International Journal of Mathematical Archive-11(3), 2020, 7-11
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A MACLAURIN'S SERIES METHOD FOR THE SOLUTION OF THE INITIAL VALUE PROBLEMS FOR NTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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(Received On: 16-01-20; Revised & Accepted On: 07-02-20)

ABSTRACT

In this paper, a numerical method for solving the linear initial problems for nth order linear differential equation with constant coefficient and analytic initial condition of independent variable is presented. The technique is based upon the Maclaurin's Theorem. Properties and the initial condition for differential equation is presented. These properties are used to reduce the differential equation to Maclaurin's theorem of the system. The Maclaurin's theorem may not converge if the solution is not analytic in whole domain, however, the present method can be applied to initial value problem for linear differential equation, when the solution is analytic in the interior of the domain. The method is computationally very attractive and applications are demonstrated through illustrative examples.

Keywords: *Maclaurin's series, nth order linear differential equation, Initial value problems.*

INTRODUCTION

Numerical solutions of the second order linear differential equations by using Maclaurin's method have been investigated by many authors. However, there are few reference on the solution of the nth order linear differential equations by using the Maclaurin's Theorem. One advantage of the method of using Maclaurin's is that differentiable approximate solution is obtained, which can be replaced in the equation by the initial condition. In this manner, the accuracy of solution can be evaluated directly. It is the solution is a function of one variable (the case ODE), the Maclaurin's theorems may be obtained directly by differentiating the right side of the differential equation written in the normal form. However, this method is suitable only for initial value problems when an explicit recurrence formula is obtained to compute the Maclaurin's theorem.

In this manner, the problem is reduced to those of solving a system of algebraic equations.

The modern theory of differential equations is based on the expansion by Maclaurin's series of the solutions of the equations in infinite series. The striking analogy existing between the theory of algebraic equations and the theory of differential equations suggested the possibility of expressing the solutions of algebraic equations in series to be obtained by an application of Maclaurin's series. After some experiment the author happened on the device of introducing a factor X into all the terms but two of the equation f (y) = 0, whereby y becomes an implicit function of x. The successive x-derivatives of y are now formed, and together with y are evaluated for x = 0. By Maclaurin's series the expansions of y in powers of x become known. If x be made unity in these expansions, the roots of f(y) = 0 are found, provided the resulting series are convergent.

MACLAURIN'S SERIES

A **Maclaurin series** is a power **series** that allows one to calculate an approximation of a function f(x) for input values close to zero, given that one knows the values of the successive derivatives of the function at zero. In many practical applications, it is equivalent to the function it represents.

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A Maclaurin series is a Taylor series expansion of a function about 0,

$$f(x) = f(0) + f^{'}(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots \ldots \ldots \ldots \ldots \ldots + \frac{f^n(0)}{n!}x^n + \cdots \ldots$$

Maclaurin series are named after the Scottish mathematician Colin Maclaurin.

The Maclaurin series of a function $f(x)$ up to order n may be found using Series $[f, \{x, 0, n\}]$. The nth term of a Maclaurin series of a function f can be computed in the Wolfram Language using Series Coefficient $[f, \{x, 0, n\}]$ and is given by the inverse Z-transform

$$a_n = z^{-1}\left[\frac{1}{x}\right](n)$$

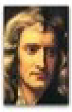
Maclaurin series are a type of series expansion in which all terms are nonnegative integer powers of the variable. Other more general types of series include the Laurent series and the Puiseux series.

DIFFERENTIAL EQUATION

In mathematics, **history of differential equations** traces the development of "differential equations" from calculus, itself independently invented by English physicist Isaac Newton and German mathematician Gottfried Leibniz.

The history of the subject of differential equations, in concise form, from a synopsis of the recent article "The History of Differential Equations, 1670-1950".

"**Differential equations** began with Leibtniz, the Bernoulli brothers, and others from the 1680s not long after Newton's "fluxional equation s in the 1670s"

Year		Mathematician	Description
1676		Isaac Newton (1643-1727) (IQ=190-200) English physicist and mathematician	Solved his first differential equation, by the use of infinite series, eleven years after his discovery of calculus in 1665.
1693		Gottfried Leibniz (1646-1716) (IQ=205) German mathematician	Solved his first differential equation, the year in which Newton first published his results.

The historical **origin** of differential equations is a mathematical tool invented independently by Isaac Newton (1676) and Gottfried Leibniz (1693).

Differential equations first came into existence with the invention of calculus by Newton and Leibniz., *Methodus fluxionum et Serierum Infinitarum,*[2] Isaac Newton listed three kinds of differential equations.

$$\frac{dy}{dx} = f(x)$$
$$\frac{dy}{dx} = f(x, y)$$

$$x_1\frac{\partial y}{\partial x_1} + x_2\frac{\partial y}{\partial x_2} = y$$

In all these cases, *y* is an unknown function of *x* (or of x_1 and x_2), and *f* is a given functions.

He solves these examples and others using infinite series and discusses the non- uniqueness of solutions.

Jacob Bernoulli proposed the Bernoulli differential equation in 1695.[3] This is an ordinary differential equation of the form

$$y^{'} + P(x)y + Q(x)^n$$

for which the following year Leibniz obtained solutions by simplifying it.[4]

Differential equations can be divided into several types. Apart from describing the properties of the equation itself, these classes of differential equations can help inform the choice of approach to a solution. Commonly used distinctions include whether the equation is: Ordinary/Partial, Linear/Non-linear, and Homogeneous/heterogeneous. This list is far from exhaustive; there are many other properties and subclasses of differential equations which can be very useful in specific contexts.

ORDINARY DIFFERENTIAL EQUATIONS

Ordinary differential equation and Linear differential equation

An ordinary differential equation (*ODE*) is an equation containing an unknown function of one real or complex variable *x*, its derivatives, and some given functions of *x*. The unknown function is generally represented by a variable (often denoted *y*), which, therefore, *depends* on *x*. Thus *x* is often called the independent variable of the equation. The term "*ordinary*" is used in contrast with the term partial differential equation, which may be with respect to *more than* one independent variable.

Linear differential equations are the differential equations that are linear in the unknown function and its derivatives. Their theory is well developed, and, in many cases, one may express their solutions in terms of integrals.

PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equation

A partial differential equation (*PDE*) is a differential equation that contains unknown multivariable functions and their partial derivatives. (This is in contrast to ordinary differential equations, which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved in closed form, or used to create a relevant computer model.

Initial value problem

In mathematics and particularly in dynamic systems, an initial condition, in some contexts called a seed value, is a value of an evolving variable at some point in time designated as the initial state (typically denoted *t* = 0).

MAIN RESULT

Nth order linear differential equation solve by Maclaurin's series method :

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. a_n y = 0 \qquad (1)$$

Let $y = e^{mx}\ be\ a\ solution\ of\ eq^n (1)$ we get auxillary equation

$$e^{mx}(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots \ldots \ldots \ldots . + a_n) = 0 \qquad (2)$$

Hence $e^{mx}\ will$ be a solution of (1) if m is the root of algebraic equation

If auxiliary equation have equal roots

If the auxiliary equation have two equal roots $m_1 = m_2$ the solution of the equation $f(D)y = 0\ is$

$$y = (c_1 + c_2 x)e^{m_1 x} \qquad (3)$$

Similarly if $f(D) = 0\ has\ three$ equal roots equal to m_1 hat the general solution of $f(D)y = 0\ is$

$$y = (c_1 + c_2 x + C_3 x^2)e^{m_3 x} + C_4 e^{m_4 x} + \cdots \ldots \ldots \ldots \ldots \ldots . + c_n e^{m^n x} \qquad (4)$$

Proceeding in the similar manner if $f(D) = 0\ has\ r\ equal\ roots$ equal to m_1 the general solution of $f(D)y = 0\ will\ be$

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 \ldots \ldots \ldots . c_{r-1} x^{r-2} + c_r x^{r-1}) \qquad (5)$$

By the use of Maclaurin's theorem

At $x = 0$

$$y(0) = c_1$$
$$y^{'}(0) = \quad (c_2 + 2c_3 x + 3c_4 x^2 + \cdots \ldots . (r-2)c_{r-1} x^{r-3} + (r-1)c_r x^{r-2}) \qquad (6)$$

At $x = 0$

$$y^{'}(0) = C_2$$
$$y^{''}(0) = (2c_3 + 3.2c_4 x + \cdots \ldots \ldots \ldots)$$

At $x = 0$

$$y^{''}(0) = 2c_3$$
$$c_3 = \frac{y^{''}(0)}{2!}$$
$$y^{'''}(0) = (3.2c_4 + \cdots \ldots \ldots \ldots)$$

At $x = 0$

$$y'''(0) = 3.2c_4$$
$$c_4 = \frac{y'''(0)}{3!}$$
$$y^r(0) = r!\,c_n$$

At $x = 0$

$$c_n = \frac{y^r(0)}{r!}$$

Substitute the value of c_1 , c_2, c_3, c_4.......,c_n in equation (5)

$$y = y(0) + y'(0)x + \frac{y''(0)x^2}{2!} + \frac{y'''(0)x^3}{3!} + \cdots \ldots \ldots .. \frac{y^r(0)x^n}{r!}$$

These are the Maclaurin's expanssion in nth order linear differential equation.

Example: $(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$

Here the auxiliary equation is:

$$m^4 - 7m^3 + 18m^2 - 20m + 8 = 0$$
$$m^4 - m^3 - 6m^3 + 6m^2 + 12m^2 - 12m + 8m + 8 = 0$$
$$m^3(m-1) - 6m^2(m-1) + 12m(m-1) - 8(m-1) = 0$$
$$(m-1)(m^3 - 6m^2 + 12m - 8) = 0$$
$$(m-1)(m-2)^3 = 0$$
$$m = 1, m = 2,2,2\ (three\ equal\ roots)$$

Hence the required solution is

$$y = c_1e^x + (c_2 + c_3x + c_4x^2)e^{2x}$$

By Maclaurin's theorem

At $x = 0$ in $(c_2 + c_3x + c_4x^2)e^{2x}$

By using of Maclaurin's theorem find the value arbitrary constant

$$y(0) = c_2 = c_1$$
$$y' = c_3 + 2c_4x$$

At $x = 0\ in$

$$y'(0) = c_3$$
$$y'' = 2c_4$$

At $x = 0$

$$c_4 = \frac{y''(0)}{2!}$$

Substitute the value of $c_1, c_2, c_3, c_4\ in\ equation(1)$

$$y = c_1 + y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \cdots ..$$

RESULT

In our main result we use the Maclaurin's series at initial condition $x = 0$ to solve nth order differential equation (auxiliary equation having equal roots) and find the value of arbitrary constant.

Maclaurin's expansion are obtain only initial condition at $x = 0$.

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Source of support: Nil, Conflict of interest: None Declared.