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A modified Newton-Raphson method for solving nonlinear equations

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Abstract

The Newton-Raphson method is a very effective numerical procedure used for solving nonlinear equations of the form f(x) = 0. With the motivation of avoiding the computation of the derivative of the function f(x), which is involved in Newton-Raphson method, we provide a linear interpolation method in solving a nonlinear equation f(x) = 0.

Key Words: Nonlinear equations, Newton-Raphson method, Secant method, Interpolation.

2010 Mathematics Subject Classification: 90C05, 49M15

1. Introduction:

The Newton-Raphson method is a Classical Optimization technique for solving nonlinear equations. In this method, we start with an initial approximation x_0 and generate a sequence of approximations. The iterative procedure terminates when the relative error for two successive approximations becomes less than or equal to the prescribed tolerance.

Newton-Raphson formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,... \tag{1}$$

Newton's method requires that the derivative be calculated directly. In most practical problems, the function in question may be given by a long and complicated formula, and hence an analytical expression for the derivative may not be easily obtainable. It is clear from the formula for Newton's method that it will fail in cases where the derivative is zero. In these situations, it may be appropriate to approximate the derivative by using the slope of a line through two points on the function. In this case, the Secant method results. These methods are discussed in many books on Numerical Analysis and Operations Research. See, for example, Ralston and Rabinowitz [2], Dennis and Schnabel [1], Stoer and Bulirsch [3] and Taha [4]. To avoid computing $f'(x)$ because $f'(x)$ may not always be available or may be costly to compute and to preserve the excellent convergence properties of the Newton-Raphson method, $f'(x_n)$ is replaced by $f[x_n, x_{n-1}]$ in equation (1)

∴ Equation (1) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]} \tag{2}$$

where $f[x_n, x_{n-1}] = (f(x_n) - f(x_{n-1}))/(x_n - x_{n-1})$

This result in equation (2) is secant method. The Newton-Raphson iteration requires two function evaluations, one of $f(x)$ and another of $f'(x)$, per iteration. The secant method on the other hand, requires only one function evaluation per iteration, namely, that of $f(x)$. The aim of this paper is to construct a new method using linear interpolation method to solve a nonlinear equation $f(x) = 0$ which is a modification of Newton-Raphson formula also for the first iteration we use secant method. Starting with two initial approximations x_0 and x_1, we compute x_2 by secant method.

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Then we apply Modified Newton-Raphson method to calculate $x_3, x_4, x_5, ..., x_n$ using interpolation table.

2. Modified Newton-Raphson Method:

We replace $f'(x_n)$ in Newton-Raphson Formula equation (1) by $g'_{n,k}(x)$. For this, we write $g_{n,k}(x)$ in Newtonian form as

$$g_{n,k}(x) = f(x_n) + \sum_{i=1}^{k} f[x_n, x_{n-1}, ..., x_{n-i}] \prod_{j=0}^{i-1} (x - x_{n-j}) \tag{3}$$

where $f[x_n, x_{n-1}, ..., x_{n-i}]$ are divided differences of $f(x)$. We can define the divided differences as

$$f[x_n] = f(x_n)$$

and $f[a,b] = \dfrac{f[a] - f[b]}{a-b}$, a ≠ b

By equation (3), $g_{n,k}(x)$ is computed by ordering the x_i as $x_n, x_{n-1}, ..., x_{n-i}$ for $i = 1,2,...,k$. Using this ordering we can compute $g'_{n,k}(x)$ easily. Differentiating $g_{n,k}(x)$ in (3), and let $x = x_n$, we obtain

$$g'_{n,k}(x) = f[x_n, x_{n-1}] + \sum_{i=2}^{k} f[x_n, x_{n-1}, ..., x_{n-i}] \prod_{j=1}^{i-1} (x_n - x_{n-j}) \tag{4}$$

The Divided Difference Table is

Table 1

x	f	I diff.	II diff.			n^{th} diff.
x_0	f_0					
		∇f_1				
x_1	f_1		$\nabla^2 f_2$			
		∇f_2				
x_2	f_2		$\nabla^2 f_n$			$\nabla^n f_n$
.	.					
.	.					
.	.					
.	.	∇f_n				
x_0	f_n					

where the first differences is:

$$\nabla f_1 = \frac{f_1 - f_0}{x_1 - x_0}, \nabla f_2 = \frac{f_2 - f_1}{x_2 - x_1}, \ldots, \nabla f_n = \frac{f_n - f_{n-1}}{x_n - x_{n-1}}$$

The second differences is:

$$\nabla^2 f_2 = \frac{\nabla f_2 - \nabla f_1}{x_2 - x_0}, \nabla^2 f_3 = \frac{\nabla f_3 - \nabla f_2}{x_3 - x_1}, \ldots,$$

$$\nabla^2 f_n = \frac{\nabla f_n - \nabla f_{n-1}}{x_n - x_{n-2}} \text{ and so on.}$$

For $k = 2$ equation (4) becomes

$$g'_{n,k}(x) = f[x_n, x_{n-1}] + f[x_n, x_{n-1}, x_{n-2}](x_n - x_{n-1}) \tag{5}$$

We write $f[x_x, x_{n-1}] = \nabla f_n$, $f[x_n, x_{n-1}, x_{n-2}] = \nabla^2 f_n$, $f[x_n, x_{n-1}, x_{n-2}] = \nabla^2 f_n$

∴ Equation (5) becomes

$$g'_{n,k}(x) = \nabla f_n + \nabla^2 f_n (x_n - x_{n-1})$$

Modified Newton-Raphson Formula for k = 2 is

$$x_{n+1} = x_n - \frac{f(x_n)}{\nabla f_n + \nabla^2 f_n (x_n - x_{n-1})} \tag{6}$$

Similarly for $k = 3$ equation (4) becomes

$$g'_{n,k}(x) = f[x_n, x_{n-1}] + f[x_n, x_{n-1}, x_{n-2}](x_n - x_{n-1}) + f[x_x, x_{n-1}, x_{n-2}, x_{n-3}](x_n - x_{n-1})(x_n - x_{n-2}) \tag{7}$$

We write $f[x_n, x_{n-1}, x_{n-2}, x_{n-3}] = \nabla^3 f_n$

Then equation (7) becomes

$$g'_{n,k}(x) = \nabla f_n + \nabla^2 f_n (x_n - x_{n-1}) + \nabla^3 f_n (x_n - x_{n-1})(x_n - x_{n-2})$$

Modified Newton-Raphson Formula for $k = 3$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{\nabla f_n + \nabla^2 f_n (x_n - x_{n-1}) + \nabla^3 f_n (x_n - x_{n-1})(x_n - x_{n-2})} \tag{8}$$

In this way we put the values in the above equations of the backward diagonal from the table. To compute next iteration, we put the values of the previous iteration in the table. In this way to compute next iteration we need only the backward diagonal of table. In many books on Computer oriented numerical methods Newton-Raphson method and Secant method is described. See, for example Rajaraman [5] and Thangaraj [6]. A computer oriented procedure is described below for implementing the Modified Newton-Raphson Method.

3. Numerical Example:

In this part we consider an example which is solved in [7]. The result of Newton-Raphson method and our Modified Newton-Raphson method of this example is presented here for comparison purpose in Table 4. Consider the following nonlinear equation:

$$x^2 - 5x + 2 = 0$$

Let $f(x) = x^2 - 5x + 2$

Now, $f(4) = -2$ i.e., negative

$f(5) = 2$ i.e., positive

Therefore, the root lies between 4 and 5.

Let $x_0 = 5$, $x_1 = 4$

Therefore, $f_0 = 2$, $f_1 = -2$

First approximation-

The Secant Formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]} \tag{9}$$

Where $f[x_n, x_{n-1}] = [f(x_n) - f(x_{n-1})]/(x_n - x_{n-1})$

Compute x_2 from above equation (8)

$x_2 = 4.5$, $f_2 = -0.25$

Second approximation-

For $k = 2$ our Modified Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{\nabla f_n + \nabla^2 f_n (x_n - x_{n-1})} \tag{10}$$

The Divided Difference Table is

Table 2

x	f	I Diff.	II Diff.
5.0	2.00		
		4.00	
4.0	-2.00		-1
		4.50	
4.5	-0.25		

Putting the values of the backward diagonal from the above table in equation (9)

$$x_3 = 4.5625.$$

Third approximation-

For $k = 3$ our modified Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{\nabla f_n + \nabla^2 f_n (x_n - x_{n-1}) + \nabla^3 f_n (x_n - x_{n-1})(x_n - x_{n-2})} \tag{11}$$

Now, Extending the Divided Difference Table

Table 3

x	f	I Diff.	II Diff.	III Diff.
5.0	2.00			
		4.00		
4.0	-2.00		-1.00	
		4.50		-0.52571428571
4.5	-0.25		-0.77	
		4.0625		
4.5625	0.00390625			

Putting the values of the backward diagonal from the above table in equation (11)

$$x_4 = 4.5625 - \frac{0.00390625}{4.0625 + (-0.77)(45625 - 4.5) + (-0.52571428571)(4.5625 - 4.5)(4.5625 - 4)}$$

$$x_4 = 4.5615224337471911940781895590619$$

Fourth approximation-

For $k = 4$ our modified Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{\nabla f_n + \nabla^2 f_n (x_n - x_{n-1}) + \nabla^3 f_n (x_n - x_{n-1})(x_n - x_{n-2}) + \nabla^4 f_n (x_n - x_{n-1})(x_n - x_{n-2})(x_n - x_{n-3})} \tag{12}$$

Now, Again Extending the Divided Difference Table

Table 4

x	f	I diff.	II diff.	III diff.	IV diff.
5	2.00				
		4.00			
			-1.00		
4	-2.00			-0.52571428571	
		4.50			-8.404160349
			-0.77		
4.5	-0.25			3.15951024042	
		4.06250			
			1.0040650		
4.5625	0.00390625				
		4.12425			
4.5615	-0.000218				

Putting the values of the backward diagonal from the above table in equation (12)

$x_5 =$ 4.5615528697373331752348716087091

Table 5: The results of different methods

Iteration	Method	x_{n+1}
1	Modified Newton-Raphson	4.5
	Newton-Raphson	4.6667
2	Modified Newton-Raphson	4.5625
	Newton-Raphson	4.5641
3	Modified Newton-Raphson	4.5615224337471911940781895590619
	Newton-Raphson	4.5616
4	Modified Newton-Raphson	4.5615528697373331752348716087091
	Newton-Raphson	4.5616

5. Conclusion:

In this paper an efficient iterative method is build up to solve Nonlinear equations. As it can be seen in Table 5, Modified Newton-Raphson Method converges rapidly to exact solution. This Modified method seems to be very easy to employ with reliable results.

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