International Journal of Mathematical Archive-2(10), 2011, Page: 1841-1848
Available online through www.ijma.info ISSN 2229 – 5046

A NOTE ON DYNAMICAL SYSTEMS ON HILBERT C^*-MODULES AND DYNAMICAL SYSTEMS ON C^*-ALGEBRAS

M. Khanehgir*

Department of Mathematics, Faculty of science, Islamic Azad University-Mashhad Branch, Mashhad, Iran, P. O. Box 413-91735.

E-mail: khanehgir@mshdiau.ac.ir

(Received on: 22-08-11; Accepted on: 30-09-11)

ABSTRACT

Suppose that C is a unitary operator on a full Hilbert C^*-module M. We describe dynamical systems on a full Hilbert C^*-module M over a C^*-algebra A as a one-parameter C-group of unitaries on M. We investigate dynamical systems on a full Hilbert module over a Frechet locally C^*-algebra. Then we discuss two-parameter dynamical systems on a full Hilbert C^*-module M over a C^*-algebra. Finally we characterize infinitesimal generator of a uniformly continuous positive $C-$semigroup on a C^*-algebra.

2000 AMS subject classification: *46L05,46L08,47B48,47B65,47D03.*

KeyWords: *Full Hilbert C^*-module , Dynamical system, Unitary operator, Frechet locally C^*-algebra, C-semigroup, Infinitesimal generator and generalized derivation.*

1.INTRODUCTION:

A pre-Hilbert C^*-module over a C^*-algebra A is an algebraic left $A-$module M equipped with an $A-$valued inner product $\langle .,. \rangle$ which is $A-$linear in the first variable and for every $x, y \in M$ satisfies the following relations:
(1) $\langle x,x \rangle \geq 0$;
(2) $\langle x,x \rangle = 0$ if and if $x = 0$;
(3) $\langle x,y \rangle^* = \langle y,x \rangle$.

We say M is a Hilbert C^*-module if it is a Banach space with respect to the norm $\| x \| = \| \langle x,x \rangle \|^{\frac{1}{2}}$. A Hilbert module M is called full if the closed linear span $\langle M, M \rangle$ of all elements of the form $\langle x,y \rangle$ $(x, y \in M)$ is equal to A. It can be easily proved that if M is a full Hilbert module over a C^*-algebra A then $a = 0$ if and only if $ax = 0$ for all $x \in M$. For more details on Hilbert modules one can see [9].

Now let M and N be Hilbert modules over C^*-algebras A and B respectively. Following [2] we call a map $\Phi: M \to N$ is a unitary operator if there exists an injective morphism of C^*-algebras $\varphi: A \to B$ such that Φ is a surjective φ-morphism. The set of all unitary operators on M together with composition of maps form a group which is denoted by $U(M)$. If M is full and $\alpha: M \to M$ is a unitary operator than there is a $*$-isomorphism $\alpha': A \to A$ such that α is an $\alpha'-$morphism ([2] Remark 2.9).

Abbaspour, Moslehian and Niknam in [1] characterized a dynamical system on a full Hilbert module M over a C^*-algebra A as a one-parameter group of unitaries on M. They proved that if $\alpha: R \to U(M)$ is a dynamical system, then it corresponds to a C^*-dynamical system α' on A such that if δ and d are the infinitesimal generators

***Corresponding author: *M.Khanehgir*,*E-mail:* khanehgir@mshdiau.ac.ir**

of α and α' respectively, then δ is a $d-$generalized derivation. In this paper we first extend this to one-parameter $C-$group of unitaries on a full Hilbert C^*-module M where C is a unitary operator on M. For more details on $C-$groups see [4]. We discuss dynamical systems on a full Hilbert module over a Frechet locally C^*-algebra. A locally C^*-algebra is a complete Hausdorff complex $*$-algebra A whose topology is determined by its continuous C^*-seminorms. The set of all continuous C^*-seminorms on A is denoted by $S(A)$. A Frechet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. Clearly any C^*-algebra is a Frechet locally C^*-algebra. In [8], Joita characterized a unitary operator $\Phi: E \to F$, where A and B are two Frechet locally C^*-algebras and E and F are full Hilbert modules over A and B respectively. We describe dynamical systems on these spaces. We also investigate two-parameter dynamical systems on a full Hilbert C^*-module. The reader is referred to [7] for more details on two-parameter dynamical systems and to [11] on C^*-dynamical systems.

In Sec. 3, we describe dynamical systems on a C^*-algebra. More precisely we characterize the infinitesimal generator of a uniformly continuous (strongly continuous) positive C-semigroups on a C^*-algebra.

2. C-GROUP OF UNITARIES ON A FULL HILBERT MODULE:

Definition: 2.1. Suppose that C is a unitary operator on a full Hilbert C^*-module M. The family of unitary operators $\{\alpha_t\}_{t\in\mathbb{R}}$ on M is a C-group if it has the following properties:

$(i)\ \alpha_0 = C$.
$(ii)\ \alpha_t\alpha_s(x) = C\alpha_{t+s}(x), t, s \in \mathbb{R}$.
$(iii)\ \alpha_t$ is strongly continuous, that is, for all $x \in M$ the map $t \to \alpha_t(x)$ form $\mathbb{R}$ into M is continuous.

The operator δ is the infinitesimal generator of C-group $\{\alpha_t\}_{t\in\mathbb{R}}$ if

$$D(\delta) = \{x \mid \lim_{t\to 0} \frac{\alpha_t(x) - C(x)}{t}\ exists\}$$

with

$$\delta(x) = C^{-1}(\lim_{t\to 0} \frac{\alpha_t(x) - C(x)}{t})$$

Note that by Theorem 3.4 of [8] C^{-1} exists.

A small modification in the proof of Theorem 4.3 in Ref. 1 gives the following result.

Theorem: 2.2. Let M be a full Hilbert A-module and C be a unitary operator on M. Let α be a C-group on M and δ be the infinitesimal generator of dynamical system α. Then there exists a derivation $d: D(d) \subseteq A \to A$ such that $D(\delta)$ is a left $D(d)$-module and

$\delta(ax) = a\delta(x) + d(a)x, a \in D(d), x \in D(\delta)$.

Proof:. Since α is a C-group on M, for each $t \in \mathbb{R}$, the mapping $\alpha_t : M \to M$ is a unitary, so there exists $*$-isomorphism $\alpha_t' : A \to A$ such that

$\langle \alpha_t(x), \alpha_t(y)\rangle = \alpha_t'(\langle x, y\rangle)$, and $\alpha_t(ax) = \alpha_t'(a)\alpha_t(x)\ (a \in A, x \in M)$.

Analogy since C is a unitary operator on M, so there exists a $*$-isomorphism $c': A \to A$ such that $\langle C(x), C(y)\rangle = c'(\langle x, y\rangle)$ and $C(ax) = c'(a)C(x)$. Now we show that $\alpha': \mathbb{R} \to Aut(A)$ is a c'-dynamical system on the C^*-algebra A. For each $a \in A, x \in M$ we have

$$c'(a)C(x) = C(ax) = \alpha_0(ax) = \alpha_0'(a)\alpha_0(x) = \alpha_0'(a)C(x)$$

and this implies that $(c'(a)-\alpha_0'(a))C(x) = 0$. Since C is surjective and M is full so, $\alpha_0'(a) = c'(a)$ for all $a \in A$. Therefore $\alpha_0' = c'$.

Also for all $t, s \in \mathbb{R}$ and $x \in M$ we have

$$\begin{aligned} c'(\alpha_{t+s}'(a))C(\alpha_{t+s}(x)) &= C(\alpha_{t+s}'(a)\alpha_{t+s}(x)) \\ &= C\alpha_{t+s}(ax) \\ &= \alpha_t\alpha_s(ax) \\ &= \alpha_t(\alpha_s'(a)\alpha_s(x)) \\ &= \alpha_t'\alpha_s'(a)\alpha_t\alpha_s(x) \\ &= \alpha_t'\alpha_s'(a)C\alpha_{t+s}(x). \end{aligned}$$

It implies that $[c'\alpha_{t+s}'(a) - \alpha_t'\alpha_s'(a)]C(\alpha_{t+s}(x)) = 0$. So we have

$$c'\alpha_{t+s}'(a) = \alpha_t'\alpha_s'(a).$$

Since for each $x \in M$, $\alpha_t(x) \to C(x)$ as $t \to 0$, we have

$$\begin{aligned} \| \alpha_t'(a)C(x) - c'(a)C(x) \| &\leq \| \alpha_t'(a)C(x) - \alpha_t'(a)\alpha_t(x) \| + \| \alpha_t'(a)\alpha_t(x) - C(ax) \| \\ &= \| \alpha_t'(a)(C(x) - \alpha_t(x)) \| + \| \alpha_t(ax) - C(ax) \| \end{aligned}$$

Thus $\lim_{t\to 0}\alpha_t'(a)x = c'(a)x$ for all $x \in$ range $C(=M)$, whence $\lim_{t\to 0}\alpha_t'(a) = c'(a)$ for all $a \in A$. Therefore $\alpha' : R \to Aut(A)$ is a c'-dynamical system on the C^*-algebra A. If d is the infintesimal generator of α' then for each $a \in D(d), x \in D(\delta)$ we have

$$\begin{aligned} \delta(ax) &= C^{-1}[\lim_{t\to 0}\frac{\alpha_t(ax) - C(ax)}{t}] \\ &= C^{-1}[\lim_{t\to 0}\frac{\alpha_t'(a)\alpha_t(x) - c'(a)\alpha_t(x)}{t} + \lim_{t\to 0}\frac{c'(a)\alpha_t(x) - c'(a)C(x)}{t}] \\ &= c'^{-1}(\lim_{t\to 0}\frac{\alpha_t'(a) - c'(a)}{t})C^{-1}(\lim_{t\to 0}\alpha_t(x)) + c'^{-1}(c'(a))C^{-1}(\lim_{t\to 0}\frac{\alpha_t(x) - C(x)}{t}) \\ &= d(a)x + a\delta(x). \end{aligned}$$

Note that by Remark 3.3 and Corollary 3.6 of [8], C^{-1} is a module map in the sense $C^{-1}(ax) = c'^{-1}(a)C^{-1}(x)$. From above relations we conclude that $ax \in D(\delta)$ and $\delta(ax) = a\delta(x) + d(a)x$. Furthermore, $D(\delta)$ is a left $D(d)$-module.

Analogue of Theorem 2.2 holds for full Hilbert module M over Frechet locally C^*-algebra A (see[8]). In this case a linear map $\Phi : E \to F$, (E, F are Hilbert modules over locally C^*-algebra A, B respectively) is a unitary operator if Φ is surjective and there is an injective morphism of locally C^*-algebra $\varphi : A \to B$ with closed range such that $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $\xi, \eta \in E$.

Now we describe two-parameter dynamical systems on a Hilbert C^*-module.

Definition: 2.3. By a two-parameter dynamical system on a Hilbert C^*-module M we mean a function from $\mathbb{R}\times\mathbb{R}$ into $U(M)$ such that

$(i)\ \alpha(s+s',t+t')=\alpha(s,t)\alpha(s',t')\ (t,t',s,s'\in\mathbb{R})$;
$(ii)\ \alpha(0,0)=I$.

It is called strongly continuous if $(s,t)\to\alpha(s,t)x$ is continuous for all x in M.

To any two-parameter dynamical system $\alpha(s,t)$ we associate two one-parameter groups $\alpha(s,0)$ and $\alpha(0,t)$, the group property of α implies that $\alpha(s,t)=\alpha(s,0)\alpha(0,t)$.

One can see that $\alpha(s,t)$ is strongly continuous if and only if $\alpha(s,0)$ and $\alpha(0,t)$ are strongly continuous (see[7]). The infinitesimal generator of $\alpha(s,0)$ and $\alpha(0,t)$ are denoted by δ_1 and δ_2, respectively. We will think of the pair (δ_1,δ_2) as the infinitesimal generator of $\alpha(s,t)$.

Theorem: 2.4. Let A be a C^*-algebra and M be a full Hilbert A-module. Also let α be a two-parameter dynamical system on M and (δ_1,δ_2) be the infinitesimal generator of α. Then $D(\delta_i)$, $i=1,2$ is a dense subspace of M and there exist derivations $d_i: D(d_i)\subseteq A\to A, i=1,2$ such that $D(\delta_i)$ is a left $D(d_i)$-module and $\delta_i(ax)=a\delta_i(x)+d_i(a)x, a\in D(d_i), x\in D(\delta_i)$.

Proof: By Hille-Yosida Theorem [10], $D(\delta_i)$ is a dense subspace of M. Since α is a two-parameter dynamical system on M, for each $t,s\in\mathbb{R}$, the mapping $\alpha(s,t)$ is a unitary, so there exist a $*$-isomorphism

$\alpha'(s,t): A\to A$ such that $\langle\alpha(s,t)x,\alpha(s,t)y\rangle=\alpha'(s,t)(\langle x,y\rangle)$ and

$\alpha(s,t)(ax)=\alpha'(s,t)(a)\alpha(s,t)(x), (a\in A, x\in M)$. Now by applying the argument similar to Theorem 4.3 of [1], it can be easily proved.

3. CHARACTERIZATION OF UNIFORMLY (STRONGLY) CONTINUOUS POSITIVE C-SEMIGROUPS ON A C^*-ALGEBRA:

In 1979 Evans and Olsen in [6] characterized infinitesimal generator of a positive uniformly continuous semigroup on a C*-algebra A. They proved L is the infinitesimal generator of this semigroup if it satisfies in the one of the following inequalities:

$(i)\ L(a^2)+aL(1)a\geq L(a)a+aL(a)$ for all self-adjoint operator $a\in A$.
$(ii)\ L(1)+u^*L(1)u\geq L(u^*)u+u^*L(u)$ for all unitary operators $u\in A$.

Then in 1981 Bratteli and Robinson generalized it for C_0-semigroup of positive operators (see [3]). In this generalization, they obtained above inequalities with resolvent $R(\lambda,L)$ instead of L.

Later the notion of $C-$semigroups was introduced by Da Prato (1967), E. B. Davies and M. M. Pang (1989) independently. $C-$semigroups are a significant generalization of strongly continuous semigroups. C may be any bounded, injective operator, where C equals the identity operator, then a $C-$semigroup is a strongly continuous semigroup (see[4]). The aim of this section is to characterize the infinitesimal generator of uniformly continuous and strongly continuous positive $C-$semigroups on a C*-algebra *A*.

In the following we recall the definition of uniformly continuous C-semigroup.

Definition: 3.1. Suppose that C is a bounded injective linear operator on a Banach space X. The family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on X is called uniformly continuous C-semigroup if in conditions (i), (ii) of Definition 2.1, we put $\alpha_t = S(t)$ and condition (iii) is replaced with the following condition

$$\lim_{t\to 0} \| S(t) - C \| = 0.$$

Infinitesimal generator A of a uniformly continuous C-semigroup $\{S(t)\}_{t\geq 0}$ is defined as Definition 2.1 with $\alpha_t = S(t)$.

For example if $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup generated by A (see[10]) then for any C that commutes with $T(t), t\geq 0, A$ is the generator of uniformly continuous C-semigroup $S(t) = T(t)C$.

Our next results are generalization of Theorem 1.2 of $[10]$.

Theorem: 3.2. If A is a bounded linear operator and C is a bounded injective linear operator on a Banach space X and A commuting with C then A is the infinitesimal generator of a uniformly continuous C-semigroup.

Proof: Set $T(t) = Ce^{tA} = C\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$. This converges in norm for every $t\geq 0$ and defines a bounded linear operator $T(t)$.

It is clear that $T(0) = C$. A straight forward calculation shows that $CT(t+s) = T(t)T(s)$. We have

$$\| T(t) - C \| \leq t \| A \| \| C \| \sum_{n=0}^{\infty} \frac{t^n \| A \|^n}{n!}.$$

The righthand side of above assertion tends to zero as t tends to infinity. Also we have

$$\| \frac{T(t) - C}{t} - CA \| \leq \| A \| \| T(t) - C \|.$$

Similarly the righthand side tends to zero when t tends to infinity by uniformly continuity of $T(t)$. So $\{T(t)\}_{t\geq 0}$ is a uniformly continuous C-semigroup of bounded linear operators on X and A is its infinitesimal generator.

Theorem: 3.3. Let $\{T(t)\}_{t\geq 0}$ be a uniformly continuous C-semigroup on a Banach space X and for each $t\geq 0$, $range\, T(t) \subseteq range(C)$ then infinitesimal generator A is bounded.

Proof.: Fix $\rho > 0$ small enough such that $\| I - \rho^{-1}\int_0^{\rho} C^{-1}T(s)ds \| \leq 1$. Since $I = \frac{1}{\rho}\int_0^{\rho} Ids$ so $I - \rho^{-1}\int_0^{\rho} C^{-1}T(s)ds = \int_0^{\rho} \frac{I - C^{-1}T(s)}{\rho} ds \to 0$ as $\rho \to 0$. Hence $\rho^{-1}\int_0^{\rho} C^{-1}T(s)ds$ is invertible and we have

$$\begin{aligned} h^{-1}(T(h) - C)\int_0^{\rho} C^{-1}T(s)ds &= h^{-1}[\int_0^{\rho} T(h+s)ds - \int_0^{\rho} T(s)ds] \\ &= h^{-1}[\int_h^{\rho+h} T(s)ds - \int_0^{\rho} T(s)ds] \end{aligned}$$

and therefore

$$h^{-1}(T(h) - C) = h^{-1}[\int_{\rho}^{\rho+h} T(s)ds - \int_0^{h} T(s)ds](\int_0^{\rho} C^{-1}T(s)ds)^{-1}.$$

Letting $h \to 0$ so $h^{-1}(T(h) - C)$ converges in norm and therefore strongly to the bounded linear operator $(T(\rho) - I)(\int_0^{\rho} C^{-1}T(s)ds)^{-1}$.

In the following theorems we describe dynamical systems of positive $C-$semigroups on a C*-algebra A. Recall that if S is a set of states on a C*-algebra A, then S is said to be full if $x\in A_h$ (set of self adjoint elements of A) and $f(x)\geq 0$ for all f in S, then $x\geq 0$. Moreover S is said to be invariant if $f\in S$ and $x\in A$ satisfies $f(x^*x)\neq 0$, then $f[x^*(.)x]/f(x^*x)\in S$.

Theorem 3.4. Let L be a bounded self adjoint linear map on a unital C*-algebra A and C be a positive injective bounded linear map on A and $CL=LC, C(ab)=C(a)C(b)$ for each a,b in A and $C(1)=1$ then the following conditions are equivalent

(i) Ce^{tL} is positive for all positive t.
(ii) $(\lambda-CL)^{-1}$ is positive for all large positive λ.
(iii) If $y\in A_+, a\in A$ satisfy $C(ya)=0$ then $a^*CL(y)a>0$.
(iv) For some full invariant set of states S on A that $y\in A_+$, $f\in S$ with $f(C(y))=0$ imply $f(CL(y))\geq 0$.
(v) $CL(x)C(x)+C(x)CL(x)\leq CL(x^2)+C(x)CL(1)C(x)$ for all self adjoint x in A.

Proof. $(iv)\Rightarrow(iii)$

Let S be a full invariant set of states satisfying (iv). Let $y\in A_+, a\in A$ satisfy $C(ya)=0$ then $f(C(a^*ya))>0$ for all f in S. So by (iv), $f(CL(a^*ya))\geq 0$ and hence $f(a^*CL(y)a)\geq 0$, since S is invariant. Thus $a^*CL(y)a>0$ since S is full.

$(iii)\Rightarrow(ii)$

Let $\lambda\leq\|CL\|$. In order to show that $(\lambda-CL)^{-1}\geq 0$ it is enough to show that if $x\in A_h$ satisfies $(\lambda-CL)x\geq 0$ then $x\geq 0$. Let $x=x^+-x^-$ where $x^+,x^-\in A_+$ and $x^+x^-=0$. Thus $C(x^+x^-)=0$ and by

(iii) $x^-CL(x^+)x^-\geq 0$. Now we have

$$0\leq\frac{x^-}{\lambda}[(\lambda-CL)x]x^-=x^-xx^--x^-[\frac{CL(x)}{\lambda}]x^-$$

$$=-(x^-)^3-x^-[\frac{CL(x^+)}{\lambda}]x^-+x^-[\frac{CL(x^-)}{\lambda}]x^-.$$

Thus

$$0\leq(x^-)^3\leq x^-[\frac{CL(x^-)}{\lambda}]x^-.$$

So we have

$$\|x^-\|^3\leq\frac{\|CL\|}{\lambda}\|x^-\|^3.$$

Hence $x^-=0$, as $\lambda\geq\|CL\|$.

$(ii)\Rightarrow(i)$

$$Ce^{tL}=\lim_{n\to\infty}C(1-\frac{t}{n}L)^{-n}=\lim_{n\to\infty}C^{n+1}(C(1)-\frac{t}{n}CL)^{-n}.$$

$(i)\Rightarrow(v)$

Let $L'(x)=L(x)-\frac{1}{2}[L(1)x+xL(1)]$. Then $e^{tL'\geq 0}$ for all $t\geq 0$ by the Lie-Trotter formula and since $Ce^{tL'}(1)=C(1)=1$ so $\|Ce^{tL'}(1)\|=1$. We have by differentiating Kadison's Schwarz inequality, namely

$$Ce^{tL'}(x^2) \geq C(e^{tL'(x)})^2$$

which is valid for all $t \geq 0,$ and all self adjoint x, that

$$C[L'e^{tL'}(x)e^{tL'(x)}] + C[e^{tL'}(x)L'e^{tL'}(x)] \leq CL'e^{tL'}(x^2).$$

Put $t = 0$ so we have

$$CL'(x)C(x) + C(x)CL'(x) \leq CL'(x^2).$$

Substituting for L' gives the desired result.
$(v) \Rightarrow (iv)$

Let $y \in A_+, f \in A_+^*$ with $f(Cy) = 0$ then by Schwartz inequality

$$| f((Cy)^{\frac{1}{2}} z) |^2 \leq f(Cy) f(z^* z).$$

So $f((Cy)^{\frac{1}{2}} z) = f(z(Cy)^{\frac{1}{2}}) = 0$ for all $z \in A$. By (v) we have

$$CL(x)C(x) + C(x)CL(x) \leq CL(x^2) + C(x)CL(1)C(x)$$

for all self adjoint x in A. Put $x = y^{\frac{1}{2}}$ so

$$CL(y^{\frac{1}{2}})C(y^{\frac{1}{2}}) + C(y^{\frac{1}{2}})CL(y^{\frac{1}{2}}) \leq CL(y) + C(y^{\frac{1}{2}})CL(1)C(y^{\frac{1}{2}}).$$

Since $C(y^{\frac{1}{2}}) = (C(y))^{\frac{1}{2}}$ the above statement implies that $f(CL(y)) \geq 0$ and the proof is complete.

Theorem 3.5. Let A be a C*-algebra with identity 1 and C be a bounded injective operator on A which is positive and for each $a, b \in A, C(ab) = C(a)C(b)$.

Let $\{T_t\}_{t\geq 0}$ be an exponentially bounded $C-$semigroup on A $(\| T_t \| \leq Me^{\omega t}$ *for some* $M > 0, \omega \in R,$ for all $t \geq 0)$ with generator L such that $T_t(a^*) = (T_t(a))^*$ for all $a \in A$ and $t \geq 0$. Then the following conditions are equivalent

(i) The $C-$semigroup T_t is positive for all $t \geq 0$.
(ii) The resolvent $(\lambda - L)^{-1}C$ is positive for all $\lambda > 0$ large enough.
(iii) $C(\lambda - L)^{-1}(x)C(x) + C(x)C(\lambda - L)^{-1}(x) \leq C(\lambda - L)^{-1}(x^2) + C(x)C(\lambda - L)^{-1}(1)C(x)$ for all self adjoint x in A and all large $\lambda > 0$.

Proof: $(i) \Leftrightarrow (ii)$

This is a standard result which follows from

$$\forall x \in A;\ T_t(x) = \lim_{n\to\infty}(I - \frac{t}{n}L)^{-1}Cx,\ t \geq 0$$

and

$$C(\lambda - L)^{-1}x = \int_0^\infty e^{-\lambda t}T_t(x)dt, \quad (x \in X).$$

If L is bounded, it is known by the previous theorem that $S_t = Ce^{tL}$ is positive if and only if $CL(x)C(x) + C(x)CL(x) \leq CL(x^2) + C(x)CL(1)C(x)$ for all self adjoint x in A. Hence to establish the theorem it is enough to show that (i) or (ii) is equivalent to the following condition

$(iv)\, Ce^{t(\lambda-L)^{-1}} \geq 0$ for all $t \geq 0$ and all large $\lambda \geq 0$.

$(ii) \Rightarrow (iv)$ This is evident from this expansion

$$Ce^{t(\lambda-L)^{-1}} = C\sum_{n=0}^{\infty}\frac{t^n}{n!}(\lambda-L)^{-1}$$

$(iv) \Rightarrow (i)$ It obtains from

$$T(t)x = \lim_{n\to\infty}\exp\{t[n^2(n-L)^{-1} - nI]\}Cx,\ t \geq 0.$$

Now proof is complete.

REFERENCES:

[1] Abbaspour Tabadkan, Gh. Moslehian, M. S. and Niknam, A.: Dynamical systems on Hilbert C^*-modules, Bulletin of the Iranian Mathematical Society, Vol. 31, No. 1(2005), pp. 25-35.

[2] Bakic, D. and Guljas, B.: On class of module maps of Hilbert C^*-modules. Math. Commun. 7(2002), 177-192.

[3] Bratteli, O. and Robinson, W.D.: Positive C_0-semigroups on C^*-algebras, Math. Scand, Vol. 49, pp. 259-274, 1981.

[4] DeLaubenfels, R.: Existence families, functional calculi and evolution equations, Lecture Notes in Math. 1570, Springer Verlag, 1994.

[5] Engle, J. K. and Nagle, R.: One-Parameter Semigroups for Linear Evaluation Equations, Springer-Verlag, New York, 2000.

[6] Evans, E. D. and Olsen, H.: The generators of positive semigroups, Journal of Functional Analysis, Vol 32, pp. 207-212, 1979.

[7] Janfada, M. and Niknam, A.: On two-parameter dynamical systems and applications. Journal of Sciences, Islamic Republic of Iran 15(2): 163169(2004).

[8] Joita, M.: A Note about full Hilbert modules over Frechet locally C^*-algebras. Novi Sad J. Math. Vol. 37, No. 1, 2007, pp. 27-32.

[9] Lance, E. C.: Hilbert C^*-modules, LMS Lecture Note Series 210, Cambridge Univ. Press, Cambridge, 1995.

[10] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci, Vol. 44, Springer-Verlag, 1983.

[11] Sakai, S.: Operator algebra in dynamical systems. Cambridge Univ. Press, Cambridge, 1991.

[12] Shaw, S.Y. and Li, Y.C.: Representation formulas for $C-$semigroups. Semigroup Forum, Vol. 46, pp. 123-125, 1993.

****