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A SHORT APPROACH TO FIND OUT THE VALUE OF EACH (n-1) EQUAL ROOTS AND A SIMPLE ROOT OF AN nth DEGREE POLYNOMIAL OF SINGLE VARIABLE

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1. ABSTRACT

T*o show that there is an alternative easy approach/formula to solve a specific type of polynomial of any degree, say n which is said to have conditionally (n-1) equal roots while there exist a conventional approach of solving the same.*

2. NOTATIONS

i. $\boldsymbol{\alpha}$ = -e = Equal root
ii. β = $\frac{-p}{m}$ = Simple root
iii. n = Degree of the polynomial/highest power of the polynomial
iv. D_n= Discriminant of the n^{th} degree polynomial
v. a_n = First term coefficient of the n^{th}degree polynomial
vi. a_{n-1}=second coefficient of the n^{th} degree polynomial
vii. a_{n-2}= Third coefficient of the n^{th}degree polynomial
viii. a_k = k^{th} coefficient of the n^{th}degree polynomial
ix. a_1 = Penultimate/last but one coefficient of the n^{th} degree polynomial
x. a_0 = Ultimate/Last term of the n^{th}degree polynomial
xi. r_1, r_2= Roots of a quadratic equation of the form a$x^2 + bx + c$ = 0
xii. S = Sum of Roots of the n^{th} degree polynomial = $\frac{-a_{n-1}}{a_n}$
xiii. $\binom{n}{r}$ = Combinations of r out of n = $\frac{(n-r)!}{n!r!}$
xiv. ∀ =For all
xv. R = Set of real numbers
xvi. ∈ =belongs to
xvii. x =a variable

3. INTRODUCTION

A polynomial is a mathematical expression consisting of variables, coefficients and the operations of addition, subtraction, multiplication and non-negative integer exponents. Below are some examples of polynomial
$x + 3,\ \ 3x^2 - 2x + 5, 2a^3b^2 - 3b^2 + 2a - 1,\ \frac{1}{2}x^2 - \frac{2}{3}x + \frac{3}{4}$

Polynomials are an important part of the language of mathematics and algebra. They are used in nearly every field of Mathematics to express numbers as a result of mathematical operations. Polynomials are also “buildings blocks” in other types of mathematical expressions such as rational expressions.

Many mathematical processes that are done in everyday life can be interpreted as polynomials. Summing the cost of items on a grocery bill can be interpreted as a polynomial. Calculating the distance travelled of a vehicle or object can be interpreted as a polynomial. Calculating perimeter, area and volume of geometric figures can be interpreted as polynomials.

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3.1 Degree of Polynomial
The degree of a polynomial is the highest of the degrees/powers of its monomials with non-zero coefficients. It is the sum of the exponents of the variables that appear in it, and thus is a non-negative integer.

"0" degree polynomials are called constants. The value of Constants don't change.

"1st" degree polynomials are called Linear polynomials. They are used to describe quantities that change at a steady rate. They are also used in many one-dimensional geometry problems involving length.

"2nd "degree polynomials are called Quadratic polynomials. They are used to describe quantities that change with some amount of acceleration or deceleration. They are also used in many two-dimensional geometry problems involving area.

"3rd "degree polynomials are also called Cubic polynomials. They are used in many three-dimensional geometry problems involving volume.

3.2 Polynomial Function
A function $f(x)$ is a polynomial function if it can be written as $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$.

In this form $a_0, a_1 \ldots a_{n-1}, a_n$ are non-variable coefficients and n is a non-negative integer.

3.3 Remainder Theorem
When a polynomial $f(x)$ is divided by $(x - a)$ the remainder is "$f(a)$".

3.4 Factor Theorem
Let $f(x)$ be a polynomial function such that $f(c)$ = 0 for some constant then $(x - c)$ is a factor of $f(x)$.

3.5 Relation between Coefficients and roots of a polynomial
According to Vieta's formula [1], the Coefficients of polynomials are related to the sums and products of their Roots as well as the products of the roots taken in groups.

3.5.1 Vieta's formula for Quadratics
Given $f(x)$ = $ax^2 + bx + c$ if the equation $f(x)$ = 0 has roots r_1 and r_2 then, Sum of roots (S) = $r_1 + r_2 = \frac{-b}{a}$ and the Product of roots (P) = $r_1 r_2 = \frac{c}{a}$

3.5.2 Generalization to Higher Degree Polynomials

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ be a polynomial with complex coefficients and degree, having complex roots $r_n, r_{n-1\ldots} \ldots\ldots r_1$ .Then for any integer o$\leq k \leq n$

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} r_{i_1} r_{i_2} r_{i_3} = (-1)^k \frac{a_{n-k}}{a_n}$$

As in the quadratic case, Viete's formula gives an equation to find the sum of roots

$$\sum_{i=1}^{n} r_1 = \frac{-a_{n-1}}{a_n}$$

Similarly, we have the following equation for the product of roots.

$$r_1 r_2 \ldots r_n = (-1)^n \frac{a_0}{a_n}$$

4. THEOREM

Let **α** be the value of each "n-1" equal roots, β be the value of the simple root, $\boldsymbol{a_1}$ and $\boldsymbol{a_0}$ be the penultimate and ultimate terms of an nth Degree Polynomial defined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 = 0$ then the polynomial f(x) is said to have "n-1" equal roots and a simple root only if the Discriminant defined by $\boldsymbol{D_n = a_{n-1}^2 - \left(\frac{2na_n a_{n-2}}{n-1}\right)}$ is a Perfect Square ∀ x ∈ ℝ and the Equal root α is given by and the Simple Root is given by **β = S-(n-1)*α,** where Sum of roots S = $\frac{-a_{n-1}}{a_n}$ (from Vieta's theorem in 3.5.1 above).

$$\boldsymbol{\alpha = \left(\frac{2}{n-2}\right) \left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}} \right] \quad \forall\ n>2}$$

4.1 Proof

Let us consider a standard polynomial form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots a_k \ldots + a_1 x + a_0 = 0 \rightarrow (1)$$

For our convenience let us assume $\boldsymbol{\alpha} = -e$ & $\beta = \frac{-p}{m}$ such that $(x+e)^{n-1}(mx+p) = 0$

Now consider,

$$(x+e)^{n-1}(mx+p) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

Now by using Binomial expansion [2] for the expression $(x+e)^{n-1}(mx+p)$ we get,

$(mx+p)\{\binom{n-1}{0}.x^{n-1}.e^0 + \binom{n-1}{1}.x^{n-2}.e + \binom{n-1}{2}.x^{n-3}.e^2 + \binom{n-1}{3}.x^{n-4}.e^3 + \cdots + \binom{n-1}{n-4}.x^3.e^{n-4} + \binom{n-1}{n-3}.x^2.e^{n-3} + \binom{n-1}{n-2}.x.e^{n-2} + \binom{n-1}{n-1}.x^0.e^{n-1}\}$.

Now from the properties of Combinations [3], we have

$$\binom{n}{r} = \binom{n}{n-r} \text{ for r} = 0,1,2 \ldots, \binom{n-1}{n-4} = \binom{n-1}{3}, \binom{n-1}{n-3} = \binom{n-1}{2}, \binom{n-1}{n-2} = \binom{n-1}{1}$$

then,

$=> (mx+p)\{x^{n-1} + (n-1).x^{n-2}e + \binom{n-1}{2}.x^{n-3}.e^2 + \binom{n-1}{3}.x^{n-4}.e^{3 \cdots\cdots} + \binom{n-1}{3}.x^3.e^{n-4} + \binom{n-1}{2}.x^2.e^{n-3} + \binom{n-1}{1}.x.e^{n-2} + e^{n-1}\}$.

$=> mx^n + x^{n-1}[p + m.(n-1).e] + x^{n-2}[p(n-1)e + m.\binom{n-1}{2}.e^2] + x^{n-3}[p.\binom{n-1}{2}.e^2 + m.\binom{n-1}{3}.e^3] + \ldots + x^{n-k}[m.\binom{n-1}{k}.e^k + p.\binom{n-1}{k-1}.e^{k-1}] + \ldots + x^3[p.\binom{n-1}{3}.e^{n-4} + \binom{n-1}{2}.m.e^{n-3}] + x^2[p.\binom{n-1}{2}.e^{n-3} + (n-1).m.e^{n-2}] + x[p.(n-1).e^{n-2} + m.e^{n-1}] + p.e^{n-1} \rightarrow (2)$

Where k^{th} coefficient of the polynomial (2) is $\boldsymbol{a_{n-k}} = [\boldsymbol{m}.\binom{\boldsymbol{n-1}}{\boldsymbol{k}}.\boldsymbol{e^k} + \boldsymbol{p}.\binom{\boldsymbol{n-1}}{\boldsymbol{k-1}}.\boldsymbol{e^{k-1}}] \rightarrow (3)$

On comparison with coefficients of the above polynomial (2)

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_k + \cdots + a_1 x + a_0 = 0$ we get,

$a_n = m$, $a_{n-1} = p + m.(n-1).e$, $a_{n-2} = p(n-1)e + m.\binom{n-1}{2}.e^2$, $a_1 = p.(n-1).e^{n-2} + m.e^{n-1}$ & $a_0 = pe^{n-1}$

So we have, $a_{n-1} = p + a_n.(n-1).e \rightarrow (4)$

$a_{n-2} = p.(n-1).e + a_n.\binom{n-1}{2}.e^2 \rightarrow (5)$

$a_1 = p.(n-1).e^{n-2} + a_n.e^{n-1} \rightarrow (6)$ &

$a_0 = p.e^{n-1} \rightarrow (7)$

Now from (4) & (5) we have,

$a_{n-2} = p.(n-1).e + a_n.\binom{n-1}{2}.e^2$

$= [a_{n-1} - a_n.(n-1).e].(n-1).e + a_n.\binom{n-1}{2}.e^2$

$= a_{n-1}.(n-1).e - a_n.(n-1)^2.e^2 + a_n.\binom{n-1}{2}.e^2$

$= a_{n-1}.(n-1).e + e^2[a_n.\binom{n-1}{2} - a_n.(n-1)^2]$

$= a_{n-1}.(n-1).e + e^2[a_n.(\frac{(n-1)(n-2)}{2}) - a_n.(n-1)^2]$ [∵from properties of Combinations[3], $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$]

$= a_{n-1}.(n-1).e - e^2.a_n.(\frac{n(n-2)}{2})$

On simplifying we get,

$=> [a_n.(\frac{n(n-2)}{2})].e^2 - [a_{n-1}.(n-1)].e + a_{n-2} = 0 \rightarrow (8)$ which is in the form of a Quadratic equation of variable "e", where

a = $a_n.(\frac{n(n-2)}{2})$, b = - $[a_{n-1}.(n-1)]$ & c = a_{n-2}

Now, on application of Quadratic formula [4] of general Quadratic equation $ax^2 + bx + c = 0$, where a, b and c are coefficients,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow (9)$$

We determine "e" value as below,

$=> e = \frac{(n-1).a_{n-1} \pm \sqrt{[a_{n-1}.(n-1)]^2 - 2.(n-1).n.a_n.a_{n-2}}}{(n-1).n.a_n}$ or;

$e = \frac{(n-1).a_{n-1} \pm (n-1).\sqrt{a_{n-1}{}^2 - \frac{2n.a_n.a_{n-2}}{n-1}}}{(n-1)n.a_n}$ or;

therefore, $e = \frac{a_{n-1} \pm \sqrt{a_{n-1}{}^2 - \frac{2n.a_n.a_{n-2}}{n-1}}}{n.a_n} \rightarrow (10)$

Now consider (6) & (7),

$a_1 = p.(n\text{-}1).\ e^{n-2} + a_n.\ e^{n-1} \rightarrow (6)$ &

$a_0 = p.e^{n-1} \rightarrow (7)$

On dividing (6) ÷ (7) we get,

$\frac{a_1}{a_0} = \frac{n-1}{e} + \frac{a_n}{p}$

$=> \frac{a_1}{a_0} = \frac{n-1}{e} + \frac{a_n}{a_{n-1} - a_n(n-1)e}$

$=> \frac{a_1}{a_0} = \frac{a_{n-1}(n-1) - a_n e(n-1)^2 + a_n e}{e[a_{n-1} - a_n(n-1)e]}$

$=> \frac{a_1}{a_0} = \frac{n a_{n-1} - a_{n-1} - a_n e(n-1)^2 + a_n e}{e a_{n-1} - a_n e^2(n-1)}$

$=> [e.\ a_{n-1}\text{-}e^2.\ a_n.(n\text{-}1)].\ a_1 = a_0.[n.\ a_{n-1}\text{-}a_{n-1}\text{-}a_n.\ e(n\text{-}1)^2 + a_n.\ e]$

$=> e.\ a_{n-1}.\ a_1\text{-}e^2.a_n.\ a_1.(n\text{-}1) = n.\ a_0.\ a_{n-1}\text{-}a_0.\ a_{n-1}\text{-}a_n.\ a_0.\ e.(n\text{-}1)^2 + a_n.\ a_0.\ e$

$=> e^2.\ a_n.\ a_1.(n\text{-}1)\text{-}\ e.\ a_{n-1}.\ a_1 + n.\ a_0.\ a_{n-1}\text{-}a_0.\ a_{n-1}\text{-}a_n.\ a_0.\ e.(\ n\text{-}1)^2 + a_n.\ a_0.\ e = 0$

$=> [a_n.\ a_1.(n\text{-}1)].\ \boldsymbol{e^2}\text{-}\ [a_{n-1}.\ a_1 + n.(n\text{-}2).\ a_n.\ a_0].\ \boldsymbol{e} + [a_{n-1}.\ a_0.\ (n-1)] = 0$

Which is also in the form of a Quadratic equation,

However, for easy simplification,

Let us consider $y = a_{n-1}.\ a_1 + n.\ (n\text{-}2).\ a_n.\ a_0$

Then, we get the quadratic equation as,

$[a_n.\ a_1.(n\text{-}1)].\ \mathbf{e}^2\text{-}\ y.\ \mathbf{e} + [a_{n-1}.\ a_0.\ (n-1)] = 0 \rightarrow (11)$

Where, a = $a_n.\ a_1.(n\text{-}1)$, b = - y & c = $a_{n-1}.\ a_0.\ (n-1)$

Now on application of Quadratic formula as in (9) we determine the value of e as,

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow (9)$

$=> \mathbf{e} = \frac{y \pm \sqrt{\mathbf{y^2} - \mathbf{4} a_n.a_1(\boldsymbol{n}-\mathbf{1})^2.a_{n-1}a_0}}{\mathbf{2}(n-\mathbf{1})a_n a_1} \rightarrow (12)$

Therefore, the two values of e (10) & (12) are

$e = \{\frac{a_{n-1} \pm \sqrt{a_{n-1}{}^{\mathbf{2}} - \frac{\mathbf{2}\boldsymbol{n}.a_n.a_{n-2}}{\boldsymbol{n}-\mathbf{1}}}}{n.a_n}$ **&** $\frac{y \pm \sqrt{\mathbf{y^2} - \mathbf{4} a_n.a_1(\boldsymbol{n}-\mathbf{1})^2.a_{n-1}a_0}}{\mathbf{2}(n-\mathbf{1})a_n a_1}\}$

Now consider, (10) = (12)

$e = \frac{a_{n-1} \pm \sqrt{a_{n-1}{}^{\mathbf{2}} - \frac{\mathbf{2}\boldsymbol{n}.a_n.a_{n-2}}{\boldsymbol{n}-\mathbf{1}}}}{n.a_n} = \frac{y \pm \sqrt{\mathbf{y^2} - \mathbf{4} a_n.a_1(\boldsymbol{n}-\mathbf{1})^2.a_{n-1}a_0}}{\mathbf{2}(n-\mathbf{1})a_n a_1}$ = k (say, k = e)

Now,

$=> \frac{a_{n-1} \pm \sqrt{a_{n-1}{}^{\mathbf{2}} - \frac{\mathbf{2}\boldsymbol{n}.a_n.a_{n-2}}{\boldsymbol{n}-\mathbf{1}}}}{n.a_n} = k$

then,

$=> \pm \sqrt{a_{n-1}{}^{\mathbf{2}} - \frac{\mathbf{2}\boldsymbol{n}.a_n.a_{n-2}}{\boldsymbol{n}-\mathbf{1}}} = n.\ k.a_n - a_{n-1}$

On squaring both sides we get,

$a_{n-1}{}^{\mathbf{2}} - \frac{\mathbf{2}\boldsymbol{n}.a_n.a_{n-2}}{\boldsymbol{n}-\mathbf{1}} = (n.k.a_n - a_{n-1})^2$

$=> a_{n-1}{}^2 - \frac{\mathbf{2}\boldsymbol{n}.a_n.a_{n-2}}{\boldsymbol{n}-\mathbf{1}} = k^2.\ n^2.a_n{}^2 + a_{n-1}{}^2 - 2.k.n.a_n a_{n-1}$

On dividing with "na_n" both sides we get,

$2.k.a_{n-1} - \frac{2a_{n-2}}{n-1} = k^2.n.\ a_n \rightarrow (13)$

Now consider,

$\frac{y \pm \sqrt{\mathbf{y^2} - \mathbf{4}a_n.a_1(\mathbf{n-1})^2.a_{n-1}a_0}}{\mathbf{2}(n-\mathbf{1})a_n a_1} = k$ then,

$=> \pm\sqrt{y^2 - 4a_n.a_1(n-1)^2.a_{n-1}a_0} = [2.k.\,a_n.\,a_1.(n\text{-}1) - y]$

On squaring both sides we get,

$y^2 - 4a_n.a_1(n-1)^2.a_{n-1}a_0 = [2.k.\,a_n.\,a_1.(n-1) - y]^2$

$=> y^2 - 4a_n.a_1(n-1)^2.a_{n-1}a_0 = 4.\,k^2.\,{a_n}^2.\,{a_1}^2.(n-1)^2 + y^2 - 4.\,k.y.\,a_n.\,a_1.(n-1)$

On dividing the above expression with "$4a_n a_1 n - 1)$" we get, $k.y - (n-1).\,a_{n-1}a_0 = k^2.a_n a_1.(n-1)$

$=> \frac{k.y}{n-1} - a_{n-1}a_0 = k^2.\,a_n.\,a_1 \rightarrow (14)$

Now in order to eliminate the term "k^2" in (13) & (14) above, to make the simplification easy, from both the equations (13) & (14), divide (14) ÷ (13) then, we get

$\frac{\frac{k.y}{n-1} - a_{n-1}a_0}{2.\text{k}.a_{n-1} - \frac{2a_{n-2}}{n-1}} = \frac{k^2.a_n.a_1}{k^2 * n.a_n}$

$=> (\frac{k.y}{n-1}).n - a_{n-1}.\,a_0.n = a_1.[2.k.\,a_{n-1} - \frac{2.a_{n-2}}{n-1}]$

$=> \frac{n.k.y}{n-1} - n.\,a_{n-1}.\,a_0 = 2.k.\,a_1.\,a_{n-1} - \frac{2.a_{n-2}.a_1}{n-1}$

$=> \frac{n.k.y}{n-1} - 2.k.\,a_1.\,a_{n-1} = n.\,a_{n-1}.\,a_0 - \frac{2.a_{n-2}.a_1}{n-1}$

$=> k.[\frac{n.y}{n-1} - 2a_1.a_{n-1}] = n.\,a_{n-1}.\,a_0 - \frac{2.a_{n-2}.a_1}{n-1}$

$=> k.[(\frac{n}{n-1}).(\,a_{n-1}.\,a_1 + a_n.\,a_0\,.n.(n-2) - 2.a_{n-1}.\,a_1] = n.\,a_{n-1}.\,a_0 - \frac{2.a_{n-2}.a_1}{n-1}$

$=> k.[\,a_n.\,a_0.\,n^2.(n-2) - (n-2).\,a_{n-1}.\,a_1] = n^2.\,a_{n-1}.\,a_0 - na_{n-1}.\,a_0 - 2a_{n-2}.\,a_1$

$=> k(n-2)[\,n^2 a_n a_0 - a_{n-1}a_1] = (n^2 - n)\,a_{n-1}a_0 - 2a_{n-2}a_1$

$=> k(n-2)[\,n^2 a_n a_0 - a_{n-1}a_1] = 2*(\frac{n(n-1)}{2})\,a_{n-1}a_0 - 2a_{n-2}a_1$

$=> k(n-2)[\,n^2 a_n a_0 - a_{n-1}a_1] = 2*\binom{n}{2}.\,a_{n-1}a_0 - 2a_{n-2}a_1$ [from properties of Combinations[3], $\binom{n}{2} = \frac{n(n-1)}{2}$]

$=> k(\frac{n-2}{2})[\,n^2 a_n a_0 - a_{n-1}a_1] = [\,\binom{n}{2}\,a_{n-1}a_0 - a_{n-2}a_1]$

$=> k = \left(\frac{2}{n-2}\right)[\,\frac{\binom{n}{2}a_0 a_{n-1} - a_1 a_{n-2}}{n^2 a_0 a_n - a_1 a_{n-1}}\,] = e$

But in the beginning we have assumed $\alpha = -e$

a_{n-1} $\forall, n > 2$

$\alpha = \left(\frac{2}{n-2}\right)[\,\frac{a_1 a_{n-2} - \binom{n}{2}\,a_0 a_{n-1}}{n^2 a_0 a_n - a_1 a_{n-1}}] \rightarrow (15)$

Now divide (1) with a_n then,

$\frac{a_{n-1}}{a_n} = \frac{p + a_n.(n-1).e}{a_n}$

$= \frac{p}{a_n} + (n-1).\,e$

$= \frac{p}{m} + (n-1).e$ [a_n = m]

On multiplying with "$-$ "we get,

$\frac{-a_{n-1}}{a_n} = \frac{-p}{m} - (n-1).e$

$=>$ S $= \beta - (\,n-1).e$ [from vieta's theorem in 7.2, we know that, s = $\frac{-a_{n-1}}{a_n}$]

$=> \beta$ = S + $(n-1).e$

$=> \beta$ = S$-(\,n-1).\boldsymbol{\alpha}$ [$\alpha = -\,e$]

β = S$-(\,n-1).\,\boldsymbol{\alpha}$

Now again consider the discriminant of the Quadratic equation

$[a_n.(\frac{n(n-2)}{2})].e^2 - [a_{n-1}.(\,n-1)].e + a_{n-2} = 0 \rightarrow (8)$

$\boldsymbol{D_n} = \boldsymbol{{a_{n-1}}^2} - \left(\frac{\mathbf{2}\boldsymbol{n a_n a_{n-2}}}{\boldsymbol{n-1}}\right) \rightarrow (16)$

Now,
On substituting the values of a_n, a_{n-1} and a_{n-2} , we get

$$D = [p + m(n-1)e]^2 - \frac{2mn\,[p(n-1)e + m*\binom{n-1}{2}*e^2]}{n-1}$$
$$= p^2 + m^2(n-1)^2e^2 + 2pme(n-1) - 2mnpe - nm^2(n-2)e^2$$
$$= p^2 + m^2(n-1)^2e^2 + 2pmne - 2pme - 2nmpe - nm^2(n-2)e^2$$
$$= p^2 + m^2(n-1)^2e^2 - 2pme - nm^2(n-2)e^2$$
$$= p^2 + m^2e^2 - 2m^2ne^2 - 2pme + 2m^2ne^2$$
$$= p^2 + m^2e^2 - 2pme$$
$$= \mathbf{(}p - me\mathbf{)^2}$$, which is a Perfect square .

5. SOME EXTRACTIONS FROM THE FORMULA WITH EXAMPLES

5.1 When a Cubic Equation $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ is said to have 2 equal roots and a simple root then,
For n=3>2, we have,

Discriminant $\boldsymbol{D_n} = {a_{n-1}}^2 - \left(\frac{2na_na_{n-2}}{n-1}\right)$

=> $D_3 = {a_2}^{\mathbf{2}} - \left(\frac{\mathbf{2*3*}a_3*a_1}{\mathbf{3-1}}\right) = {a_2}^{\mathbf{2}} - \mathbf{3}a_3a_1$

Equal root $\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right)\left[\frac{a_1a_{n-2} - \binom{n}{2}a_0a_{n-1}}{\boldsymbol{n^2}a_0a_n - a_1a_{n-1}}\right]$

$= \left(\frac{2}{3-2}\right)\left[\frac{{a_1}^2 - \binom{3}{2}a_0a_2}{3^2a_0a_3 - a_1a_2}\right]$

$= 2\left[\frac{{a_1}^2 - 3_{C_2}\binom{3}{2}a_0a_2}{9a_0a_3 - a_1a_2}\right]$

$\boldsymbol{\alpha} = \left[\frac{2{a_1}^2 - 6a_0a_2}{9a_0a_3 - a_1a_2}\right]$

And the Simple root β = **S - (n-1)* α**

= S - (3-1)*$\boldsymbol{\alpha} = \mathbf{S} - \mathbf{2}\boldsymbol{\alpha}$ where **α** is the Equal root and Sum of roots **S** = $\frac{-a_2}{a_3}$.

5.1.1 Example: Solve $x^3 + 6x^2 + 9x + 4 = 0$

Solution: Given that,

a_3=1, a_2= 6, a_1= 9, a_0= 4

S = $\frac{-a_2}{a_3}$ = -6

For n=3>2, we have,
Discriminant $D_3 = {a_2}^{\mathbf{2}}$ - $\mathbf{3}a_3a_1$= 6^2-3*1*9 = **9(Perfect Square)**

The given equation is said to have a double equal root

$x_{eq} = \left[\frac{2{a_1}^2 - 6a_0a_2}{9a_0a_3 - a_1a_2}\right] = \left[\frac{2*9*9-6*6*4}{9*1*4-6*9}\right] = -1$

And the simple root β = **S-(n-1)* α** = S-2* **α** = - 6 + 2 = -4

Hence, the roots are x= **(-1,-1,-4)**

5.2 When a Quartic Equation $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ is said to have 3 equal roots and a simple root then,
For n = 4 > 2, we have,

Discriminant $\boldsymbol{D_n} = {a_{n-1}}^2 - \left(\frac{2na_na_{n-2}}{n-1}\right)$

=> $D_4 = {a_3}^{\mathbf{2}} - \left(\frac{\mathbf{2*4*}a_4*a_2}{\mathbf{4-1}}\right) = {a_3}^2 - \frac{\mathbf{8}a_4a_2}{\mathbf{3}}$

Equal root $\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right)\left[\frac{a_1a_{n-2} - \binom{n}{2}a_0a_{n-1}}{\boldsymbol{n^2}a_0a_n - a_1a_{n-1}}\right]$

$= \left(\frac{2}{4-2}\right)\left[\frac{a_1a_2 - \binom{4}{2}a_0a_3}{4^2a_0a_4 - a_1a_3}\right]$

$= \left[\frac{a_1a_2 - \binom{4}{2}a_0a_3}{16a_0a_4 - a_1a_3}\right]$

$\boldsymbol{\alpha} = \left[\frac{\boldsymbol{a_1a_2 - 6a_0a_3}}{\boldsymbol{16a_0a_4 - a_1a_3}}\right]$

And the Simple root β = **S-(n-1)* α**

= S-(4-1)*$\boldsymbol{\alpha} = \mathbf{S} - \mathbf{3}\boldsymbol{\alpha}$ where **α** is the Equal root and Sum of roots **S** = $\frac{-a_3}{a_4}$.

5.2.1 Example: Solve $x^4 - x^3 - 3x^2 + 5x - 2 = 0$

Solution: Given that,

a_4=1, a_3=-1, a_2=-3, a_1=5, a_0=-2

s= $\frac{b}{a}\frac{-a_3}{a_4}$ = 1

For n=4>2, we have,

Discriminant $D_4 = {a_3}^{\mathbf{2}} - \frac{\mathbf{8ac}}{\mathbf{3}}\frac{\mathbf{8}a_4a_3}{\mathbf{3}} = (-1)^2 - \frac{8*1*(-3)}{3}$

= 9(Perfect Square)

The given equation is said to have a triple equal root

$$x_{eq} = [\frac{\boldsymbol{a_1a_2 - 6a_0a_3}}{\boldsymbol{16a_0a_4 - a_1a_3}}]$$
$$= [\frac{(-3)*5-6*(-1)*(-2)}{16*1*(-2) -(-1)*5}]$$
$$=1$$

And the simple root β = **S-(n-1)* α** = S-3*1x_{eq} = 1-3 = -2

Hence, the roots are x= **(1, 1, 1,-2).**

5.3 When a Quintic Equation $a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0= 0$ is said to have 4 equal roots and a simple root then,

For n=5>2, we have,

Discriminant $\boldsymbol{D_n} = {a_{n-1}}^2 - \left(\frac{2na_na_{n-2}}{n-1}\right)$

$=> D_5 = {a_4}^{\mathbf{2}} - \left(\frac{\mathbf{2*5}*a_5*a_3}{\mathbf{5-1}}\right) = {a_4}^{\mathbf{2}} - \frac{\mathbf{5}a_5a_3}{\mathbf{2}}$

Equal root $\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right) [\frac{a_1a_{n-2}-\binom{n}{2}a_0a_{n-1}}{\boldsymbol{n^2}a_0a_n - a_1a_{n-1}}]$

$$= \left(\frac{2}{5-2}\right) [\frac{a_1a_3-\binom{5}{2}a_0a_4}{5^2a_0a_5-a_1a_4}]$$
$$= \frac{2}{3}[\frac{a_1*a_3- 10*a_0*a_4}{25*a_0*a_5-a_1*a_4}]$$
$$\boldsymbol{\alpha} = [\frac{\mathbf{2}a_3a_1-\mathbf{20}a_4a_0}{\mathbf{75}a_5a_0 -\mathbf{3}a_4a_1}]$$

And the Simple root β = **S-(n-1)* α**

= S-(5-1)*$\boldsymbol{\alpha} = \mathbf{S - 4\alpha}$ where **α** is the Equal root and Sum of roots **S** = $\frac{-a_4}{a_5}$.

5.3.1 Example: Solve $x^5-2x^4-2x^3+8x^2-7x+2 = 0$

Solution: Given that,

a_5=1, a_4=-2, a_3= -2, a_2= 8, a_1 = -7, a_0 = 2

S = $\frac{-a_4}{a_5}$ = 2

For n=5>2, we have,

Discriminant $D_5 = {a_4}^{\mathbf{2}} - \frac{\mathbf{5}a_5a_3}{\mathbf{2}}$

$= (-2)^2 - \frac{5*1*(-2)}{2}$

= 9(Perfect Square)

The given equation is said to have a quadruple or 4 equal roots

$$\boldsymbol{\alpha} = [\frac{\mathbf{2}a_3a_1-\mathbf{20}a_4a_0}{\mathbf{75}a_5a_0 -\mathbf{3}a_4a_1}]$$
$$= [\frac{2*(-2)*(-7)-20*(-2)*2}{75*1*2 -3*(-2)*(-7)}]$$
$$= 1$$

And the simple root β = **S-(n-1)* α** =S-4* **α** = 2-4 = -2

Hence, the roots are x = **(1, 1, 1, 1,-2)**

5.4 When a sextic or hexic Equation $a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0= 0$ is said to have 5 equal roots and a simple root then,
For n = 6 > 2, we have,
Discriminant $\boldsymbol{D_n} = {a_{n-1}}^2 - \left(\frac{2na_n a_{n-2}}{n-1}\right)$

$\Rightarrow D_6 = {a_5}^{\mathbf{2}} - \left(\frac{\mathbf{2*6*}a_6*a_4}{\mathbf{6-1}}\right) = {a_5}^{\mathbf{2}} - \frac{\mathbf{12}a_6a_4}{\mathbf{5}}$

Equal root $\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right)\left[\frac{a_1a_{n-2}-\binom{n}{2}a_0a_{n-1}}{\boldsymbol{n^2}a_0a_n-a_1a_{n-1}}\right]$

$= \left(\frac{2}{6-2}\right)\left[\frac{a_1a_4-\binom{6}{2}a_0a_5}{6^2a_0a_6-a_1a_5}\right]$

$= \frac{1}{2}\left[\frac{a_1a_4-15a_0a_5}{36a_0a_6-a_1a_5}\right]$

$\boldsymbol{\alpha} = \left[\frac{a_1a_4-15a_0a_5}{72a_0a_6-2a_1a_5}\right]$

And the Simple root β = **S-(n-1)* α**

= S-(6-1)*$\boldsymbol{\alpha} = \mathbf{S-5}\boldsymbol{\alpha}$ where **α** is the Equal root and Sum of roots **S =** $\frac{-a_5}{a_6}$.

5.4.1 Example: Solve $x^6+3x^5-10x^3-15x^2-9x-2 = 0$

Solution: Given that,

$a_6=1$, $a_5=3$, $a_4=0$, $a_3=-10$, $a_2=-15$, $a_1=-9$, $a_0= -2$

$S = \frac{-a_5}{a_6} = -3$

For n = 6 > 2, we have,

Discriminant $D_6 = {a_5}^{\mathbf{2}} - \frac{\mathbf{12}a_6a_4}{\mathbf{5}}$

$= 3^2 - \frac{12*1*0}{5}$

= 9(Perfect Square)

The given equation is said to have 5 equal roots

$\boldsymbol{\alpha} = \left[\frac{a_1a_4-15a_0a_5}{72a_0a_6-2a_1a_5}\right]$

$= \left[\frac{0*(-9)-15*3*(-2)}{72*1*(-2)-2*3*(-9)}\right]$

$= -1$

And the simple root β = **S-(n-1)* α** =S-5* **α** = -3+5= 2
Hence, the roots are x= **(-1,-1,-1,-1,-1, 2)**

5.5 When a Septic or heptic Equation $a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ is said to have 6 equal roots and a simple root then,

For n=7>2, we have,
Discriminant $\boldsymbol{D_n} = {a_{n-1}}^2 - \left(\frac{2na_n a_{n-2}}{n-1}\right)$

$\Rightarrow D_7 = {a_6}^{\mathbf{2}} - \left(\frac{\mathbf{2*7*}a_7*a_5}{\mathbf{7-1}}\right) = {a_6}^{\mathbf{2}} - \frac{\mathbf{7}a_7a_5}{\mathbf{3}}$

Equal root $\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right)\left[\frac{a_1a_{n-2}-\binom{n}{2}a_0a_{n-1}}{\boldsymbol{n^2}a_0a_n-a_1a_{n-1}}\right]$

$= \left(\frac{2}{7-2}\right)\left[\frac{a_1a_5-\binom{7}{2}a_0a_6}{7^2a_0a_7-a_1a_6}\right]$

$= \frac{2}{5}\left[\frac{a_1a_5-21a_0a_6}{49a_0a_7-a_1a_6}\right]$

$\boldsymbol{\alpha} = \frac{\mathbf{2}}{\mathbf{5}}\left[\frac{a_1a_5-21a_0a_6}{49a_0a_7-a_1a_6}\right]$

And the Simple root β = S-(n-1)***α**

= S-(7-1)*$\boldsymbol{\alpha} = \mathbf{S-6}\boldsymbol{\alpha}$ where **α** is the Equal root and Sum of roots **S =** $\frac{-a_6}{a_7}$

5.5.1 Example: Solve $x^7+x^6-15x^5-55x^4-85x^3-69x^2-29x-5=0$

Solution: Given that,

$a_7=1$, $a_6=1$, $a_5= -15$, $a_4 = -55$, $a_3=-85$, $a_2=-69$, $a_1=-29$, $a_0=-5$

$s= \frac{-a_6}{a_7} = -1$

For n=7>2, we have,

Discriminant $D_7 = a_6^{\mathbf{2}} - \frac{\mathbf{7}a_7a_5}{\mathbf{3}}$

$= 1^2 - \frac{7*1*(-15)}{3}$

= 36(Perfect Square)

The given equation is said to have 6 equal roots

$$\boldsymbol{\alpha} = \frac{\mathbf{2}}{\mathbf{5}}\left[\frac{a_1a_5 - 21a_0a_6}{49a_0a_7 - a_1a_6}\right]$$

$$= \frac{2}{5}\left[\frac{(-15)*(-29)-21*1*(-5)}{49*1*(-5) - 1*(-29)}\right]$$

$= -1$

And the simple root β = **S-(n-1)* α** = S - 6* **α** = -1 + 6 = 5

Hence, the roots are x= **(-1,-1,-1,-1,-1,-1, 5)**

5.6 When a **25th Degree** equation

$a_{25}x^{25}+a_{24}x^{24}+a_{23}x^{23}+a_{22}x^{22}+a_{21}x^{21}+a_{20}x^{20}+a_{19}x^{19}+a_{18}x^{18}+a_{17}x^{17}+a_{16}x^{16}+a_{15}x^{15}+a_{14}x^{14}$ $+a_{13}x^{13}+a_{12}x^{12}+a_{11}x^{11} + a_{10}x^{10}+a_9x^9+a_8x^8+a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ is said to have 24 equal roots and a simple root then

For n=25>2 we have,

Discriminant $\boldsymbol{D_n} = a_{n-1}{}^2 - \left(\frac{2na_na_{n-2}}{n-1}\right)$

$\Rightarrow D_{25} = a_{24}{}^{\mathbf{2}} - \left(\frac{\mathbf{2*25}*a_{25}*a_{23}}{\mathbf{25-1}}\right) = a_{24}{}^{\mathbf{2}} - \frac{\mathbf{25}a_{25}a_{23}}{\mathbf{12}}$

Equal root $\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right)\left[\frac{a_1a_{n-2}-\binom{n}{2}a_0a_{n-1}}{\boldsymbol{n^2}a_0a_n - a_1a_{n-1}}\right]$

$= \left(\frac{2}{25-2}\right)\left[\frac{a_1a_{23}-\binom{25}{2}a_0a_{24}}{25^2a_0a_{25}-a_1a_{24}}\right]$

$= \left(\frac{2}{23}\right)\left[\frac{a_1a_{23}-\binom{25}{2}a_0a_{24}}{25^2a_0a_{25}-a_1a_{24}}\right]$

$\boldsymbol{\alpha} = \left\{\left(\frac{\mathbf{2}}{\mathbf{23}}\right)\left[\frac{a_{23}a_1-\mathbf{300}a_{24}a_0}{\mathbf{625}a_{25}a_0 - a_{24}a_1}\right] \text{ or } \left[\frac{\mathbf{2}a_{23}a_1-\mathbf{600}a_{24}a_0}{\mathbf{14375}a_{25}a_0 - a_{24}a_1}\right]\right\}$

And the Simple root β = **S-(n-1)* α**

= S-(25-1)*$\boldsymbol{\alpha} = S - 24\boldsymbol{\alpha}$ where **α** is the Equal root and Sum of roots **S** = $\frac{-a_{24}}{a_{25}}$.

5.6.1 Example:

$X^{25}+26x^{24}+324x^{23}+2576x^{22}+14674x^{21}+63756x^{20}+219604x^{19}+615296x^{18}+1427679x^{17}+2778446x^{16}+4576264x^{15}$ $+6418656x^{14}+7696444x^{13}+7904456x^{12}+6953544x^{11}+5230016x^{10}+3350479x^9+1817046x^8+826804x^7+311696x^6$ $+95634x^5+23276x^4+4324x^3+576x^2+49x+2=0$

Solution: Given that,

$a_{25}=1$, $a_{24}=26$, $a_{23}=324$, $a_1=49$ & $a_0=2$

$s = \frac{-a_{24}}{a_{25}} = -26$

For n = 25>2, we have,

Discriminant $D_{25} = a_{24}{}^{\mathbf{2}} - \frac{\mathbf{25}a_{25}a_{23}}{\mathbf{12}}$

$= 26^2 - \frac{25*1*324}{12}$

= 1(Perfect Square)

The given equation is said to have 24 equal roots

$\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\mathbf{23}}\right)\left[\frac{a_1a_{23}-{}^{25}C_2a_0a_{24}}{25^2a_0a_{25}-a_1a_{24}}\right]$

$= \left(\frac{2}{23}\right)\left[\frac{324*49-300*26*2}{625*1*2 - 26*49}\right]$

$= \left(\frac{2}{23}\right)*\left(\frac{276}{-24}\right)$

$= -1$

And the simple root β = **S-(n-1)* α**

= S-24* **α** = -26+24 = -2

Hence, the roots are

X= **(-1, -1, -1, -1, -1, -1, -1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-2)**

5.7 Likewise, for any higher degree such as **n=999**, we can easily find its discriminant, **998 equal roots** and the Simple root as below,
Let the 999th Degree equation be as

$$f(x) = a_{999}x^{999} + a_{998}x^{998} + a_{997}x^{997} + a_{996}x^{996} + \ldots + a_1 x + a_0 = 0$$

Then the Discriminant

$$\boldsymbol{D_n} = {a_{n-1}}^2 - \left(\frac{2n a_n a_{n-2}}{n-1}\right)$$

$$\Rightarrow D_{999} = {a_{998}}^{\mathbf{2}} - \left(\frac{\mathbf{2*999}*a_{999}*a_{997}}{\mathbf{999-1}}\right)$$

$$= {a_{998}}^{\mathbf{2}} - \left(\frac{\mathbf{999}a_{999}a_{997}}{\mathbf{499}}\right) \text{ which is a Perfect Square.}$$

Now the value of each 998 equal roots is given by

$$\text{Equal root } \boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\boldsymbol{n-2}}\right)\left[\frac{a_1 a_{n-2} - \binom{n}{2} a_0 a_{n-1}}{\boldsymbol{n^2} a_0 a_n - a_1 a_{n-1}}\right]$$

$$= \left(\frac{2}{999-2}\right)\left[\frac{a_1 a_{997} - \binom{999}{2} a_0 a_{998}}{999^2 a_0 a_{999} - a_1 a_{998}}\right]$$

$$\Rightarrow \boldsymbol{\alpha} = \begin{cases} \left(\frac{\mathbf{2}}{\mathbf{997}}\right)\left[\frac{a_1 a_{997} - 49850\, a_0 a_{998}}{998001\, a_0 a_{999} - a_1 a_{998}}\right] \\ \textbf{or} \\ \left[\frac{\mathbf{2}a_0 a_{997} - \mathbf{997002}a_0 a_{998}}{\mathbf{995006997}a_0 a_{999} - \mathbf{997}a_1 a_{998}}\right] \end{cases}$$

And the Simple root β = S-(n-1)* $\boldsymbol{\alpha}$

$= S-(999-1)*\boldsymbol{\alpha} = \mathbf{S - 998\alpha}$ where $\boldsymbol{\alpha}$ is the Equal root and Sum of roots $\mathbf{S} = \frac{-a_{998}}{a_{999}}$.

5.7.1 Example

Let us first frame a 999th degree equation which is having 998 equal roots of "**-1**"and a simple root "**-2**" as below:
$(x+1)^{998}(x+2)=0$

As we know that the kth term of the expression $(x+e)^{n-1}(mx+p)$ from above (x) we have,

$$\boldsymbol{a_{n-k}} = [\mathbf{m.}\binom{\boldsymbol{n-1}}{\boldsymbol{k}}\mathbf{.\ e^k} + \mathbf{p.}\binom{\boldsymbol{n-1}}{\boldsymbol{k-1}}\mathbf{.\ e^{k-1}}]$$

e=1, m=1,p=2,n=999

We have,

$$a_{999} = a_{999-0} = = [1*\binom{999-1}{0}*1^0 + 2*\binom{999-1}{0-1}*1^{(0-1)}] = 1$$

$$a_{998} = a_{999-1} = [1*\binom{999-1}{1}*1^1 + 2*\binom{999-1}{1-1}*1^{1-1}] = 1000$$

$$a_{997} = a_{999-2} = [1*\binom{999-1}{2}*1^2 + 2*\binom{999-1}{2-1}*1^{2-1}] = 499499$$

$$a_1 = a_{999-998} = [1*\binom{999-1}{998}*1^{998} + 2*\binom{999-1}{997}*1^{997}] = 1997$$

$$a_0 = a_{999-999} = [1*\binom{999-1}{999}*1^{999} + 2*\binom{999-1}{998}*1^{998}] = 2$$

So the equation is

$$\mathbf{x^{999} + 1000x^{998} + 499499x^{997} + \ldots + 1997x + 2 = 0}$$

Solution: Given that,

a_{999}=1, a_{998}= 1000, a_{997}= 499499, a_1= 1997, a_0 =2

$$S = \frac{-a_{998}}{a_{999}} = -1000$$

For n=999>2, we have,

$$\text{Discriminant } D_{999} = {a_{998}}^{\mathbf{2}} - \left(\frac{\mathbf{999}a_{999}a_{997}}{\mathbf{499}}\right)$$

$$= (1000)^2 - \frac{999*1*499499}{499}$$

$$= 10,00,000 - 999,999$$

$$= \mathbf{1(Perfect\ Square)}$$

The given equation is said to have a 998 equal roots

$$\boldsymbol{\alpha} = \left(\frac{\mathbf{2}}{\mathbf{997}}\right)\left[\frac{a_1 a_{997} - 49850 a_0 a_{998}}{998001 a_0 a_{999} - a_1 a_{998}}\right]$$

$$= \frac{2}{997}\left[\frac{499499*1997 - 498501*1000*2}{998001*1*2 - 1000*1997}\right]$$

$$= \frac{2}{997}\left[\frac{497503}{-998}\right]$$

$$= -1$$

And the simple root β = **S-(n-1)*** $\boldsymbol{\alpha}$

= S-998* $\boldsymbol{\alpha}$ = -1000+998= -2

Hence, the roots are

x= **[-1, -1, -1, -1, ...,-1(998 times), -2]**

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