

**A SHORT APPROACH TO FIND OUT THE VALUE OF EACH (n-1) EQUAL ROOTS AND
A SIMPLE ROOT OF AN nth DEGREE POLYNOMIAL OF SINGLE VARIABLE**

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1. ABSTRACT

To show that there is an alternative easy approach/formula to solve a specific type of polynomial of any degree, say n which is said to have conditionally (n-1) equal roots while there exist a conventional approach of solving the same.

2. NOTATIONS

- i. $\alpha = -e =$ Equal root
- ii. $\beta = \frac{-p}{m} =$ Simple root
- iii. n = Degree of the polynomial/highest power of the polynomial
- iv. $D_n =$ Discriminant of the n^{th} degree polynomial
- v. $a_n =$ First term coefficient of the n^{th} degree polynomial
- vi. $a_{n-1} =$ second coefficient of the n^{th} degree polynomial
- vii. $a_{n-2} =$ Third coefficient of the n^{th} degree polynomial
- viii. $a_k = k^{th}$ coefficient of the n^{th} degree polynomial
- ix. $a_1 =$ Penultimate/last but one coefficient of the n^{th} degree polynomial
- x. $a_0 =$ Ultimate/Last term of the n^{th} degree polynomial
- xi. $r_1, r_2 =$ Roots of a quadratic equation of the form $ax^2 + bx + c = 0$
- xii. S = Sum of Roots of the n^{th} degree polynomial = $\frac{-a_{n-1}}{a_n}$
- xiii. $\binom{n}{r} =$ Combinations of r out of n = $\frac{(n-r)!}{n!r!}$
- xiv. $\forall =$ For all
- xv. R = Set of real numbers
- xvi. $\in =$ belongs to
- xvii. x = a variable

3. INTRODUCTION

A polynomial is a mathematical expression consisting of variables, coefficients and the operations of addition, subtraction, multiplication and non-negative integer exponents. Below are some examples of polynomial

$$x + 3, 3x^2 - 2x + 5, 2a^3b^2 - 3b^2 + 2a - 1, \frac{1}{2}x^2 - \frac{2}{3}x + \frac{3}{4}$$

Polynomials are an important part of the language of mathematics and algebra. They are used in nearly every field of Mathematics to express numbers as a result of mathematical operations. Polynomials are also “buildings blocks” in other types of mathematical expressions such as rational expressions.

Many mathematical processes that are done in everyday life can be interpreted as polynomials. Summing the cost of items on a grocery bill can be interpreted as a polynomial. Calculating the distance travelled of a vehicle or object can be interpreted as a polynomial. Calculating perimeter, area and volume of geometric figures can be interpreted as polynomials.

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3.1 Degree of Polynomial

The degree of a polynomial is the highest of the degrees/powers of its monomials with non-zero coefficients. It is the sum of the exponents of the variables that appear in it, and thus is a non-negative integer.

“0” degree polynomials are called constants. The value of Constants don’t change.

“1st” degree polynomials are called Linear polynomials. They are used to describe quantities that change at a steady rate. They are also used in many one-dimensional geometry problems involving length.

“2nd” degree polynomials are called Quadratic polynomials. They are used to describe quantities that change with some amount of acceleration or deceleration. They are also used in many two-dimensional geometry problems involving area.

“3rd” degree polynomials are also called Cubic polynomials. They are used in many three-dimensional geometry problems involving volume.

3.2 Polynomial Function

A function $f(x)$ is a polynomial function if it can be written as $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$.

In this form $a_0, a_1 \dots a_{n-1}, a_n$ are non-variable coefficients and n is a non-negative integer.

3.3 Remainder Theorem

When a polynomial $f(x)$ is divided by $(x - a)$ the remainder is “ $f(a)$ ”.

3.4 Factor Theorem

Let $f(x)$ be a polynomial function such that $f(c) = 0$ for some constant then $(x - c)$ is a factor of $f(x)$.

3.5 Relation between Coefficients and roots of a polynomial

According to Vieta’s formula [1], the Coefficients of polynomials are related to the sums and products of their Roots as well as the products of the roots taken in groups.

3.5.1 Vieta’s formula for Quadratics

Given $f(x) = ax^2 + bx + c$ if the equation $f(x) = 0$ has roots r_1 and r_2 then, Sum of roots (S) = $r_1 + r_2 = \frac{-b}{a}$ and the Product of roots (P) = $r_1 r_2 = \frac{c}{a}$

3.5.2 Generalization to Higher Degree Polynomials

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$ be a polynomial with complex coefficients and degree, having complex roots r_n, r_{n-1}, \dots, r_1 . Then for any integer $0 \leq k \leq n$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} r_{i_1} r_{i_2} r_{i_3} \dots = (-1)^k \frac{a_{n-k}}{a_n}$$

As in the quadratic case, Vieta’s formula gives an equation to find the sum of roots

$$\sum_{i=1}^n r_i = \frac{-a_{n-1}}{a_n}$$

Similarly, we have the following equation for the product of roots.

$$r_1 r_2 \dots r_n = (-1)^n \frac{a_0}{a_n}$$

4. THEOREM

Let α be the value of each “n-1” equal roots, β be the value of the simple root, a_1 and a_0 be the penultimate and ultimate terms of an n^{th} Degree Polynomial defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$ then the polynomial $f(x)$ is said to have “n-1” equal roots and a simple root only if the Discriminant defined by

$D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1} \right)$ is a Perfect Square $\forall x \in \mathbb{R}$ and the Equal root α is given by and the Simple Root is given by $\beta = S - (n-1)\alpha$, where Sum of roots $S = \frac{-a_{n-1}}{a_n}$ (from Vieta’s theorem in 3.5.1 above).

$$\alpha = \left(\frac{2}{n-2} \right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right] \quad \forall n > 2$$

4.1 Proof

Let us consider a standard polynomial form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_k \dots + a_1 x + a_0 = 0 \rightarrow (1)$$

For our convenience let us assume $\alpha = -e$ & $\beta = \frac{-p}{m}$ such that $(x + e)^{n-1}(mx + p) = 0$

Now consider,

$$(x + e)^{n-1}(mx + p) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$$

Now by using Binomial expansion [2] for the expression $(x + e)^{n-1}(mx + p)$ we get,

$$(mx + p) \left\{ \binom{n-1}{0} \cdot x^{n-1} \cdot e^0 + \binom{n-1}{1} \cdot x^{n-2} \cdot e + \binom{n-1}{2} \cdot x^{n-3} \cdot e^2 + \binom{n-1}{3} \cdot x^{n-4} \cdot e^3 + \dots + \binom{n-1}{n-4} \cdot x^3 \cdot e^{n-4} + \binom{n-1}{n-3} \cdot x^2 \cdot e^{n-3} + \binom{n-1}{n-2} \cdot x \cdot e^{n-2} + \binom{n-1}{n-1} \cdot x^0 \cdot e^{n-1} \right\}.$$

Now from the properties of Combinations [3], we have

$$\binom{n}{r} = \binom{n}{n-r} \text{ for } r = 0, 1, 2, \dots, \binom{n-1}{n-4} = \binom{n-1}{3}, \binom{n-1}{n-3} = \binom{n-1}{2}, \binom{n-1}{n-2} = \binom{n-1}{1}$$

then,

$$\Rightarrow (mx + p) \left\{ x^{n-1} + (n-1) \cdot x^{n-2} e + \binom{n-1}{2} \cdot x^{n-3} \cdot e^2 + \binom{n-1}{3} \cdot x^{n-4} \cdot e^3 + \dots + \binom{n-1}{3} \cdot x^3 \cdot e^{n-4} + \binom{n-1}{2} \cdot x^2 \cdot e^{n-3} + \binom{n-1}{1} \cdot x \cdot e^{n-2} + e^{n-1} \right\}.$$

$$\Rightarrow mx^n + x^{n-1} [p + m \cdot (n-1) \cdot e] + x^{n-2} [p(n-1)e + m \cdot \binom{n-1}{2} \cdot e^2] + x^{n-3} [p \cdot \binom{n-1}{2} \cdot e^2 + m \cdot \binom{n-1}{3} \cdot e^3] + \dots + x^{n-k} [m \cdot \binom{n-1}{k} \cdot e^k + p \cdot \binom{n-1}{k-1} \cdot e^{k-1}] + \dots + x^3 [p \cdot \binom{n-1}{3} \cdot e^{n-4} + \binom{n-1}{2} \cdot m \cdot e^{n-3}] + x^2 [p \cdot \binom{n-1}{2} \cdot e^{n-3} + (n-1) \cdot m \cdot e^{n-2}] + x [p \cdot (n-1) \cdot e^{n-2} + m \cdot e^{n-1}] + p \cdot e^{n-1} \rightarrow (2)$$

Where k^{th} coefficient of the polynomial (2) is $a_{n-k} = [m \cdot \binom{n-1}{k} \cdot e^k + p \cdot \binom{n-1}{k-1} \cdot e^{k-1}] \rightarrow (3)$

On comparison with coefficients of the above polynomial (2)

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_k + \dots + a_1 x + a_0 = 0 \text{ we get,}$$

$$a_n = m, a_{n-1} = p + m \cdot (n-1) \cdot e, a_{n-2} = p(n-1)e + m \cdot \binom{n-1}{2} \cdot e^2, a_1 = p \cdot (n-1) \cdot e^{n-2} + m \cdot e^{n-1} \text{ \& } a_0 = p e^{n-1}$$

So we have, $a_{n-1} = p + a_n \cdot (n-1) \cdot e \rightarrow (4)$

$$a_{n-2} = p \cdot (n-1) \cdot e + a_n \cdot \binom{n-1}{2} \cdot e^2 \rightarrow (5)$$

$$a_1 = p \cdot (n-1) \cdot e^{n-2} + a_n \cdot e^{n-1} \rightarrow (6) \text{ \& }$$

$$a_0 = p \cdot e^{n-1} \rightarrow (7)$$

Now from (4) & (5) we have,

$$\begin{aligned} a_{n-2} &= p \cdot (n-1) \cdot e + a_n \cdot \binom{n-1}{2} \cdot e^2 \\ &= [a_{n-1} - a_n \cdot (n-1) \cdot e] \cdot (n-1) \cdot e + a_n \cdot \binom{n-1}{2} \cdot e^2 \\ &= a_{n-1} \cdot (n-1) \cdot e - a_n \cdot (n-1)^2 \cdot e^2 + a_n \cdot \binom{n-1}{2} \cdot e^2 \\ &= a_{n-1} \cdot (n-1) \cdot e + e^2 [a_n \cdot \binom{n-1}{2} - a_n \cdot (n-1)^2] \\ &= a_{n-1} \cdot (n-1) \cdot e + e^2 [a_n \cdot (\frac{(n-1)(n-2)}{2}) - a_n \cdot (n-1)^2] \text{ [}\cdot\text{:from properties of Combinations[3], } \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}] \\ &= a_{n-1} \cdot (n-1) \cdot e - e^2 \cdot a_n \cdot \binom{n(n-2)}{2} \end{aligned}$$

On simplifying we get,

$$\Rightarrow [a_n \cdot \binom{n(n-2)}{2}] \cdot e^2 - [a_{n-1} \cdot (n-1)] \cdot e + a_{n-2} = 0 \rightarrow (8) \text{ which is in the form of a Quadratic equation of variable "e",}$$

where

$$a = a_n \cdot \binom{n(n-2)}{2}, b = -[a_{n-1} \cdot (n-1)] \text{ \& } c = a_{n-2}$$

Now, on application of Quadratic formula [4] of general Quadratic equation $ax^2 + bx + c = 0$, where a, b and c are coefficients,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow (9)$$

We determine "e" value as below,

$$\Rightarrow e = \frac{(n-1) \cdot a_{n-1} \pm \sqrt{[a_{n-1} \cdot (n-1)]^2 - 2 \cdot (n-1) \cdot n \cdot a_n \cdot a_{n-2}}}{(n-1) \cdot n \cdot a_n} \text{ or;}$$

$$e = \frac{(n-1) \cdot a_{n-1} \pm (n-1) \cdot \sqrt{a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1}}}{(n-1) \cdot n \cdot a_n} \text{ or;}$$

$$\text{therefore, } e = \frac{a_{n-1} \pm \sqrt{a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1}}}{n \cdot a_n} \rightarrow (10)$$

Now consider (6) & (7),

$$a_1 = p \cdot (n-1) \cdot e^{n-2} + a_n \cdot e^{n-1} \rightarrow (6) \text{ \& } \\ a_0 = p \cdot e^{n-1} \rightarrow (7)$$

On dividing (6) ÷ (7) we get,

$$\begin{aligned} \frac{a_1}{a_0} &= \frac{n-1}{e} + \frac{a_n}{p} \\ \Rightarrow \frac{a_1}{a_0} &= \frac{n-1}{e} + \frac{a_n}{a_{n-1} - a_n(n-1)e} \\ \Rightarrow \frac{a_1}{a_0} &= \frac{a_{n-1}(n-1) - a_n e(n-1)^2 + a_n e}{e[a_{n-1} - a_n(n-1)e]} \\ \Rightarrow \frac{a_1}{a_0} &= \frac{na_{n-1} - a_{n-1} - a_n e(n-1)^2 + a_n e}{e a_{n-1} - a_n e^2(n-1)} \\ \Rightarrow [e \cdot a_{n-1} - e^2 \cdot a_n \cdot (n-1)] \cdot a_1 &= a_0 \cdot [n \cdot a_{n-1} - a_{n-1} - a_n \cdot e(n-1)^2 + a_n \cdot e] \\ \Rightarrow e \cdot a_{n-1} \cdot a_1 - e^2 \cdot a_n \cdot a_1 \cdot (n-1) &= n \cdot a_0 \cdot a_{n-1} - a_0 \cdot a_{n-1} - a_n \cdot a_0 \cdot e \cdot (n-1)^2 + a_n \cdot a_0 \cdot e \\ \Rightarrow e^2 \cdot a_n \cdot a_1 \cdot (n-1) - e \cdot a_{n-1} \cdot a_1 + n \cdot a_0 \cdot a_{n-1} - a_0 \cdot a_{n-1} - a_n \cdot a_0 \cdot e \cdot (n-1)^2 + a_n \cdot a_0 \cdot e &= 0 \\ \Rightarrow [a_n \cdot a_1 \cdot (n-1)] \cdot e^2 - [a_{n-1} \cdot a_1 + n \cdot (n-2) \cdot a_n \cdot a_0] \cdot e + [a_{n-1} \cdot a_0 \cdot (n-1)] &= 0 \end{aligned}$$

Which is also in the form of a Quadratic equation,

However, for easy simplification,

$$\text{Let us consider } y = a_{n-1} \cdot a_1 + n \cdot (n-2) \cdot a_n \cdot a_0$$

Then, we get the quadratic equation as,

$$[a_n \cdot a_1 \cdot (n-1)] \cdot e^2 - y \cdot e + [a_{n-1} \cdot a_0 \cdot (n-1)] = 0 \rightarrow (11)$$

$$\text{Where, } a = a_n \cdot a_1 \cdot (n-1), b = -y \text{ \& } c = a_{n-1} \cdot a_0 \cdot (n-1)$$

Now on application of Quadratic formula as in (9) we determine the value of e as,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow (9) \\ \Rightarrow e = \frac{y \pm \sqrt{y^2 - 4a_n \cdot a_1 \cdot (n-1)^2 \cdot a_{n-1} \cdot a_0}}{2(n-1)a_n a_1} \rightarrow (12)$$

Therefore, the two values of e (10) & (12) are

$$e = \left\{ \frac{a_{n-1} \pm \sqrt{a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1}}}{n \cdot a_n} \text{ \& } \frac{y \pm \sqrt{y^2 - 4a_n \cdot a_1 \cdot (n-1)^2 \cdot a_{n-1} \cdot a_0}}{2(n-1)a_n a_1} \right\}$$

Now consider, (10) = (12)

$$e = \frac{a_{n-1} \pm \sqrt{a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1}}}{n \cdot a_n} = \frac{y \pm \sqrt{y^2 - 4a_n \cdot a_1 \cdot (n-1)^2 \cdot a_{n-1} \cdot a_0}}{2(n-1)a_n a_1} = k \text{ (say, } k = e)$$

Now,

$$\Rightarrow \frac{a_{n-1} \pm \sqrt{a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1}}}{n \cdot a_n} = k$$

then,

$$\Rightarrow \pm \sqrt{a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1}} = n \cdot k \cdot a_n - a_{n-1}$$

On squaring both sides we get,

$$a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1} = (n \cdot k \cdot a_n - a_{n-1})^2 \\ \Rightarrow a_{n-1}^2 - \frac{2n \cdot a_n \cdot a_{n-2}}{n-1} = k^2 \cdot n^2 \cdot a_n^2 + a_{n-1}^2 - 2 \cdot k \cdot n \cdot a_n \cdot a_{n-1}$$

On dividing with “na_n” both sides we get,

$$2 \cdot k \cdot a_{n-1} - \frac{2a_{n-2}}{n-1} = k^2 \cdot n \cdot a_n \rightarrow (13)$$

Now consider,

$$\frac{y \pm \sqrt{y^2 - 4a_n \cdot a_1 (n-1)^2 \cdot a_{n-1} a_0}}{2(n-1)a_n a_1} = k \text{ then,}$$

$$\Rightarrow \pm \sqrt{y^2 - 4a_n \cdot a_1 (n-1)^2 \cdot a_{n-1} a_0} = [2 \cdot k \cdot a_n \cdot a_1 \cdot (n-1) - y]$$

On squaring both sides we get,

$$y^2 - 4a_n \cdot a_1 (n-1)^2 \cdot a_{n-1} a_0 = [2 \cdot k \cdot a_n \cdot a_1 \cdot (n-1) - y]^2$$

$$\Rightarrow y^2 - 4a_n \cdot a_1 (n-1)^2 \cdot a_{n-1} a_0 = 4 \cdot k^2 \cdot a_n^2 \cdot a_1^2 \cdot (n-1)^2 + y^2 - 4 \cdot k \cdot y \cdot a_n \cdot a_1 \cdot (n-1)$$

On dividing the above expression with “ $4a_n a_1 n - 1$ ” we get, $k \cdot y - (n-1) \cdot a_{n-1} a_0 = k^2 \cdot a_n a_1 \cdot (n-1)$

$$\Rightarrow \frac{k \cdot y}{n-1} - a_{n-1} a_0 = k^2 \cdot a_n \cdot a_1 \rightarrow (14)$$

Now in order to eliminate the term “ k^2 ” in (13) & (14) above, to make the simplification easy, from both the equations (13) & (14), divide (14) ÷ (13) then, we get

$$\frac{\frac{k \cdot y}{n-1} - a_{n-1} a_0}{2 \cdot k \cdot a_{n-1} - \frac{2a_{n-2}}{n-1}} = \frac{k^2 \cdot a_n \cdot a_1}{k^2 \cdot n \cdot a_n}$$

$$\Rightarrow \left(\frac{k \cdot y}{n-1}\right) \cdot n - a_{n-1} \cdot a_0 \cdot n = a_1 \cdot [2 \cdot k \cdot a_{n-1} - \frac{2 \cdot a_{n-2}}{n-1}]$$

$$\Rightarrow \frac{n \cdot k \cdot y}{n-1} - n \cdot a_{n-1} \cdot a_0 = 2 \cdot k \cdot a_1 \cdot a_{n-1} - \frac{2 \cdot a_{n-2} \cdot a_1}{n-1}$$

$$\Rightarrow \frac{n \cdot k \cdot y}{n-1} - 2 \cdot k \cdot a_1 \cdot a_{n-1} = n \cdot a_{n-1} \cdot a_0 - \frac{2 \cdot a_{n-2} \cdot a_1}{n-1}$$

$$\Rightarrow k \cdot \left[\frac{n \cdot y}{n-1} - 2a_1 \cdot a_{n-1}\right] = n \cdot a_{n-1} \cdot a_0 - \frac{2 \cdot a_{n-2} \cdot a_1}{n-1}$$

$$\Rightarrow k \cdot \left[\left(\frac{n}{n-1}\right) \cdot (a_{n-1} \cdot a_1 + a_n \cdot a_0 \cdot n \cdot (n-2) - 2 \cdot a_{n-1} \cdot a_1)\right] = n \cdot a_{n-1} \cdot a_0 - \frac{2 \cdot a_{n-2} \cdot a_1}{n-1}$$

$$\Rightarrow k \cdot [a_n \cdot a_0 \cdot n^2 \cdot (n-2) - (n-2) \cdot a_{n-1} \cdot a_1] = n^2 \cdot a_{n-1} \cdot a_0 - n a_{n-1} \cdot a_0 - 2a_{n-2} \cdot a_1$$

$$\Rightarrow k(n-2)[n^2 a_n a_0 - a_{n-1} a_1] = (n^2 - n) a_{n-1} a_0 - 2a_{n-2} a_1$$

$$\Rightarrow k(n-2)[n^2 a_n a_0 - a_{n-1} a_1] = 2 \cdot \binom{n(n-1)}{2} a_{n-1} a_0 - 2a_{n-2} a_1$$

$$\Rightarrow k(n-2)[n^2 a_n a_0 - a_{n-1} a_1] = 2 \cdot \binom{n}{2} \cdot a_{n-1} a_0 - 2a_{n-2} a_1 \quad [\text{from properties of Combinations}[3], \binom{n}{2} = \frac{n(n-1)}{2}]$$

$$\Rightarrow k \left(\frac{n-2}{2}\right) [n^2 a_n a_0 - a_{n-1} a_1] = \left[\binom{n}{2} a_{n-1} a_0 - a_{n-2} a_1\right]$$

$$\Rightarrow k = \left(\frac{2}{n-2}\right) \left[\frac{\binom{n}{2} a_0 a_{n-1} - a_1 a_{n-2}}{n^2 a_0 a_n - a_1 a_{n-1}}\right] = e$$

But in the beginning we have assumed $\alpha = -e$

$$a_{n-1} \quad \forall, n > 2$$

$$\alpha = \left(\frac{2}{n-2}\right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}}\right] \rightarrow (15)$$

Now divide (1) with a_n then,

$$\frac{a_{n-1}}{a_n} = \frac{p + a_n \cdot (n-1) \cdot e}{a_n}$$

$$= \frac{p}{a_n} + (n-1) \cdot e$$

$$= \frac{p}{m} + (n-1) \cdot e \quad [a_n = m]$$

On multiplying with “-” we get,

$$\frac{-a_{n-1}}{a_n} = \frac{-p}{m} - (n-1) \cdot e$$

$$\Rightarrow S = \beta - (n-1) \cdot e \quad [\text{from vieta's theorem in 7.2, we know that, } s = \frac{-a_{n-1}}{a_n}]$$

$$\Rightarrow \beta = S + (n-1) \cdot e$$

$$\Rightarrow \beta = S - (n-1) \cdot \alpha \quad [\alpha = -e]$$

$$\beta = S - (n-1) \cdot \alpha$$

Now again consider the discriminant of the Quadratic equation

$$[a_n \cdot \left(\frac{n(n-2)}{2}\right)] \cdot e^2 - [a_{n-1} \cdot (n-1)] \cdot e + a_{n-2} = 0 \rightarrow (8)$$

$$D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1}\right) \rightarrow (16)$$

Now,

On substituting the values of a_n, a_{n-1} and a_{n-2} , we get

$$\begin{aligned} D &= [p + m(n-1)e]^2 - \frac{2mn[p(n-1)e + m\binom{n-1}{2}e^2]}{n-1} \\ &= p^2 + m^2(n-1)^2e^2 + 2pme(n-1) - 2mnpe - nm^2(n-2)e^2 \\ &= p^2 + m^2(n-1)^2e^2 + 2pmne - 2pme - 2nmpe - nm^2(n-2)e^2 \\ &= p^2 + m^2(n-1)^2e^2 - 2pme - nm^2(n-2)e^2 \\ &= p^2 + m^2e^2 - 2m^2ne^2 - 2pme + 2m^2ne^2 \\ &= p^2 + m^2e^2 - 2pme \\ &= (p - me)^2, \text{ which is a Perfect square.} \end{aligned}$$

5. SOME EXTRACTIONS FROM THE FORMULA WITH EXAMPLES

5.1 When a Cubic Equation $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ is said to have 2 equal roots and a simple root then, For $n=3>2$, we have,

$$\begin{aligned} \text{Discriminant } D_n &= a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1}\right) \\ \Rightarrow D_3 &= a_2^2 - \left(\frac{2 \cdot 3 \cdot a_3 \cdot a_1}{3-1}\right) = a_2^2 - 3a_3 a_1 \end{aligned}$$

$$\begin{aligned} \text{Equal root } \alpha &= \left(\frac{2}{n-2}\right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right] \\ &= \left(\frac{2}{3-2}\right) \left[\frac{a_1^2 - \binom{3}{2} a_0 a_2}{3^2 a_0 a_3 - a_1 a_2} \right] \\ &= 2 \left[\frac{a_1^2 - 3a_0 a_2}{9a_0 a_3 - a_1 a_2} \right] \\ \alpha &= \left[\frac{2a_1^2 - 6a_0 a_2}{9a_0 a_3 - a_1 a_2} \right] \end{aligned}$$

And the Simple root $\beta = S - (n-1) \alpha$

$$= S - (3-1) \alpha = S - 2\alpha \text{ where } \alpha \text{ is the Equal root and Sum of roots } S = \frac{-a_2}{a_3}.$$

5.1.1 Example: Solve $x^3 + 6x^2 + 9x + 4 = 0$

Solution: Given that,

$$\begin{aligned} a_3 &= 1, a_2 = 6, a_1 = 9, a_0 = 4 \\ S &= \frac{-a_2}{a_3} = -6 \end{aligned}$$

For $n=3>2$, we have,

$$\text{Discriminant } D_3 = a_2^2 - 3a_3 a_1 = 6^2 - 3 \cdot 1 \cdot 9 = 9 \text{ (Perfect Square)}$$

The given equation is said to have a double equal root

$$x_{eq} = \left[\frac{2a_1^2 - 6a_0 a_2}{9a_0 a_3 - a_1 a_2} \right] = \left[\frac{2 \cdot 9 \cdot 9 - 6 \cdot 6 \cdot 4}{9 \cdot 1 \cdot 4 - 6 \cdot 9} \right] = -1$$

And the simple root $\beta = S - (n-1) \alpha = S - 2 \alpha = -6 + 2 = -4$

Hence, the roots are $x = (-1, -1, -4)$

5.2 When a Quartic Equation $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ is said to have 3 equal roots and a simple root then, For $n = 4 > 2$, we have,

$$\begin{aligned} \text{Discriminant } D_n &= a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1}\right) \\ \Rightarrow D_4 &= a_3^2 - \left(\frac{2 \cdot 4 \cdot a_4 \cdot a_2}{4-1}\right) = a_3^2 - \frac{8a_4 a_2}{3} \end{aligned}$$

$$\begin{aligned} \text{Equal root } \alpha &= \left(\frac{2}{n-2}\right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right] \\ &= \left(\frac{2}{4-2}\right) \left[\frac{a_1 a_2 - \binom{4}{2} a_0 a_3}{4^2 a_0 a_4 - a_1 a_3} \right] \\ &= \left[\frac{a_1 a_2 - \binom{4}{2} a_0 a_3}{16a_0 a_4 - a_1 a_3} \right] \\ \alpha &= \left[\frac{a_1 a_2 - 6a_0 a_3}{16a_0 a_4 - a_1 a_3} \right] \end{aligned}$$

And the Simple root $\beta = S - (n-1) \alpha$

$$= S - (4-1) \alpha = S - 3\alpha \text{ where } \alpha \text{ is the Equal root and Sum of roots } S = \frac{-a_3}{a_4}.$$

5.2.1 Example: Solve $x^4 - x^3 - 3x^2 + 5x - 2 = 0$

Solution: Given that,

$$a_4=1, a_3=-1, a_2=-3, a_1=5, a_0=-2$$

$$s = \frac{b - a_3}{a - a_4} = 1$$

For $n=4>2$, we have,

$$\text{Discriminant } D_4 = a_3^2 - \frac{8ac}{3} - \frac{8a_4a_3}{3} = (-1)^2 - \frac{8*1*(-3)}{3}$$

$$= 9(\text{Perfect Square})$$

The given equation is said to have a triple equal root

$$x_{eq} = \left[\frac{a_1 a_2 - 6a_0 a_3}{16a_0 a_4 - a_1 a_3} \right]$$

$$= \left[\frac{(-3)*5 - 6*(-1)*(-2)}{16*1*(-2) - (-1)*5} \right]$$

$$= 1$$

And the simple root $\beta = S - (n-1) * \alpha = S - 3*1x_{eq} = 1 - 3 = -2$

Hence, the roots are $x = (1, 1, 1, -2)$.

5.3 When a Quintic Equation $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ is said to have 4 equal roots and a simple root then,

For $n=5>2$, we have,

$$\text{Discriminant } D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1} \right)$$

$$\Rightarrow D_5 = a_4^2 - \left(\frac{2*5*a_5*a_3}{5-1} \right) = a_4^2 - \frac{5a_5a_3}{2}$$

$$\text{Equal root } \alpha = \left(\frac{2}{n-2} \right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right]$$

$$= \left(\frac{2}{5-2} \right) \left[\frac{a_1 a_3 - \binom{5}{2} a_0 a_4}{5^2 a_0 a_5 - a_1 a_4} \right]$$

$$= \frac{2}{3} \left[\frac{a_1 a_3 - 10 a_0 a_4}{25 a_0 a_5 - a_1 a_4} \right]$$

$$\alpha = \left[\frac{2a_3 a_1 - 20a_4 a_0}{75a_5 a_0 - 3a_4 a_1} \right]$$

And the Simple root $\beta = S - (n-1) * \alpha$

$$= S - (5-1) * \alpha = S - 4\alpha \text{ where } \alpha \text{ is the Equal root and Sum of roots } S = \frac{-a_4}{a_5}.$$

5.3.1 Example: Solve $x^5 - 2x^4 - 2x^3 + 8x^2 - 7x + 2 = 0$

Solution: Given that,

$$a_5=1, a_4=-2, a_3=-2, a_2=8, a_1=-7, a_0=2$$

$$S = \frac{-a_4}{a_5} = 2$$

For $n=5>2$, we have,

$$\text{Discriminant } D_5 = a_4^2 - \frac{5a_5a_3}{2}$$

$$= (-2)^2 - \frac{5*1*(-2)}{2}$$

$$= 9(\text{Perfect Square})$$

The given equation is said to have a quadruple or 4 equal roots

$$\alpha = \left[\frac{2a_3 a_1 - 20a_4 a_0}{75a_5 a_0 - 3a_4 a_1} \right]$$

$$= \left[\frac{2*(-2)*(-7) - 20*(-2)*2}{75*1*2 - 3*(-2)*(-7)} \right]$$

$$= 1$$

And the simple root $\beta = S - (n-1) * \alpha = S - 4 * \alpha = 2 - 4 = -2$

Hence, the roots are $x = (1, 1, 1, -2)$

5.4 When a sextic or hexic Equation $a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ is said to have 5 equal roots and a simple root then,

For $n = 6 > 2$, we have,

$$\text{Discriminant } D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1} \right)$$

$$\Rightarrow D_6 = a_5^2 - \left(\frac{2*6*a_6*a_4}{6-1} \right) = a_5^2 - \frac{12a_6a_4}{5}$$

$$\text{Equal root } \alpha = \left(\frac{2}{n-2} \right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right]$$

$$= \left(\frac{2}{6-2} \right) \left[\frac{a_1 a_4 - \binom{6}{2} a_0 a_5}{6^2 a_0 a_6 - a_1 a_5} \right]$$

$$= \frac{1}{2} \left[\frac{a_1 a_4 - 15 a_0 a_5}{36 a_0 a_6 - a_1 a_5} \right]$$

$$\alpha = \left[\frac{a_1 a_4 - 15 a_0 a_5}{72 a_0 a_6 - 2 a_1 a_5} \right]$$

And the Simple root $\beta = S - (n-1) * \alpha$

$$= S - (6-1) * \alpha = S - 5\alpha \text{ where } \alpha \text{ is the Equal root and Sum of roots } S = \frac{-a_5}{a_6}$$

5.4.1 Example: Solve $x^6+3x^5-10x^3-15x^2-9x-2=0$

Solution: Given that,

$$a_6=1, a_5=3, a_4=0, a_3=-10, a_2=-15, a_1=-9, a_0=-2$$

$$S = \frac{-a_5}{a_6} = -3$$

For $n = 6 > 2$, we have,

$$\text{Discriminant } D_6 = a_5^2 - \frac{12a_6a_4}{5}$$

$$= 3^2 - \frac{12*1*0}{5}$$

$$= 9 \text{ (Perfect Square)}$$

The given equation is said to have 5 equal roots

$$\alpha = \left[\frac{a_1 a_4 - 15 a_0 a_5}{72 a_0 a_6 - 2 a_1 a_5} \right]$$

$$= \left[\frac{0*(-9) - 15*3*(-2)}{72*1*(-2) - 2*3*(-9)} \right]$$

$$= -1$$

And the simple root $\beta = S - (n-1) * \alpha = S - 5 * \alpha = -3 + 5 = 2$

Hence, the roots are $x = (-1, -1, -1, -1, -1, 2)$

5.5 When a Septic or heptic Equation $a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ is said to have 6 equal roots and a simple root then,

For $n=7>2$, we have,

$$\text{Discriminant } D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1} \right)$$

$$\Rightarrow D_7 = a_6^2 - \left(\frac{2*7*a_7*a_5}{7-1} \right) = a_6^2 - \frac{7a_7a_5}{3}$$

$$\text{Equal root } \alpha = \left(\frac{2}{n-2} \right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right]$$

$$= \left(\frac{2}{7-2} \right) \left[\frac{a_1 a_5 - \binom{7}{2} a_0 a_6}{7^2 a_0 a_7 - a_1 a_6} \right]$$

$$= \frac{2}{5} \left[\frac{a_1 a_5 - 21 a_0 a_6}{49 a_0 a_7 - a_1 a_6} \right]$$

$$\alpha = \frac{2}{5} \left[\frac{a_1 a_5 - 21 a_0 a_6}{49 a_0 a_7 - a_1 a_6} \right]$$

And the Simple root $\beta = S - (n-1) * \alpha$

$$= S - (7-1) * \alpha = S - 6\alpha \text{ where } \alpha \text{ is the Equal root and Sum of roots } S = \frac{-a_6}{a_7}$$

5.5.1 Example: Solve $x^7+x^6-15x^5-55x^4-85x^3-69x^2-29x-5=0$

Solution: Given that,

$$a_7=1, a_6=1, a_5=-15, a_4=-55, a_3=-85, a_2=-69, a_1=-29, a_0=-5$$

$$s = \frac{-a_6}{a_7} = -1$$

5.7 Likewise, for any higher degree such as $n=999$, we can easily find its discriminant, **998 equal roots** and the Simple root as below,

Let the 999th Degree equation be as

$$f(x) = a_{999}x^{999} + a_{998}x^{998} + a_{997}x^{997} + a_{996}x^{996} + \dots + a_1x + a_0 = 0$$

Then the Discriminant

$$D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1} \right)$$

$$\Rightarrow D_{999} = a_{998}^2 - \left(\frac{2 \cdot 999 \cdot a_{999} \cdot a_{997}}{999-1} \right)$$

$$= a_{998}^2 - \left(\frac{999 a_{999} a_{997}}{499} \right) \text{ which is a Perfect Square.}$$

Now the value of each 998 equal roots is given by

$$\text{Equal root } \alpha = \left(\frac{2}{n-2} \right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right]$$

$$= \left(\frac{2}{999-2} \right) \left[\frac{a_1 a_{997} - \binom{999}{2} a_0 a_{998}}{999^2 a_0 a_{999} - a_1 a_{998}} \right]$$

$$\Rightarrow \alpha = \left\{ \begin{array}{l} \left(\frac{2}{997} \right) \left[\frac{a_1 a_{997} - 49850 a_0 a_{998}}{998001 a_0 a_{999} - a_1 a_{998}} \right] \\ \text{or} \\ \left[\frac{2 a_0 a_{997} - 997002 a_0 a_{998}}{995006997 a_0 a_{999} - 997 a_1 a_{998}} \right] \end{array} \right.$$

And the Simple root $\beta = S - (n-1) \cdot \alpha$

$$= S - (999-1) \cdot \alpha = S - 998\alpha \text{ where } \alpha \text{ is the Equal root and Sum of roots } S = \frac{-a_{998}}{a_{999}}.$$

5.7.1 Example

Let us first frame a 999th degree equation which is having 998 equal roots of “-1” and a simple root “-2” as below:
 $(x+1)^{998}(x+2)=0$

As we know that the k^{th} term of the expression $(x+e)^{n-1}(mx+p)$ from above (x) we have,

$$a_{n-k} = [m \cdot \binom{n-1}{k} \cdot e^k + p \cdot \binom{n-1}{k-1} \cdot e^{k-1}]$$

$$e=1, m=1, p=2, n=999$$

We have,

$$a_{999} = a_{999-0} = [1 \cdot \binom{999-1}{0} \cdot 1^0 + 2 \cdot \binom{999-1}{0-1} \cdot 1^{(0-1)}] = 1$$

$$a_{998} = a_{999-1} = [1 \cdot \binom{999-1}{1} \cdot 1^1 + 2 \cdot \binom{999-1}{1-1} \cdot 1^{1-1}] = 1000$$

$$a_{997} = a_{999-2} = [1 \cdot \binom{999-1}{2} \cdot 1^2 + 2 \cdot \binom{999-1}{2-1} \cdot 1^{2-1}] = 499499$$

$$a_1 = a_{999-998} = [1 \cdot \binom{999-1}{998} \cdot 1^{998} + 2 \cdot \binom{999-1}{997} \cdot 1^{997}] = 1997$$

$$a_0 = a_{999-999} = [1 \cdot \binom{999-1}{999} \cdot 1^{999} + 2 \cdot \binom{999-1}{998} \cdot 1^{998}] = 2$$

So the equation is

$$x^{999} + 1000x^{998} + 499499x^{997} + \dots + 1997x + 2 = 0$$

Solution: Given that,

$$a_{999}=1, a_{998}=1000, a_{997}=499499, a_1=1997, a_0=2$$

$$S = \frac{-a_{998}}{a_{999}} = -1000$$

For $n=999 > 2$, we have,

$$\text{Discriminant } D_{999} = a_{998}^2 - \left(\frac{999 a_{999} a_{997}}{499} \right)$$

$$= (1000)^2 - \frac{999 \cdot 1 \cdot 499499}{499}$$

$$= 10,00,000 - 999,999$$

$$= 1 \text{ (Perfect Square)}$$

The given equation is said to have a **998 equal roots**

$$\alpha = \left(\frac{2}{997} \right) \left[\frac{a_1 a_{997} - 49850 a_0 a_{998}}{998001 a_0 a_{999} - a_1 a_{998}} \right]$$

$$= \frac{2}{997} \left[\frac{499499 \cdot 1997 - 498501 \cdot 1000 \cdot 2}{998001 \cdot 1 \cdot 2 - 1000 \cdot 1997} \right]$$

$$= \frac{2}{997} \left[\frac{497503}{-998} \right]$$

$$= -1$$

And the simple root $\beta = S - (n-1) \cdot \alpha$

$$= S - 998 \cdot \alpha = -1000 + 998 = -2$$

Hence, the roots are

$$x = [-1, -1, -1, -1, \dots, -1(998 \text{ times}), -2]$$

6 REFERENCES

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