A NOTE ON DYNAMICAL SYSTEMS ON HILBERT C^* – MODULES AND DYNAMICAL SYSTEMS ON C^* – ALGEBRAS

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ABSTRACT

Suppose that C is a unitary operator on a full Hilbert C^* – module M . We describe dynamical systems on a full Hilbert C^* – module M over a C^* – algebra A as a one-parameter C -group of unitaries on M . We investigate dynamical systems on a full Hilbert module over a Frechet locally C^* – algebra. Then we discuss two-parameter dynamical systems on a full Hilbert C^* – module M over a C^* – algebra. Finally we characterize infinitesimal generator of a uniformly continuous positive C – semigroup on a C^* – algebra.

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1.INTRODUCTION:

A pre-Hilbert C^*-module over a C^*-algebra A is an algebraic left A – module M equipped with an A – valued inner product $\langle .,. \rangle$ which is A – linear in the first variable and for every $x, y \in M$ satisfies the following relations:

(1) $\langle x, x \rangle \geq 0$;

(2) $\langle x, x \rangle = 0$ if and if $x = 0$;

(3) $\langle x, y \rangle^* = \langle y, x \rangle$.

We say M is a Hilbert C^*-module if it is a Banach space with respect to the norm $\| x \| = \| \langle x, x \rangle \|^{\frac{1}{2}}$. A Hilbert module M is called full if the closed linear span $\langle M, M \rangle$ of all elements of the form $\langle x, y \rangle$ $(x, y \in M)$ is equal to A . It can be easily proved that if M is a full Hilbert module over a C^*-algebra A then $a = 0$ if and only if $ax = 0$ for all $x \in M$. For more details on Hilbert modules one can see [9].

Now let M and N be Hilbert modules over C^*-algebras A and B respectively. Following [2] we call a map $\Phi : M \to N$ is a unitary operator if there exists an injective morphism of C^*-algebras $\varphi : A \to B$ such that Φ is a surjective φ -morphism. The set of all unitary operators on M together with composition of maps form a group which is denoted by $U(M)$. If M is full and $\alpha : M \to M$ is a unitary operator than there is a $*$ -isomorphism $\alpha^{'} : A \to A$ such that α is an $\alpha^{'}$ – morphism ([2] Remark 2.9).

Abbaspour, Moslehian and Niknam in [1] characterized a dynamical system on a full Hilbert module M over a C^* - algebra A as a one-parameter group of unitaries on M . They proved that if $\alpha : R \to U(M)$ is a dynamical system, then it corresponds to a C^* -dynamical system $\alpha^{'}$ on A such that if δ and d are the infinitesimal generators

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of α and α' respectively, then δ is a $d-$ generalized derivation. In this paper we first extend this to one-parameter $C-$ group of unitaries on a full Hilbert C^*-module M where C is a unitary operator on M. For more details on $C-$ groups see [4]. We discuss dynamical systems on a full Hilbert module over a Frechet locally C^*-algebra. A locally C^*-algebra is a complete Hausdorff complex *-algebra A whose topology is determined by its continuous C^*-seminorms. The set of all continuous C^*-seminorms on A is denoted by $S(A)$. A Frechet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. Clearly any C^*-algebra is a Frechet locally C^*-algebra. In [8], Joita characterized a unitary operator $\Phi : E \rightarrow F$, where A and B are two Frechet locally C^*-algebras and E and F are full Hilbert modules over A and B respectively. We describe dynamical systems on these spaces. We also investigate two-parameter dynamical systems on a full Hilbert C^*-module. The reader is referred to [7] for more details on two-parameter dynamical systems and to [11] on C^*-dynamical systems.

In Sec. 3, we describe dynamical systems on a C^*-algebra. More precisely we characterize the infinitesimal generator of a uniformly continuous (strongly continuous) positive C-semigroups on a C^*-algebra.

2. C-GROUP OF UNITARIES ON A FULL HILBERT MODULE:

Definition: 2.1. Suppose that C is a unitary operator on a full Hilbert C^*-module M. The family of unitary operators $\{\alpha_t\}_{t \in \mathbb{R}}$ on M is a C-group if it has the following properties:

$(i)\ \alpha_0 = C$.
$(ii)\ \alpha_t \alpha_s(x) = C\alpha_{t+s}(x), t, s \in \mathbb{R}$.
$(iii)\ \alpha_t$ is strongly continuous, that is, for all $x \in M$ the map $t \rightarrow \alpha_t(x)$ form $\mathbb{R}$ into M is continuous.

The operator δ is the infinitesimal generator of C-group $\{\alpha_t\}_{t \in \mathbb{R}}$ if

$$D(\delta) = \{x \mid \lim_{t \to 0} \frac{\alpha_t(x) - C(x)}{t} \ exists\}$$

with

$$\delta(x) = C^{-1}(\lim_{t \to 0} \frac{\alpha_t(x) - C(x)}{t})$$

Note that by Theorem 3.4 of [8] C^{-1} exists.

A small modification in the proof of Theorem 4.3 in Ref. 1 gives the following result.

Theorem: 2.2. Let M be a full Hilbert A-module and C be a unitary operator on M. Let α be a C-group on M and δ be the infinitesimal generator of dynamical system α. Then there exists a derivation $d : D(d) \subseteq A \rightarrow A$ such that $D(\delta)$ is a left $D(d)$-module and

$\delta(ax) = a\delta(x) + d(a)x, a \in D(d), x \in D(\delta)$.

Proof:. Since α is a C-group on M , for each $t \in \mathbb{R}$, the mapping $\alpha_t : M \rightarrow M$ is a unitary, so there exists *-isomorphism $\alpha_t' : A \rightarrow A$ such that

$\langle \alpha_t(x), \alpha_t(y) \rangle = \alpha_t'(\langle x, y \rangle)$, and $\alpha_t(ax) = \alpha_t'(a)\alpha_t(x)\ (a \in A, x \in M)$.

Analogy since C is a unitary operator on M , so there exists a *-isomorphism $c' : A \rightarrow A$ such that $\langle C(x), C(y) \rangle = c'(\langle x, y \rangle)$ and $C(ax) = c'(a)C(x)$. Now we show that $\alpha' : \mathbb{R} \rightarrow Aut(A)$ is a c'-dynamical system on the C^*-algebra A . For each $a \in A, x \in M$ we have

$$c'(a)C(x) = C(ax) = \alpha_0(ax) = \alpha_0'(a)\alpha_0(x) = \alpha_0'(a)C(x)$$

and this implies that $(c'(a) - \alpha_0'(a))C(x) = 0$. Since C is surjective and M is full so, $\alpha_0'(a) = c'(a)$ for all $a \in A$. Therefore $\alpha_0' = c'$.

Also for all $t, s \in \mathbb{R}$ and $x \in M$ we have

$$
\begin{aligned}
c'(\alpha_{t+s}'(a))C(\alpha_{t+s}(x)) &= C(\alpha_{t+s}'(a)\alpha_{t+s}(x)) \\
&= C\alpha_{t+s}(ax) \\
&= \alpha_t \alpha_s(ax) \\
&= \alpha_t(\alpha_s'(a)\alpha_s(x)) \\
&= \alpha_t'\alpha_s'(a)\alpha_t\alpha_s(x) \\
&= \alpha_t'\alpha_s'(a)C\alpha_{t+s}(x).
\end{aligned}
$$

It implies that $[c'\alpha_{t+s}'(a) - \alpha_t'\alpha_s'(a)]C(\alpha_{t+s}(x)) = 0$. So we have

$$c'\alpha_{t+s}'(a) = \alpha_t'\alpha_s'(a).$$

Since for each $x \in M$, $\alpha_t(x) \to C(x)$ as $t \to 0$, we have

$$
\begin{aligned}
\| \alpha_t'(a)C(x) - c'(a)C(x) \| &\leq \| \alpha_t'(a)C(x) - \alpha_t'(a)\alpha_t(x) \| + \| \alpha_t'(a)\alpha_t(x) - C(ax) \| \\
&= \| \alpha_t'(a)(C(x) - \alpha_t(x)) \| + \| \alpha_t(ax) - C(ax) \|
\end{aligned}
$$

Thus $\lim_{t \to 0}\alpha_t'(a)x = c'(a)x$ for all $x \in$ range $C(= M)$, whence $\lim_{t \to 0}\alpha_t'(a) = c'(a)$ for all $a \in A$. Therefore $\alpha' : R \to Aut(A)$ is a c'-dynamical system on the C^*-algebra A. If d is the infintesimal generator of α' then for each $a \in D(d), x \in D(\delta)$ we have

$$
\begin{aligned}
\delta(ax) &= C^{-1}[\lim_{t \to 0}\frac{\alpha_t(ax) - C(ax)}{t}] \\
&= C^{-1}[\lim_{t \to 0}\frac{\alpha_t'(a)\alpha_t(x) - c'(a)\alpha_t(x)}{t} + \lim_{t \to 0}\frac{c'(a)\alpha_t(x) - c'(a)C(x)}{t}] \\
&= c'^{-1}(\lim_{t \to 0}\frac{\alpha_t'(a) - c'(a)}{t})C^{-1}(\lim_{t \to 0}\alpha_t(x)) + c'^{-1}(c'(a))C^{-1}(\lim_{t \to 0}\frac{\alpha_t(x) - C(x)}{t}) \\
&= d(a)x + a\delta(x).
\end{aligned}
$$

Note that by Remark 3.3 and Corollary 3.6 of [8], C^{-1} is a module map in the sense $C^{-1}(ax) = c'^{-1}(a)C^{-1}(x)$. From above relations we conclude that $ax \in D(\delta)$ and $\delta(ax) = a\delta(x) + d(a)x$. Furthermore, $D(\delta)$ is a left $D(d)$-module.

Analogue of Theorem 2.2 holds for full Hilbert module M over Frechet locally C^*-algebra A (see[8]). In this case a linear map $\Phi : E \to F$, (E, F are Hilbert modules over locally C^*-algebra A, B respectively) is a unitary operator if Φ is surjective and there is an injective morphism of locally C^*-algebra $\varphi : A \to B$ with closed range such that $\varphi(\langle \xi, \eta \rangle) = \langle \Phi(\xi), \Phi(\eta) \rangle$ for all $\xi, \eta \in E$.

Now we describe two-parameter dynamical systems on a Hilbert C^*-module.

Definition: 2.3. By a two-parameter dynamical system on a Hilbert C^*-module M we mean a function from $\mathbb{R} \times \mathbb{R}$ into $U(M)$ such that

$(i)\ \alpha(s+s', t+t') = \alpha(s,t)\alpha(s',t')\ (t,t',s,s' \in \mathbb{R})$;
$(ii)\ \alpha(0,0) = I$.

It is called strongly continuous if $(s,t) \to \alpha(s,t)x$ is continuous for all x in M .

To any two-parameter dynamical system $\alpha(s,t)$ we associate two one-parameter groups $\alpha(s,0)$ and $\alpha(0,t)$, the group property of α implies that $\alpha(s,t) = \alpha(s,0)\alpha(0,t)$.

One can see that $\alpha(s,t)$ is strongly continuous if and only if $\alpha(s,0)$ and $\alpha(0,t)$ are strongly continuous (see[7]). The infinitesimal generator of $\alpha(s,0)$ and $\alpha(0,t)$ are denoted by δ_1 and δ_2, respectively. We will think of the pair (δ_1, δ_2) as the infinitesimal generator of $\alpha(s,t)$.

Theorem: 2.4. Let A be a C^*-algebra and M be a full Hilbert A-module. Also let α be a two-parameter dynamical system on M and (δ_1, δ_2) be the infinitesimal generator of α . Then $D(\delta_i)$, $i = 1, 2$ is a dense subspace of M and there exist derivations $d_i : D(d_i) \subseteq A \to A, i = 1, 2$ such that $D(\delta_i)$ is a left $D(d_i)$-module and $\delta_i(ax) = a\delta_i(x) + d_i(a)x, a \in D(d_i), x \in D(\delta_i)$.

Proof: By Hille-Yosida Theorem [10], $D(\delta_i)$ is a dense subspace of M . Since α is a two-parameter dynamical system on M , for each $t, s \in \mathbb{R}$, the mapping $\alpha(s,t)$ is a unitary, so there exist a $*$-isomorphism

$\alpha'(s,t) : A \to A$ such that $\langle \alpha(s,t)x, \alpha(s,t)y \rangle = \alpha'(s,t)(\langle x, y \rangle)$ and

$\alpha(s,t)(ax) = \alpha'(s,t)(a)\alpha(s,t)(x), (a \in A, x \in M)$. Now by applying the argument similar to Theorem 4.3 of [1], it can be easily proved.

3. CHARACTERIZATION OF UNIFORMLY (STRONGLY) CONTINUOUS POSITIVE C-SEMIGROUPS ON A C^*-ALGEBRA:

In 1979 Evans and Olsen in [6] characterized infinitesimal generator of a positive uniformly continuous semigroup on a C*-algebra A . They proved L is the infinitesimal generator of this semigroup if it satisfies in the one of the following inequalities:

$(i)\ L(a^2) + aL(1)a \geq L(a)a + aL(a)$ for all self-adjoint operator $a \in A$.
$(ii)\ L(1) + u^*L(1)u \geq L(u^*)u + u^*L(u)$ for all unitary operators $u \in A$.

Then in 1981 Bratteli and Robinson generalized it for C_0 − semigroup of positive operators (see [3]). In this generalization, they obtained above inequalities with resolvent $R(\lambda, L)$ instead of L .

Later the notion of C − semigroups was introduced by Da Prato (1967), E. B. Davies and M. M. Pang (1989) independently. C − semigroups are a significant generalization of strongly continuous semigroups. C may be any bounded, injective operator, where C equals the identity operator, then a C − semigroup is a strongly continuous semigroup (see[4]). The aim of this section is to characterize the infinitesimal generator of uniformly continuous and strongly continuous positive C − semigroups on a C*-algebra A.

In the following we recall the definition of uniformly continuous C -semigroup.

Definition: 3.1. Suppose that C is a bounded injective linear operator on a Banach space X . The family of bounded

linear operators $\{S(t)\}_{t\geq 0}$ on X is called uniformly continuous C-semigroup if in conditions $(i),(ii)$ of Definition 2.1, we put $\alpha_t = S(t)$ and condition (iii) is replaced with the following condition

$$\lim_{t\to 0}\| S(t) - C \|= 0.$$

Infinitesimal generator A of a uniformly continuous C-semigroup $\{S(t)\}_{t\geq 0}$ is defined as Definition 2.1 with $\alpha_t = S(t)$.

For example if $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup generated by A (see[10]) then for any C that commutes with $T(t), t\geq 0, A$ is the generator of uniformly continuous C-semigroup $S(t) = T(t)C$.

Our next results are generalization of Theorem 1.2 of $[10]$.

Theorem: 3.2. If A is a bounded linear operator and C is a bounded injective linear operator on a Banach space X and A commuting with C then A is the infinitesimal generator of a uniformly continuous C-semigroup.

Proof: Set $T(t) = Ce^{tA} = C\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}$. This converges in norm for every $t\geq 0$ and defines a bounded linear operator $T(t)$.

It is clear that $T(0) = C$. A straight forward calculation shows that $CT(t+s) = T(t)T(s)$. We have

$$\| T(t) - C \|\leq t\| A \|\| C \|\sum_{n=0}^{\infty}\frac{t^n\| A \|^n}{n!}.$$

The righthand side of above assertion tends to zero as t tends to infinity. Also we have

$$\|\frac{T(t) - C}{t} - CA \|\leq\| A \|\| T(t) - C \|.$$

Similarly the righthand side tends to zero when t tends to infinity by uniformly continuity of $T(t)$. So $\{T(t)\}_{t\geq 0}$ is a uniformly continuous C-semigroup of bounded linear operators on X and A is its infinitesimal generator.

Theorem: 3.3. Let $\{T(t)\}_{t\geq 0}$ be a uniformly continuous C-semigroup on a Banach space X and for each $t\geq 0$, $range\, T(t) \subseteq range(C)$ then infinitesimal generator A is bounded.

Proof: Fix $\rho > 0$ small enough such that $\| I - \rho^{-1}\int_0^{\rho}C^{-1}T(s)ds \|\leq 1$. Since $I = \frac{1}{\rho}\int_0^{\rho}Ids$ so

$$I - \rho^{-1}\int_0^{\rho}C^{-1}T(s)ds = \int_0^{\rho}\frac{I - C^{-1}T(s)}{\rho}ds \to 0 \text{ as } \rho\to 0. \text{ Hence } \rho^{-1}\int_0^{\rho}C^{-1}T(s)ds \text{ is invertible and we have}$$

$$h^{-1}(T(h) - C)\int_0^{\rho}C^{-1}T(s)ds = h^{-1}[\int_0^{\rho}T(h+s)ds - \int_0^{\rho}T(s)ds]$$

$$= h^{-1}[\int_h^{\rho+h}T(s)ds - \int_0^{\rho}T(s)ds]$$

and therefore

$$h^{-1}(T(h) - C) = h^{-1}[\int_{\rho}^{\rho+h}T(s)ds - \int_0^{h}T(s)ds](\int_0^{\rho}C^{-1}T(s)ds)^{-1}.$$

Letting $h\to 0$ so $h^{-1}(T(h) - C)$ converges in norm and therefore strongly to the bounded linear operator $(T(\rho) - I)(\int_0^{\rho}C^{-1}T(s)ds)^{-1}$.

In the following theorems we describe dynamical systems of positive C − semigroups on a C*-algebra A. Recall that if S is a set of states on a C*-algebra A, then S is said to be full if $x \in A_h$ (set of self adjoint elements of A) and $f(x) \geq 0$ for all f in S, then $x \geq 0$. Moreover S is said to be invariant if $f \in S$ and $x \in A$ satisfies $f(x^*x) \neq 0$, then $f[x^*(.)x] / f(x^*x) \in S$.

Theorem 3.4. Let L be a bounded self adjoint linear map on a unital C*-algebra A and C be a positive injective bounded linear map on A and $CL = LC, C(ab) = C(a)C(b)$ for each a,b in A and $C(1) = 1$ then the following conditions are equivalent

(i) Ce^{tL} is positive for all positive t.

(ii) $(\lambda - CL)^{-1}$ is positive for all large positive λ.

(iii) If $y \in A_+, a \in A$ satisfy $C(ya) = 0$ then $a^*CL(y)a > 0$.

(iv) For some full invariant set of states S on A that $y \in A_+, f \in S$ with $f(C(y)) = 0$ imply $f(CL(y)) \geq 0$.

(v) $CL(x)C(x) + C(x)CL(x) \leq CL(x^2) + C(x)CL(1)C(x)$ for all self adjoint x in A.

Proof. $(iv) \Rightarrow (iii)$

Let S be a full invariant set of states satisfying (iv). Let $y \in A_+, a \in A$ satisfy $C(ya) = 0$ then $f(C(a^*ya)) > 0$ for all f in S. So by (iv), $f(CL(a^*ya)) \geq 0$ and hence $f(a^*CL(y)a) \geq 0$, since S is invariant. Thus $a^*CL(y)a > 0$ since S is full.

$(iii) \Rightarrow (ii)$

Let $\lambda \leq \| CL \|$. In order to show that $(\lambda - CL)^{-1} \geq 0$ it is enough to show that if $x \in A_h$ satisfies $(\lambda - CL)x \geq 0$ then $x \geq 0$. Let $x = x^+ - x^-$ where $x^+, x^- \in A_+$ and $x^+x^- = 0$. Thus $C(x^+x^-) = 0$ and by

(iii) $x^-CL(x^+)x^- \geq 0$. Now we have

$$0 \leq \frac{x^-}{\lambda}[(\lambda - CL)x]x^- = x^-xx^- - x^-[\frac{CL(x)}{\lambda}]x^-$$

$$= -(x^-)^3 - x^-[\frac{CL(x^+)}{\lambda}]x^- + x^-[\frac{CL(x^-)}{\lambda}]x^-.$$

Thus

$$0 \leq (x^-)^3 \leq x^-[\frac{CL(x^-)}{\lambda}]x^-.$$

So we have

$$\| x^- \|^3 \leq \frac{\| CL \|}{\lambda} \| x^- \|^3.$$

Hence $x^- = 0$, as $\lambda \geq \| CL \|$.

$(ii) \Rightarrow (i)$

$$Ce^{tL} = \lim_{n \to \infty} C(1 - \frac{t}{n}L)^{-n} = \lim_{n \to \infty} C^{n+1}(C(1) - \frac{t}{n}CL)^{-n}.$$

$(i) \Rightarrow (v)$

Let $L'(x) = L(x) - \frac{1}{2}[L(1)x + xL(1)]$. Then $e^{tL' \geq 0}$ for all $t \geq 0$ by the Lie-Trotter formula and since $Ce^{tL'}(1) = C(1) = 1$ so $\| Ce^{tL'}(1) \| = 1$. We have by differentiating Kadison's Schwarz inequality, namely

$$Ce^{tL'}(x^2) \geq C(e^{tL'(x)})^2$$

which is valid for all $t \geq 0,$ and all self adjoint x, that

$$C[L'e^{tL'}(x)e^{tL'(x)}] + C[e^{tL'}(x)L'e^{tL'}(x)] \leq CL'e^{tL'}(x^2).$$

Put $t = 0$ so we have

$$CL'(x)C(x) + C(x)CL'(x) \leq CL'(x^2).$$

Substituting for L' gives the desired result.
$(v) \Rightarrow (iv)$

Let $y \in A_+, f \in A_+^*$ with $f(Cy) = 0$ then by Schwartz inequality

$$| f((Cy)^{\frac{1}{2}} z) |^2 \leq f(Cy)f(z^*z).$$

So $f((Cy)^{\frac{1}{2}} z) = f(z(Cy)^{\frac{1}{2}}) = 0$ for all $z \in A$. By (v) we have

$$CL(x)C(x) + C(x)CL(x) \leq CL(x^2) + C(x)CL(1)C(x)$$

for all self adjoint x in A. Put $x = y^{\frac{1}{2}}$ so

$$CL(y^{\frac{1}{2}})C(y^{\frac{1}{2}}) + C(y^{\frac{1}{2}})CL(y^{\frac{1}{2}}) \leq CL(y) + C(y^{\frac{1}{2}})CL(1)C(y^{\frac{1}{2}}).$$

Since $C(y^{\frac{1}{2}}) = (C(y))^{\frac{1}{2}}$ the above statement implies that $f(CL(y)) \geq 0$ and the proof is complete.

Theorem 3.5. Let A be a C*-algebra with identity 1 and C be a bounded injective operator on A which is positive and for each $a, b \in A, C(ab) = C(a)C(b)$.

Let $\{T_t\}_{t \geq 0}$ be an exponentially bounded C −semigroup on A ($\| T_t \| \leq Me^{\omega t}$ $for\ someM > 0, \omega \in R,$ for all $t \geq 0$) with generator L such that $T_t(a^*) = (T_t(a))^*$ for all $a \in A$ and $t \geq 0$. Then the following conditions are equivalent

(i) The C −semigroup T_t is positive for all $t \geq 0$.

(ii) The resolvent $(\lambda − L)^{-1}C$ is positive for all $\lambda > 0$ large enough.

(iii) $C(\lambda − L)^{-1}(x)C(x) + C(x)C(\lambda − L)^{-1}(x) \leq C(\lambda − L)^{-1}(x^2) + C(x)C(\lambda − L)^{-1}(1)C(x)$ for all self adjoint x in A and all large $\lambda > 0$.

Proof: $(i) \Leftrightarrow (ii)$

This is a standard result which follows from

$$\forall x \in A;\ T_t(x) = \lim_{n \to \infty}(I − \frac{t}{n}L)^{-1}Cx,\ t \geq 0$$

and

$$C(\lambda − L)^{-1}x = \int_0^\infty e^{-\lambda t}T_t(x)dt,\ (x \in X).$$

If L is bounded, it is known by the previous theorem that $S_t = Ce^{tL}$ is positive if and only if $CL(x)C(x) + C(x)CL(x) \leq CL(x^2) + C(x)CL(1)C(x)$ for all self adjoint x in A. Hence to establish the theorem it is enough to show that (i) or (ii) is equivalent to the following condition

$(iv)\ Ce^{t(\lambda-L)^{-1}} \geq 0$ for all $t \geq 0$ and all large $\lambda \geq 0$.

$(ii) \Rightarrow (iv)$ This is evident from this expansion

$$Ce^{t(\lambda-L)^{-1}} = C\sum_{n=0}^{\infty} \frac{t^n}{n!}(\lambda-L)^{-1}$$

$(iv) \Rightarrow (i)$ It obtains from

$$T(t)x = \lim_{n\to\infty} \exp\{t[n^2(n-L)^{-1} - nI]\}Cx,\ t \geq 0.$$

Now proof is complete.

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