$. then T reduces to

$$S := \left(\frac{r-1}{r+1}\right)^2\left(\frac{pe^{(r-1)t} + q}{pe^{(r+1)t} + q}\right)\left(\frac{e^{(r+1)t} - 1}{e^{(r-1)t} - 1}\right)^2 - exp\left(\frac{-4}{r} - \frac{2pe^{rt}t}{pe^{rt} + q} + \frac{4te^{rt}}{e^{rt} - 1}\right)$$

Developing in $t \in R$ and some laborious calculation, leads to;

$$\left(\frac{r-1}{r+1}\right)^2 \left(\frac{pe^{(r-1)t} + q}{pe^{(r+1)t} + q}\right)\left(\frac{e^{(r+1)t} - 1}{e^{(r-1)t} - 1}\right)^2 = 1 + t + \frac{1}{6}(3-r)t^2 + O[t]^3$$

and

$$exp\left(\frac{-4}{r} - \frac{2pe^{rt}t}{pe^{rt} + q} + \frac{4te^{rt}}{e^{rt} - 1}\right) = 1 + t + \frac{1}{6}(3-r)t^2 + O[t]^3$$

Therefore, the above equations together implies that;

$$S = \frac{1}{3}(r-2)(r+1)(2r-1)t^3 + O(t^4).$$

Hence S is a definite sign and sufficiently for very small real number t, only if $r = s = 1/2$.

The concept used to conclude the homogeneous function $N_{p,q}(r, s; a, b)$ is logarithmic-convexis based on the famous theorem in [8] by Feng Qi.

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Source of support: Nil, Conflict of interest: None Declared.