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GEOMETRIC EXPLOSION OF THE REAL NUMBERS AND THE CARDINAL NUMBER $\aleph_0$

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ABSTRACT

Theory of exploded numbers raises the question: Is it possible to compare the cardinal number $\aleph_0$ to the exploded number $\overbrace{1}$? For example, both are greater than any arbitrary natural number n. On the other hand, the first is the cardinality of the set of natural number ℕ while the second is a „number" modelled on the exploded number line. To answer the question, we explode the real numbers and their fundamental rules. Finally, we discover the alteregos of natural numbers beyond $\overbrace{1}$.

INTRODUCTION

The set of exploded numbers is an extension of the set of real numbers with the following Postulates and Requirements, detailed in [1]. (See [1], Chapter 2.)

POSTULATE OF EXTENSION

The set of real numbers $\mathbb{R}$ is a proper subset of the *set of exploded numbers* $\check{\mathbb{R}}$. For any real number x there exists one *exploded number* $\check{x}$ which is called *exploded* x or the *exploded of* x. Moreover, the set of exploded x is called the set of exploded numbers.

POSTULATE OF UNAMBIGUITY

For any pair of real numbers x and y, their *explodeds are equal* (in $\check{\mathbb{R}}$) if and only if x is equal to y (in $\mathbb{R}$). Shortly,

$$\check{x} =^{\check{R}} \check{y} \Leftrightarrow x = y.$$

POSTULATE OF ORDERING

For any pair of real numbers x and y, the *exploded x is smaller* (in $\check{\mathbb{R}}$) *than exploded y* if and only if *x* smaller than *y* (in $\mathbb{R}$). Shortly,

$$\check{x} <^{\check{R}} \check{y} \Leftrightarrow x < y.$$

POSTULATE OF SUPER-ADDITION

For any pair of real numbers x and y, the *super - sum* of their explodeds is the exploded of their sum. To put it by symbols:

(0.1) $$\check{x} \overline{\oplus} \check{y} =^{\check{R}} \check{\overline{x+y}}.$$

POSTULATE OF SUPER-MULTIPLICATION

For any pair of real numbers x and y, the *super - product* of their explodeds is the exploded of their product. Expressed by symbols:

(0.2) $$\check{x} \overline{\odot} \check{y} =^{\check{R}} \check{\overline{x \cdot y}}.$$

REQUIREMENT OF EQUALITY FOR EXPLODED NUMBERS

If x and y are real numbers then x as an exploded number equals to y as an exploded number if they are equal in the traditional sense. Shortly,

$$x =^{\check{R}} y \Leftrightarrow x = y$$

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REQUIREMENT OF ORDERING FOR EXPLODED NUMBERS

If x and y are real numbers then x as an exploded number is smaller than y as an exploded number if x is smaller than y in the traditional sense. Shortly,

$$x <^{\breve{R}} y \Leftrightarrow x < y$$

After the requirements of equality and ordering, the distinguishing between the equalities and orderings in $\breve{\mathbb{R}}$ and $\mathbb{R}$ is unnecessary, so instead of „$=^{\breve{\mathbb{R}}}$" we can write „=" and instead of „$<^{\breve{\mathbb{R}}}$" we can write „<".

REQUIREMENT FOR ZERO

The exploded of 0 is itself. Expressed by symbols

(0.3) $$\breve{0} = 0.$$

After the latter requirement the following definitions are given.

Definition 0.4: The exploded real number $\breve{x}$ is called positive if $0 < \breve{x}$.

By the Postulate of ordering and (0.3) it is fulfilled if and only if $x > 0$.

Definition 0.5: The exploded real number $\breve{x}$ is called negative if $\breve{x} < 0$.
By the Postulate of ordering and (0.3) it is fulfilled if and only if $x < 0$.

REQUIREMENT FOR EXPLOSION

The set $\breve{\mathbb{R}} \setminus \mathbb{R}$ has both positive and negative element and has negative elements.

REQUIREMENT OF MONOTONITY OF SUPER-ADDITION

If $\breve{x}$ and $\breve{y}$ are arbitrary exploded numbers and $\breve{x}$ is smaller than $\breve{y}$ then, for any exploded number $\breve{z}$, $\breve{x}\overline{\oplus}\breve{z}$ is smaller than $\breve{y}\overline{\oplus}\breve{z}$.

REQUIREMENT OF MONOTONITY OF SUPER-MULTIPLICATION

If $\breve{x}$ and $\breve{y}$ are arbitrary exploded numbers and $\breve{x}$ is smaller than $\breve{y}$ then, for any positive exploded number $\breve{z}$, $\breve{x}\overline{\odot}\breve{z}$ is smaller than $\breve{y}\overline{\odot}\breve{z}$.
By isomorphism

$$x \leftrightarrow \breve{x} \quad , x \in \mathbb{R}$$

we can find that the set of exploded real number $\breve{\mathbb{R}}$ is an ordered field with respect to super-addition and super-multiplication . It is isomorphic with the set of real numbers $\mathbb{R}$, but super-operations are not extensions of traditional operations.

Considering the Postulates and Reqirements, their independence from each other is not investigated. That is the reason why they are not mentioned as axioms. At this stage, the theory of exploded numbers is not an axiomatical construction, but it is contradictory if and only if the axiomatical construction of real numbers is contradictory.

Considering an arbitrary element u of $\breve{\mathbb{R}}$, the Postulate of unambiguity says that there exists only one real number such that its exploded is u. This real number is called *compressed of u*, or *compressed u* and denoted by $\underline{u}$. Hence, we have the *first inversion formula*

(0.6) $$\breve{(\underline{u})} = u \quad , \quad u \in \breve{\mathbb{R}}.$$

Denoting $x = \underline{u}$, the first inversion formula says that $\breve{x} = u$. Hence, we have the *second inversion formula*

(0.7) $$\underline{(\breve{x})} = x \quad , \quad x \in \mathbb{R}.$$

Moreover, instead of (0.1) we can write

(0.8) $$u\overline{\oplus}v = \overset{\smile}{\underline{u} + \underline{v}} \quad ; u, v \in \breve{\mathbb{R}}$$

or

(0.9) $$\underline{u\overline{\oplus}v} = \underline{u} + \underline{v} \quad ; u, v \in \breve{\mathbb{R}}.$$

Similarly, instead of (0.2) we can write

(0.10) $$u\overline{\odot}v = \overset{\smile}{\underline{u} \cdot \underline{v}} \quad ; u, v \in \breve{\mathbb{R}}$$

or

(0.11) $$\underline{u\overline{\odot}v} = \underline{u} \cdot \underline{v} \quad ; u, v \in \breve{\mathbb{R}}.$$

1. THE GEOMETRIC EXPLOSION

Considering that the hyperbolic model of exploded number (see [2], (0.1)) was dicussed in [2], for the geometric model, instead of mark $(\breve{\ })$ we use $\widehat{(\ )}$.

The geometric compressor formula already had been mentioned in [3] (See [3], (1.3).) Now, we give a model of exploded numbers constructed by the geometric exploder –formula

(1.1) $$\widehat{x} = \left((sgn\ x)\frac{\{|x|\}}{1-\{|x|\}}, (sgn\ x)[|x|] \right), \quad x \in \mathbb{R}.$$

The pair of real numbers $\widehat{x}$ is called exploded number and the set of exploded numbers is denoted by $\widehat{\mathbb{R}}$. The real number x is considered as a pair having the second member 0, that is $x = (x, 0)$. With respect to the Requirement of equality for exploded numbers we say that

$$\widehat{x} = \widehat{y} \text{ if and only if } (sgn\ x)\frac{\{|x|\}}{1-\{|x|\}} = (sgn\ y)\frac{\{|y|\}}{1-\{|y|\}} \text{ and } (sgn\ x)[|x|] = (sgn\ y)[|y|].$$

By (1.1) we can see that if $-1 < x < 1$ then $\widehat{x} = \frac{x}{1-|x|}$ overruns the set of real numbers, so $\mathbb{R} \subset \widehat{\mathbb{R}}$.

Hence, the Postulate of extension is fulfilled. Moreover, (1.1) shows that Reqirement for zero is satisfied, such that $0 = (0,0)$.

For the Postulate of unambiguity we prove the following theorem.

***Theorem* 1.2:** (Theorem of unambiguity.) If x and y are real numbers, then $\widehat{x} = \widehat{y}$ if and only if $x = y$.

Proof: Obviously, if $x\ and\ y \in \mathbb{R}$ then (1.1) gives that $\widehat{x} = \widehat{y}$. Conversely, we assume that $\widehat{x} = \widehat{y}$.
If $\widehat{x} = 0 = \widehat{y}$, then (1.1) gives that $x = 0 = y$., so we assume that If $\widehat{x} \neq 0$ and $\widehat{y} \neq 0$. Using (1.1) we have

$$(sgn\ x) \cdot \frac{\{|x|\}}{1-\{|x|\}} = (sgn\ y) \cdot \frac{\{|y|\}}{1-\{|y|\}}.$$

As $\frac{\{|x|\}}{1-\{|x|\}}$ and $\frac{\{|y|\}}{1-\{|y|\}}$ are positive numbers and $sgn\ x$ and $sgn\ y$ are different from 0, we have

(1.3) $$sgn\ x = sgn\ y\ .$$

So,

$$\frac{\{|x|\}}{1-\{|x|\}} = \frac{\{|y|\}}{1-\{|y|\}}.$$

Hence

(1.4) $$\{|x|\} = \{|y|\}.$$

Moreover, by (1.1) gives that

$$(sgn\ x) \cdot [|x|] = (sgn\ y) \cdot [|y|]$$

and by (1.3)

(1.5) $$[|x|] = [|y|]$$

is obtained. As $|x| = [|x|] + \{|x|\}$ and $|y| = [|y|] + \{|y|\}$ the (1.5) with (1.4) yield $|x| = |y|$. Finally, by (1.3) $x = (sgn\ x) \cdot |x| = (sgn\ y) \cdot |y| = y$ is obtained.■ Istennek Hála, 2019. 10. 13.,17.04, Szalay István.

Let us consider the following subset of the set $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$

(1.6) $$\mathbb{S} = \left\{ u = (x, y) \in \mathbb{R}^2 | \begin{Bmatrix} y \in \mathbb{Z} \\ x \cdot y \geq 0 \end{Bmatrix} \right\}.$$

By (1.1) we can see that $\widehat{x} \in \mathbb{S}$. On the other hand we prove

***Theorem* 1.7:** (Theorem of completeness.) If $u \in \mathbb{S}$ then we have

(1.8) $$\widehat{y + \frac{x}{1+|x|}} = u.$$

Proof: With respect to (1.6) we write y=n, that is

$$u = (x, n).$$

We distinguish three cases

***Case* I:** $n = 1,2,3 \cdots$.

By (1.6) we have $x \geq 0$. that We apply the geometric exploder formula (1.1)

$$\widehat{y + \frac{x}{1+|x|}} = \left(\frac{\frac{x}{1+x}}{1-\frac{x}{1+x}}, n \right) = u\ .$$

***Case* II:** $n = 0$.
Now u is a real number and $u = (x, 0)$. Applying the geometric exploder formula (1.1) we obtain

$$\overbrace{\left(\frac{x}{1+|x|}\right)} = \left(\left(sgn\ \frac{x}{1+|x|}\right)\left(\frac{\left\{\left|\frac{x}{1+|x|}\right|\right\}}{1-\left\{\left|\frac{x}{1+|x|}\right|\right\}}\right), 0\right) = \left((sgn\, x\,)\frac{\frac{|x|}{1+|x|}}{1-\frac{|x|}{1+|x|}}, 0\right) = u$$

***Case* III:** $n = -1, -2, -3, \dots$.
By (1.6) we have $x \le 0$. We apply the geometric exploder formula (1.1)

$$\overbrace{y+\frac{x}{1+|x|}} = \left(-\left(\frac{\frac{-x}{1-x}}{1+\frac{x}{1-x}}\right), -(-n)\right) = u\,.$$

Theorem of completeness says that the elemnents of the set $\mathbb{S}$ are exploded numbers. Earlier we have already seen that exploded numbers are elements of $\mathbb{S}$. So, $\mathbb{S}$ is the set of exploded numbers. Casting a glance at (1.6) we write

(1.9) $$\overset{\frown\frown}{\mathbb{R}} = \left\{u = (x, y) \in \mathbb{R}^2 \,|\, \left\{\begin{matrix} y \in \mathbb{Z} \\ x \cdot y \ge 0\end{matrix}\right.\right\}.$$

Introducing the geometric compressor formula

(1.10) $$\underbrace{u} = y + \frac{x}{1+|x|} \qquad , u \in \overset{\frown\frown}{\mathbb{R}}\,,$$

by (1.8) we have the first inversion formula

(1.11) $$\overbrace{\left(\underbrace{u}\right)} = u \qquad , \quad u \in \overset{\frown\frown}{\mathbb{R}}\,. \qquad \text{(See (0.6).)}$$

Of course, $\underbrace{u}$ is a real number. (See (1.10).) Having Theorem 1.2, the real number $\underbrace{u}$ is unambiguously determined. Moreover, $\underbrace{u}$ is called as compressed of u. Denoting $x = \underbrace{u}$ by (1.11) we have $\overset{\frown\frown}{x} = u$. Hence, (1.11) gives the second inversion formula

(1.12) $$\underbrace{\left(\overset{\frown\frown}{x}\right)} = x \qquad x \in \mathbb{R}\,. \qquad \text{(See (0.7).)}$$

Considering (0.8) and (0.10) we define the super – addition and super multiplication

(1.13) $$u\,\mathcal{A}\,v = \overbrace{\underbrace{u} + \underbrace{v}} \qquad , u, v \in \overset{\frown\frown}{\mathbb{R}}$$

and

(1.14) $$u\,\mathcal{M}\,v = \overbrace{\underbrace{u} \cdot \underbrace{v}} \qquad , u, v \in \overset{\frown\frown}{\mathbb{R}}\,,$$

respectively.
Writing $\underbrace{u} = x\,, u = \overset{\frown\frown}{x}$ and $\underbrace{v} = y\,, v = \overset{\frown\frown}{y}$ and using Theorem 1.2 with (1.13) and (1.14) give that the mapping

$$x \leftrightarrow \overset{\frown\frown}{x} \qquad , x \in \mathbb{R}$$

is izomorphism between the algebraic structures $(\mathbb{R}\,, +\,,\cdot)$ and $\left(\overset{\frown\frown}{\mathbb{R}}, \mathcal{A}, \mathcal{M}\right)$. As $(\mathbb{R}\,, +\,,\cdot)$ is a field, $\left(\overset{\frown\frown}{\mathbb{R}}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{M}}\right)$ **is a field**, too. Important properties of the field $\left(\overset{\frown\frown}{\mathbb{R}}, \mathcal{A}, \mathcal{M}\right)$:

$$x, y \text{ and } z \in \mathbb{R}$$

Associative laws: $\left(\overset{\frown\frown}{x}\ \mathcal{A}\ \overset{\frown\frown}{y}\right)\mathcal{A}\ \overset{\frown\frown}{z} = \overset{\frown\frown}{x}\ \mathcal{A}\left(\overset{\frown\frown}{y}\,\mathcal{A}\ \overset{\frown\frown}{z}\right)$ and $\left(\overset{\frown\frown}{x}\ \mathcal{M}\ \overset{\frown\frown}{y}\right)\mathcal{M}\ \overset{\frown\frown}{z} = \overset{\frown\frown}{x}\ \mathcal{M}\left(\overset{\frown\frown}{y}\,\mathcal{M}\ \overset{\frown\frown}{z}\right)$
Commutative laws: $\overset{\frown\frown}{x}\ \mathcal{A}\ \overset{\frown\frown}{y} = \overset{\frown\frown}{y}\ \mathcal{A}\ \overset{\frown\frown}{x}$ and $\overset{\frown\frown}{x}\ \mathcal{M}\ \overset{\frown\frown}{y} = \overset{\frown\frown}{y}\ \mathcal{M}\ \overset{\frown\frown}{x}$
Distributive law: $\left(\overset{\frown\frown}{x}\ \mathcal{A}\,\overset{\frown\frown}{y}\right)\mathcal{M}\ \overset{\frown\frown}{z} = \left(\overset{\frown\frown}{x}\ \mathcal{M}\ \overset{\frown\frown}{z}\right)\mathcal{A}\left(\overset{\frown\frown}{y}\,\mathcal{M}\ \overset{\frown\frown}{z}\right)$.
Additive unit element: $\overset{\frown\frown}{x}\,\mathcal{A}\ 0 = \overset{\frown\frown}{x}$, uniqueness of additive unit element can be proved.
(See the Requirement of zero.)
Additive inverse element: $\overset{\frown\frown}{x}\,\mathcal{A}\ \overbrace{(-x)} = 0$, uniqueness of additive inverse element can be proved. Moreover, $\overset{\frown\frown}{1} = (0,1)$. (See (1.1).)

Multiplicative inverse element: If $x \ne 0$ *then* $\overset{\frown\frown}{x}\,\mathcal{M}\overbrace{\left(\frac{1}{x}\right)} = \overset{\frown\frown}{1}$, uniqueness of multiplicative inverse element can be proved. Moreover, $\overbrace{\left(\frac{1}{x}\right)} = \left((sgn\ x)\left(\frac{\left\{\frac{1}{|x|}\right\}}{1-\left\{\frac{1}{|x|}\right\}} + i\left[\frac{1}{|x|}\right]\right), (sgn\ x)\left[\frac{1}{|x|}\right]\right)$.

***Definition* 1.15:** (Ordering of exploded numbers.)
For any pair $x\,, y \in \mathbb{R}$ we say, that $\overset{\frown\frown}{x} <^{in\ \overset{\frown\frown}{\mathbb{R}}} \overset{\frown\frown}{y}$ if

$$(sgnx)\cdot[|x|] < (sgnx)\cdot[|y|]$$

or

if $(sgnx)\cdot[|x|] = (sgnx)\cdot[|y|]$ then $(sgn\ x)\frac{\{|x|\}}{1-\{|x|\}} < (sgn\ y)\frac{\{|y|\}}{1-\{|y|\}}$.

***Lemma* 1.16:** If $\widehat{x}$ and $\widehat{y}$ are real numbers then $\widehat{x} <^{in\,\widehat{\mathbb{R}}} \widehat{y} \Leftrightarrow \widehat{x} < \widehat{y}$.

Proof: If $\widehat{x}$ is a real number then $[|x|] = 0$. So, $\{|x|\} = |x|$ and $\widehat{x} = \left(\frac{x}{1-|x|}, 0\right) = \frac{x}{1-|x|}$. By Definition 1.15 we have $\widehat{x} <^{in\,\widehat{\mathbb{R}}} \widehat{y} \Leftrightarrow \frac{x}{1-|x|} < \frac{y}{1-|y|}$. In the next we distinguish three cases:

- if $0 \le \widehat{x} < \widehat{y}$ then $0 \le x < 1$ and $0 < y$ <1, so, $\frac{x}{1-x} < \frac{y}{1-y} \Leftrightarrow x < y$,
- if $\widehat{x} \le 0 < \widehat{y}$ then $-1 < x \le 0$ and $0 < y < 1$, so , $x < y$,
- if $\widehat{x} < \widehat{y} \le 0$ then $-1 < x < 0$ and $-1 < y \le 0$, $\frac{x}{1+x} < \frac{y}{1+y} \Leftrightarrow x < y$.

After Lemma 1.16 we use „<” instead of „$<^{in\,\widehat{\mathbb{R}}}$ ". We remark that the Requirement of ordering for exploded numbers is fulfilled.

***Theorem* 1.17:** (Theorem of ordering.) For any pair $x , y \in \mathbb{R}$, $\widehat{x} < \widehat{y}$ if and only if $x < y$.

Proof:

Necessity.

Let us assume that , $\widehat{x} < \widehat{y}$.

By Definition 1.15 we have two cases:

Case I:

$$(sgn\, x)[|x|] < (sgn\, y)[|y|]$$

Because the function $(sgn\ \cdot)[|\cdot|]$ is monotonic increasing in wider sense, if $x \ge y$ then we get a contradiction, so, $x < y$.

***Case* II:**

(1.18) $$(sgn\, x)[|x|] = (sgn\, y)[|y|]$$

and

(1.19) $$(sgn\, x)\frac{\{|x|\}}{1-\{|x|\}} < (sgn\, y)\frac{\{|y|\}}{1-\{|y|\}}.$$

First of all we remark that $x \ne 0$. (If $x = 0$ then (1.18) gives that $y = 0$ which contradicts (1.19).)

Using that for any real number $x \ne 0$ we have that $[|x|] > 0$ or $\frac{\{|x|\}}{1-\{|x|\}} > 0$, three possibilities are considered

Possibility a) $sgn\, x = sgn\, y = 1.$ Now, *x* and *y* are positive numbers. So, $|x| = x$, $\{|x|\} = \{x\}$, $|y| = y$, $\{|y|\} = \{y\}$. Moreover, (1.18) and (1.19) yield $[x] = [y]$ and $\{x\} < \{y\}$ .Hence, $x < y$.

Possibility b) $sgn\, x = -1$ *and* $sgn\, y = 1$. Now, $x < 0 < y$.

Possibility c) $sgn\, x = sgn\, y = -1.$ Now, *x* and *y* are negative numbers. So, $|x| = -x$, and $|y| = -y$, Now, (1.19) yields $\frac{\{|x|\}}{1-\{|x|\}} > \frac{\{|y|\}}{1-\{|y|\}} \Leftrightarrow \{|x|\} > \{|y|\}$. As (1.18) gives $[|x|] = [|y|]$, $|x| > |y|$ is obtained. Hence, , $x < y$.

Sufficiency.

Let us assume that $x < y$.

Because the function $(sgn\ \cdot)[|\cdot|]$ is monotonic increasing in wider sense, we have two possibilities.

Possibility I:

$$(sgn\, x)[|x|] < (sgn\, y)[|y|],$$

then Definition 1.15 says that $\widehat{x} < \widehat{y}$,

or

Possibility II:

$$(sgn\, x)[|x|] = (sgn\, y)[|y|].$$

Let us denote by $n = (sgn\, x)[|x|]$ where $n = 0, \pm1, \pm2, \ldots$.

By (1.1) we have that $\widehat{x} = \left((sgn\, x)\frac{\{|x|\}}{1-\{|x|\}}, n\right)$ and $\widehat{y} = \left((sgn\, y)\frac{\{|y|\}}{1-\{|y|\}}, n\right)$. Considering the intervals $[n , n+1[$,$n = 1,2,3, \ldots$, $]-1 , 1[$, $]n , n+1]$, $n = -1, -2, -3, \ldots$ we may state that x *and* y belong to the same interval. As on the above listed intervals the function $(sgn\, x)\frac{\{|x|\}}{1-\{|x|\}}$ is strictly increasing $(sgn\, x)\frac{\{|x|\}}{1-\{|x|\}} < (sgn\, x)\frac{\{|x|\}}{1-\{|x|\}}$. So, By definition 1.15 relation $\widehat{x} < \widehat{y}$ holds.

After the Theorem of ordering we can say that the geometric model of the exploded number satisfies the Postulate of ordering. Consequently, the ***transitivity and trichotomy properties remain valid for extended ordering, too***

***Definition* 1.20:** The exploded number *u* is called negative if $u < 0$ and positive if $u > 0$, respectively.

Using Definition 1.20, by the inversion formulas and Reqirement for zero we have

***Corollary* 1.21:** The exploded number *u* is negative if the real number $\underset{\smile}{u} < 0$ and positive if $\underset{\smile}{u} > 0$. Consequently **if $u \neq 0$ then $u \nless u$.**

For example $\overset{\frown}{-1} < 0 \; and \; \overset{\frown}{1} > 0$. So, the Reqiuirement for explosion is fulfilled.

***Lemma* 1.22:** If *u* and *v* are arbitrary exploded numbers and $u < v$, then for any exploded number *w*

$$u \,\mathcal{A}\, w < v \,\mathcal{A}\, w.$$

Proof: Considering the real numbers $\underset{\smile}{u}$, $\underset{\smile}{v}$ and $\underset{\smile}{w}$ by (1.13) and Theorem 1.17 we can write

$$u \,\mathcal{A}\, w = \overset{\frown}{\underset{\smile}{u} + \underset{\smile}{w}} < \overset{\frown}{\underset{\smile}{v} + \underset{\smile}{w}} = v \,\mathcal{A}\, w.$$

***Definition* 1.23:** If $u \neq 0$ and $u \nless v$ then $u > v$.

***Lemma* 1.24:** If *u* and *v* are arbitrary exploded numbers and $u < v$, then for any positive exploded number *w*

$$u \,\mathcal{M}\, w < v \,\mathcal{M}\, w.$$

Moreover, for any negative exploded number *w*

$$u \,\mathcal{M}\, w > v \,\mathcal{M}\, w$$

holds.

Proof: Considering the real numbers $\underset{\smile}{u}$, $\underset{\smile}{v}$ and $\underset{\smile}{w}$ using Corollary 1.21, by (1.13) and Theorem 1.17 for posituve *w* we can write

$$u \,\mathcal{M}\, w = \overset{\frown}{\underset{\smile}{u} \cdot \underset{\smile}{w}} < \overset{\frown}{\underset{\smile}{v} \cdot \underset{\smile}{w}} = v \,\mathcal{M}\, w.$$

Moreover, using Definition 1.23 for any negative exploded number *w*

$$u \,\mathcal{M}\, w = \overset{\frown}{\underset{\smile}{u} \cdot \underset{\smile}{w}} > \overset{\frown}{\underset{\smile}{v} \cdot \underset{\smile}{w}} = v \,\mathcal{M}\, w$$

is obtained.

Now we already have that $\left(\overset{\frown}{\mathbb{R}}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{M}}, <\right)$ **is an ordered field**. Moreover, we introdoce the super – subtraction and super - division

(1.25) $$u \,\mathcal{S}\, v = \overset{\frown}{\underset{\smile}{u} - \underset{\smile}{v}} \qquad , u, v \in \overset{\frown}{\mathbb{R}}$$

and

(1.26) $$u \,\mathcal{D}\, v = \overset{\frown}{\left(\frac{\underset{\smile}{u}}{\underset{\smile}{v}}\right)} \qquad , u, v (\neq 0) \in \overset{\frown}{\mathbb{R}},$$

respectively.

Of course, for any pair $u, v \in \overset{\frown}{\mathbb{R}}$ indentities $v \,\mathcal{A}\, (u \,\mathcal{S}\, v) = u$ and $v \,\mathcal{M}\, (u \,\mathcal{D}\, v) = u$, where $v \neq 0$, are valid.

Finally, we give some useful signs and symbols.

The extension of the sign „minus"

$$- \overset{\frown}{x} =^{def} \overset{\frown}{(-x)} \quad , x \in \mathbb{R}.$$

If $\overset{\frown}{x} \in \mathbb{R}$ then $[|x|] = 0$, so, $\overset{\frown}{(-x)} = \left(sgn(-x)\right) \frac{|-x|}{1-|-x|} = -(sgn\, x) \frac{|x|}{1-|x|} = -\overset{\frown}{x}$.

The definition of the super - function „$\overset{\frown}{sgn}$" for any $u \in \overset{\frown}{\mathbb{R}}$,

$$\overset{\frown}{sgn}\; u = \overset{\frown}{sgn\; \underset{\smile}{u}} = \begin{cases} \overset{\frown}{1} \;\; if \;\; u > 0 \\ 0 \;\;\; if \;\; u = 0 \\ -\overset{\frown}{1} \; if \;\; u < 0 \end{cases}.$$

The extension of „absolute value" for any $u \in \overset{\frown}{\mathbb{R}}$,

(1.27) $$|u| = \begin{cases} u \;\; if \;\; u > 0 \\ 0 \;\;\; if \;\; u = 0 \\ -u \; if \;\; u < 0 \end{cases}.$$

Hence, by the Requirement of zero, Theorem 1.17, the first inversion formula (1.11) and the extension of sign „minus" yield

(1.28) $$|u| = \overset{\frown}{\left|\underset{\smile}{u}\right|},$$

which is an extension of absolute value of real numbers for exploded numbers, because if $u \in \mathbb{R}$ then $u = (u, 0)$, so by (1.10) $\underbrace{u} = \frac{u}{1+|u|}$. $Hence,$ $\left|\underbrace{u}\right| = \frac{|u|}{1+|u|}$, so by (1.1) $\overbrace{\left|\underbrace{u}\right|} = \left(\frac{\frac{|u|}{1+|u|}}{1-\frac{|u|}{1+|u|}}, 0\right) = |u|$ is obtained. Moreover, by (1.28) the second inversion formula (1.12) gives

(1.29) $$\underbrace{|u|} = \left|\underbrace{u}\right| \quad , \quad u \in \overbrace{\mathbb{R}}.$$

2. TRADITIONAL ADDITION AND MULTIPLICATION FOR REAL NUMBERS EXPRESSED BY SUPER – OPERATORS

***Theorem* 2.1:** If u and v are real numbers, then

(2.2) $$u + v = \left(u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)|\right).$$

Proof: Using (1.13) , (1.25) , (1.14) and (1.12) we can write

$$u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|) = \overbrace{\underbrace{u} + \underbrace{v} - \underbrace{u\mathcal{M}|v|} - \underbrace{v\mathcal{M}|u|}} =$$

$$\overbrace{\underbrace{u} + \underbrace{v} - \underbrace{\left(\overbrace{\underbrace{u} \cdot \underbrace{|v|}}\right)} - \underbrace{\left(\overbrace{\underbrace{v} \cdot \underbrace{|u|}}\right)}} = \overbrace{\underbrace{u} + \underbrace{v} - \underbrace{u} \cdot \underbrace{|v|} - \underbrace{v} \cdot \underbrace{|u|}}.$$

Using (1.25), (1.13) , (1.14) , (1.29) and (1.12)we can write

$$\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)| =$$

$$= \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + \underbrace{|u\mathcal{M}v|} + \underbrace{|u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)|}} =$$

$$= \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + \left|\underbrace{u\mathcal{M}v}\right| + \left|\underbrace{u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)}\right|} =$$

$$= \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + \left|\underbrace{\left(\overbrace{\underbrace{u} \cdot \underbrace{v}}\right)}\right| + \left|\underbrace{\left(\overbrace{\underbrace{u} + \underbrace{v} - \underbrace{u\mathcal{M}|v|} - \underbrace{v\mathcal{M}|u|}}\right)}\right|} =$$

$$= \overbrace{1 - \underbrace{|u|} - \underbrace{|v|} + \left|\underbrace{u} \cdot \underbrace{v}\right| + \left|\underbrace{u} + \underbrace{v} - \underbrace{u} \cdot \underbrace{|v|} - \underbrace{v} \cdot \underbrace{|u|}\right|}.$$

Moreover, by (1.12) , (1.26) and (1.29) we have

$$\left(u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\overbrace{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{A}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{S}(v\mathcal{M}|u|)|\right) =$$

$$= \overbrace{\frac{\underbrace{u} + \underbrace{v} - \underbrace{u} \cdot \underbrace{|v|} - \underbrace{v} \cdot \underbrace{|u|}}{1 - \underbrace{|u|} - \underbrace{|v|} + \left|\underbrace{u} \cdot \underbrace{v}\right| + \left|\underbrace{u} + \underbrace{v} - \underbrace{u} \cdot \underbrace{|v|} - \underbrace{v} \cdot \underbrace{|u|}\right|}} =$$

$$= \overbrace{\left(\frac{\underbrace{u} + \underbrace{v} - \underbrace{u} \cdot \left|\underbrace{v}\right| - \underbrace{v} \cdot \left|\underbrace{u}\right|}{1 - \left|\underbrace{u}\right| - \left|\underbrace{v}\right| + \left|\underbrace{u} \cdot \underbrace{v}\right| + \left|\underbrace{u} + \underbrace{v} - \underbrace{u} \cdot \left|\underbrace{v}\right| - \underbrace{v} \cdot \left|\underbrace{u}\right|\right|}\right)}.$$

The following should be for the simplicity $\underbrace{u} = x$ and $\underbrace{v} = y$. As u and v are real numbers we have that $-1 < x < 1$ and $-1 < y < 1$. We distinquish four cases:

I. $x \geq 0$ and $y \geq 0$.

Having that $0 \leq x + y - 2xy$ and $0 \leq \frac{x+y-2xy}{1-xy} < 1$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{x + y - 2xy}{1 - x - y + xy + |x + y - 2xy|}\right)} = \overbrace{\left(\frac{x + y - 2xy}{1 - xy}\right)} = \frac{x}{1 - x} + \frac{y}{1 - y} = \overbrace{x} + \overbrace{y} = u + v$$

is obtained.

II/ a. $x \leq 0$ and $y \geq 0$ $such\ that$ $x + y \geq 0$. $(Of\ course, x \neq -1\ and\ y \neq 1.)$

Having that $1+2x - xy > 0$ and $0 \leq \frac{x+y}{1+2x-xy} < 1$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{x + y}{1 + x - y - xy + |x + y|}\right)} = \overbrace{\left(\frac{x + y}{1 + 2x - xy}\right)} = \frac{x}{1 + x} + \frac{y}{1 - y} = \overbrace{x} + \overbrace{y} = u + v$$

is obtained.

II/ b. $x \leq 0 \; and \; y \geq 0 \;\; such \; that \; x+y \;\leq 0\,.\,(Of\; course, x \neq -1 \; and \; y \neq 1.)$

Having that 1-2$y - xy > 0$ and $-1 < \frac{x+y}{1-2y-xy} \leq 0$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{x+y}{1+x-y-xy+|x+y|}\right)} = \overbrace{\left(\frac{x+y}{1-2y-xy}\right)} = \frac{x}{1+x} + \frac{y}{1-y} = \overset{\frown}{x} + \overset{\frown}{y} = u+v$$

is obtained.

III. $x \leq 0 \; and \; y \leq 0.$

Having that $-1 < x+y+2xy \leq 0$ and $-1 \leq \frac{x+y+2xy}{1-xy} \leq 0$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{x+y+2xy}{1+x+y+xy+|x+y+2xy|}\right)} = \overbrace{\left(\frac{x+y+2xy}{1-xy}\right)} = \frac{x}{1+x} + \frac{y}{1+y} = \overset{\frown}{x} + \overset{\frown}{y} = u+v$$

is obtained.

IV/ a. $x \geq 0 \; and \; y \leq 0 \;\; such \; that \; x+y \;\geq 0.\,(Of\; course, x \neq 1 \; and \; y \neq -1.)$

Having that 1+2$y - xy > 0$ and $0 \leq \frac{x+y}{1+2y-xy} < 1$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{x+y}{1-x+y-xy+|x+y|}\right)} = \overbrace{\left(\frac{x+y}{1+2y-xy}\right)} = \frac{x}{1-x} + \frac{y}{1+y} = \overset{\frown}{x} + \overset{\frown}{y} = u+v$$

is obtained.

IV/ b. $x \geq 0 \; and \; y \leq 0 \;\; such \; that \; x+y \;\leq 0.\,(Of\; course, x \neq 1 \; and \; y \neq -1.)$

Having that 1-2$x - xy > 0$ and $-1 < \frac{x+y}{1-2x-xy} \leq 0$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{x+y}{1-x+y-xy+|x+y|}\right)} = \overbrace{\left(\frac{x+y}{1-2x-xy}\right)} = \frac{x}{1-x} + \frac{y}{1+y} = \overset{\frown}{x} + \overset{\frown}{y} = u+v$$

is obtained. Istennek Hála! 2019.11.02,13:55 Szalay István

***Theorem* 2.3:** If *u* and *v* are real numbers, then

(2.4) $$u \cdot v = (u\mathcal{M}v)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\,\overset{\frown}{2}\,\mathcal{M}|u\mathcal{M}v|\right).$$

Proof: Using (1.14) , (1.25) , (1.26) and (1.29) we can write

$$(u\mathcal{M}v)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}\,\overset{\frown}{2}\,\mathcal{M}|u\mathcal{M}v|\right) =$$

$$= \left(\overbrace{\underbrace{u}\cdot\underbrace{v}}\right)\mathcal{D}\left(\overbrace{1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+\underbrace{\overset{\frown}{2}\,\mathcal{M}|u\mathcal{M}v|}}\right) =$$

$$= \left(\overbrace{\underbrace{u}\cdot\underbrace{v}}\right)\mathcal{D}\left(\overbrace{1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+\underbrace{\left(\overbrace{2\cdot\left|\underbrace{u}\cdot\underbrace{v}\right|}\right)}}\right) = \left(\overbrace{\underbrace{u}\cdot\underbrace{v}}\right)\mathcal{D}\left(\overbrace{1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+2\cdot\left|\underbrace{u}\cdot\underbrace{v}\right|}\right) =$$

$$= \overbrace{\left(\frac{\underbrace{u}\cdot\underbrace{v}}{1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+2\cdot\left|\underbrace{u}\cdot\underbrace{v}\right|}\right)} = \overbrace{\left(\frac{\underbrace{u}\cdot\underbrace{v}}{1-\left|\underbrace{u}\right|-\left|\underbrace{v}\right|+2\cdot\left|\underbrace{u}\cdot\underbrace{v}\right|}\right)}.$$

The following should be for the simplicity $\underbrace{u} = x \; and \; \underbrace{v} = y$. As *u* and *v* are real numbers we have that $-1 < x < 1$ and $-1 < y < 1$. We distinquish four cases:

I. $x \geq 0 \; and \; y \geq 0.$

Having that $0 \leq xy$ and $0 \leq \frac{xy}{1-x-y+2xy} < 1$, by (1.1) and (1.11) we have

$$\overbrace{\left(\frac{xy}{1-x-y+2xy}\right)} = \frac{x}{1-x} \cdot \frac{y}{1-y} = \overset{\frown}{x} \cdot \overset{\frown}{y} = u \cdot v.$$

II. $x \leq 0 \; and \; y \geq 0.$

Having that $xy \leq 0$ and $-1 < \frac{xy}{1+x-y+2xy} \leq 0$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{xy}{1+x-y-2xy}\right)} = \frac{x}{1+x} \cdot \frac{y}{1-y} = \overset{\frown}{x} \cdot \overset{\frown}{y} = u \cdot v$$

is obtained.

III. $x \leq 0 \; and \; y \leq 0.$

Having that $xy \geq 0$ and $0 \leq \frac{xy}{1+x-y+2xy} < 1$, by (1.1) and (1.11) we have

$$\overbrace{\left(\frac{xy}{1+x+y+2xy}\right)} = \frac{x}{1+x} \cdot \frac{y}{1+y} = \widehat{x} \cdot \widehat{y} = u \cdot v\,.$$

IV. $x \geq 0$ *and* $y \leq 0$.

Having that $xy \leq 0$ and $-1 < \frac{xy}{1-x+y+2xy} \leq 0$, by (1.1) and (1.11)

$$\overbrace{\left(\frac{xy}{1-x+y-2xy}\right)} = \frac{x}{1-x} \cdot \frac{y}{1+y} = \widehat{x} \cdot \widehat{y} = u \cdot v$$

is obtained.

***Theorem* 2.5:** If u and v are real numbers, then

(2.6) $u - v = \left(u\mathcal{S}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{A}(v\mathcal{M}|u|)\right)\mathcal{D}\left(\widehat{1}\,\mathcal{S}|u|\mathcal{S}|v|\mathcal{A}|u\mathcal{M}v|\mathcal{A}|u\mathcal{S}v\mathcal{S}(u\mathcal{M}|v|)\mathcal{A}(v\mathcal{M}|u|)|\right)$.

The proof is similar to the proof of Theorem 2.1, so, we omit it.

***Theorem* 2.7:** If u and $v(\neq 0)$ are real numbers, then

(2.8) $\frac{u}{v} = \left(u\mathcal{M}\left(\widehat{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\widehat{sgn}\left(v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right)\right)\mathcal{D}\left(\left|u\mathcal{M}\left(\widehat{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right|\right)$.

Proof: Starting from the definition of $\widehat{sgn}$ for any pair u , v of real numbers we have

(2.9) $$\widehat{sgn}\left(v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right) = \widehat{sgn\ v}.$$

Hence, because if $v \neq 0$ then $|sgn\ v| = 1$ and by (1.14),(1.10),(1.25),(2.9) and (1.12) we can write

(2.10) $$u\mathcal{M}\left(\widehat{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\widehat{sgn}\left(v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right) = \overbrace{\left(\frac{u \cdot sgn\ v}{(1+|u|)(1+|v|)}\right)}\,.$$

By (1.13) , (1.14) , (1.25) and (1.12) we can write

(2.11) $$\left|u\mathcal{M}\left(\widehat{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right| = \overbrace{\left(\frac{|u|+|v|}{(1+|u|)(1+|v|)}\right)}\,.$$

Finally, by (1.26), (2.10), (2.11) and (1.1)

$$\left(u\mathcal{M}\left(\widehat{1}\,\mathcal{S}|v|\right)\mathcal{M}\,\widehat{sgn}\left(v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right)\right)\mathcal{D}\left(\left|u\mathcal{M}\left(\widehat{1}\,\mathcal{S}|v|\right)\right|\mathcal{A}\left|v\mathcal{M}\left(\widehat{1}\,\mathcal{S}|u|\right)\right|\right) =$$

$$= \overbrace{\left(\frac{u \cdot sgn\ v}{|u|+|v|}\right)} = \frac{u \cdot sgn\ v}{|v|} = \frac{u}{v}.$$

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Formulas (2.2),(2.4),(2.6) and (2.8) are less complicated in the case of non-negative real numbers.

Theorems 2.1, 2.3, 2.5 and 2.7 yield the following

***Corollary* 2.12:** If x and y are non – negative real numbers, then

(2.13) $$u + v = \left((u\,\mathcal{A}\,v)\,\mathcal{S}\,\left(\widehat{2}\,\mathcal{M}\,u\,\mathcal{M}\,v\right)\right)\mathcal{D}\,\left(\widehat{1}\,\mathcal{S}\,(u\,\mathcal{M}\,v)\right),$$

(2.14) for the case $u \geq v \geq 0$ we give $u - v = (u\,\mathcal{S}\,v\,)\mathcal{D}\,\left(\widehat{1}\,\mathcal{S}\,\left(\widehat{2}\,\mathcal{M}\,v\right)\mathcal{A}\,(u\,\mathcal{M}\,v)\right),$

(2.15) $$u \cdot v = (u\,\mathcal{M}\,v)\,\mathcal{D}\,\left(\widehat{1}\,\mathcal{S}\,(u\,\mathcal{A}\,v)\,\mathcal{A}\,\left(\widehat{2}\,\mathcal{M}\,u\,\mathcal{M}\,v\right)\right)$$

and assuming that $v \neq 0$

(2.16) $$\frac{u}{v} = (u\,\mathcal{S}\,(u\,\mathcal{M}\,v)\,)\,\mathcal{D}\,\left((u\,\mathcal{A}\,v)\,\mathcal{S}\,\left(\widehat{2}\,\mathcal{M}\,u\,\mathcal{M}\,v\right)\right)$$

are valid.

3. THE BEHAVIOR OF $_0\aleph$ AND $\widehat{1}$ WITH SOME OPERATORS OF NATURAL NUMBERS

Considering a set H we denote its cardinal number by $|H|$. The empty set is denoted by $\{\ \ \}$. The basic of our comparison is that $|\{\ \ \}| = 0$ and any positive natural number n (based on the Peano axioms) is the cardinal number of the set $H_n = \{a_0, a_1, \ldots, a_{n-1}\}$, that is $|H_n| = n$. On the other hand $_0\aleph$ is not natural number but it is the cardinal number of the set of natural numbers $\mathbb{N}$, that is $_0\aleph = |\mathbb{N}|$. By the relation „≤” of the set theory

(3.1) $$n < {_0\aleph}\quad, n \in \mathbb{N}$$

is well- known.

The exploded number $\overset{\frown}{1}$ is not natural number, but by Definition 1.15

(3.2) $$n < \overset{\frown}{1} \quad , n \in \mathbb{N},$$

is given.

Considering the basic – set $\mathbb{N}$ and using the operations

$$|A| + |B| = |A \cup B| \quad , \quad where \;\; A \cap B = \{ \quad \},$$
$$|A| \cdot |B| = |A \times B| \,, where \; A \times B \; is \; the \; Cartesian - product,$$
$$|A| - |B| = |A \setminus B| \quad , \qquad where \; B \subset A,$$

we have

(3.3) $$n + {_0\aleph} = {_0\aleph} = {_0\aleph} + n \qquad , n \in \mathbb{N} \,,$$

(3.4) $${_0\aleph} + {_0\aleph} = {_0\aleph} \qquad ,$$

(3.5) $$n \cdot {_0\aleph} = {_0\aleph} = {_0\aleph} \cdot n \qquad , n \in 1, 2 \,, 3, \ldots \;\; ,$$

(3.6) $$0 \cdot {_0\aleph} = 0 = {_0\aleph} \cdot 0 \,,$$

(3.7) $${_0\aleph} \cdot {_0\aleph} = {_0\aleph}$$

and

(3.8) $${_0\aleph} - n = {_0\aleph} \quad , \;\; n \in \mathbb{N}.$$

Considering the formulas (3.3) – (3.8) we try to substitute $_0\aleph$ by $\overset{\frown}{1}$**.** Using Corollary 2.12 we can write

Ad (3.3)

Extending the formula (2.13) for $v = \overset{\frown}{1}$ we use it as the definition of $u + \overset{\frown}{1}$ for any $u \in \mathbb{R}$. So, for any natural number *n* we have

$$n + \overset{\frown}{1} = \left(\left(n \,\mathcal{A}\, \overset{\frown}{1}\right) \mathcal{S} \left(\overset{\frown}{2} \,\mathcal{M}\, n\right)\right) \mathcal{D} \left(\overset{\frown}{1} \,\mathcal{S}\, n\right) = \left(\overset{\frown}{1} \,\mathcal{S}\, n\right) \mathcal{D} \left(\overset{\frown}{1} \,\mathcal{S}\, n\right) = \overset{\frown}{1}.$$

By (2.13) the commutativity $n + \overset{\frown}{1} = \overset{\frown}{1} + n$ is obvious,

(3.3)* $$n + \overset{\frown}{1} = \overset{\frown}{1} = \overset{\frown}{1} + n \quad , \;\; n \in \mathbb{N}$$

is obtained.

Ad (3.4)

We are not able to extend the formula (2.13) for $u = \overset{\frown}{1}$ and $v = \overset{\frown}{1}$ because $\overset{\frown}{1} \,\mathcal{S} \left(\overset{\frown}{1} \,\mathcal{M}\, \overset{\frown}{1}\right) = 0$. (It is true that $\left(\overset{\frown}{1} \,\mathcal{A}\, \overset{\frown}{1}\right) \mathcal{S} \left(\overset{\frown}{2} \,\mathcal{M}\, \overset{\frown}{1} \,\mathcal{M}\, \overset{\frown}{1}\right) = 0$, too.) So,

(3.4)* $\overset{\frown}{1} + \overset{\frown}{1}$ is undetermined.

Ad (3.5) and (3.6)

Extending formula (2.15) for $v = \overset{\frown}{1}$ we use it as the definition of $u \cdot \overset{\frown}{1}$ for any $u \in \mathbb{R}$. So, for any positive natural number *n* we have

$$n \cdot \overset{\frown}{1} = n \,\mathcal{D} \left(\overset{\frown}{1} \,\mathcal{S} \left(n \,\mathcal{A}\, \overset{\frown}{1}\right) \mathcal{A} \left(2 \,\mathcal{M}\, n\right)\right) = n \,\mathcal{D}\, n = \overset{\frown}{1}.$$

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By (2.15) the commutativity $n \cdot \overset{\frown}{1} = \overset{\frown}{1} \cdot n$ is obvious, so,

(3.5)* $$n \cdot \overset{\frown}{1} = \overset{\frown}{1} = \overset{\frown}{1} \cdot n \quad , \;\; n = 1,2,3, \ldots,$$

is obtained. In the case $n = 0$ we can use the formula (2.15) again, but now, $\overset{\frown}{1} \,\mathcal{S} \left(0 \,\mathcal{A}\, \overset{\frown}{1}\right) = 0$, so

(3.6)* $0 \cdot \overset{\frown}{1}$ and $\overset{\frown}{1} \cdot 0$ are undetermined.

Ad (3.7)

Extending formula (2.15) for $u = \overset{\frown}{1}$ and $v = \overset{\frown}{1}$ we use it as the definition of $\overset{\frown}{1} \cdot \overset{\frown}{1}$.

$$\overset{\frown}{1} \cdot \overset{\frown}{1} = \overset{\frown}{1} \,\mathcal{D} \left(\overset{\frown}{1} \,\mathcal{S} \left(\overset{\frown}{1} \,\mathcal{A}\, \overset{\frown}{1}\right) \mathcal{A}\, \overset{\frown}{2}\right) = \overset{\frown}{1} \,\mathcal{D}\, \overset{\frown}{1} = \overset{\frown}{1} \,,$$

so, we have

(3.7)* $$\overset{\frown}{1} \cdot \overset{\frown}{1} = \overset{\frown}{1} \,.$$

Ad (3.8)

Extending the formula (2.14) for $u = \overset{\frown}{1}$ we use it as the definition of $\overset{\frown}{1} - v$ for any $v \in \mathbb{R}$. So, for any natural number *n* we have

$$\overset{\frown}{1} - n = \left(\overset{\frown}{1} \,\mathcal{S}\, n\right) \mathcal{D} \left(\overset{\frown}{1} \,\mathcal{S} \left(\overset{\frown}{2} \,\mathcal{M}\, n\right) \mathcal{A}\, n\right) = \left(\overset{\frown}{1} \,\mathcal{S}\, n\right) \mathcal{D} \left(\overset{\frown}{1} \,\mathcal{S}\, n\right) = \overset{\frown}{1},$$

so, we have

(3.8)* $$\overset{\frown}{1} - n = \overset{\frown}{1} \quad , n \in \mathbb{N} \,.$$

The pairs (3.3) – (3.3)* ; (3.5) – (3.5)* ; (3.7) – (3.7)* and (3.8) – (3.8)* show consonance between $_0\aleph$ and $\overset{\frown}{1}$,while (3.4) – (3.4)* and (3.6) – (3.6)* give dissonance. The most important dissonance that „$- {_0\aleph}$ ” does not exist, while $\left(-\overset{\frown}{1}\right) = \overset{\frown}{(-1)} = (0, -1)$. (See the extension of sign „minus” and (1.1)).

If we extend the formula (2.2) for $u = \overset{\frown}{1}$ and $v = \overset{\frown}{(-1)}$ then using (1.27) we have

$$\overset{\frown}{1}\,\mathcal{S}\left|\overset{\frown}{1}\right|\mathcal{S}\left|\overset{\frown}{(-1)}\right|\mathcal{A}\left|\overset{\frown}{(-1)}\right|\mathcal{A}\left|\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{(-1)}\,\mathcal{S}\left(\left|\overset{\frown}{(-1)}\right|\right)\mathcal{S}\left(\overset{\frown}{(-1)}\,\mathcal{M}\left|\overset{\frown}{1}\right|\right)\right| =$$
$$= \overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{1}\,\mathcal{A}\left|\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{(-1)}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\left(\overset{\frown}{(-1)}\,\mathcal{M}\,\overset{\frown}{1}\right)\right| = 0.$$

Moreover,

$$\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{(-1)}\,\mathcal{S}\left|\overset{\frown}{(-1)}\right|\mathcal{S}\left(\overset{\frown}{(-1)}\,\mathcal{M}\left|\overset{\frown}{1}\right|\right) = 0\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{(-1)} = 0,$$

so,

(3.9) $\qquad \overset{\frown}{1} + \overset{\frown}{(-1)}$ is undetermined.

If we extend the formula (2.6) for $u = \overset{\frown}{1}$ and $v = \overset{\frown}{1}$ then using (1.27) we have

$$\overset{\frown}{1}\,\mathcal{S}\left|\overset{\frown}{1}\right|\mathcal{S}\left|\overset{\frown}{1}\right|\mathcal{A}\left|\overset{\frown}{1}\,\mathcal{M}\,\overset{\frown}{1}\right|\mathcal{A}\left|\overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\left(\overset{\frown}{1}\,\mathcal{M}\left|\overset{\frown}{1}\right|\right)\mathcal{A}\left(\overset{\frown}{1}\,\mathcal{M}\left|\overset{\frown}{1}\right|\right)\right| = \overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{1}\,\mathcal{A}\left|0\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{1}\right| = 0.$$

Moreover,

$$\overset{\frown}{1}\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\left(\overset{\frown}{1}\,\mathcal{M}\left|\overset{\frown}{1}\right|\right)\mathcal{A}\left(\overset{\frown}{1}\,\mathcal{M}\left|\overset{\frown}{1}\right|\right) = 0\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{A}\,\overset{\frown}{1} = 0\,,$$

so,

(3.10) $\qquad \overset{\frown}{1} - \overset{\frown}{1}$ is undetermined.

We remark that the trying for definition ${}_0\aleph - {}_0\aleph$ by the way

$$|A| - |B| = |A \setminus B| \quad , \qquad where\ B \subseteq A$$

is not uniquely defined, because by performing this subtraction can get finite every cardinal number and also ${}_0\aleph$. This statement show consonance with (3.9) and (3.10).

If we extend the formula (2.6) for $v = \overset{\frown}{1}$ then using (1.27) we write

$$n - \overset{\frown}{1} = \left(n\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\,n\,\mathcal{A}\,n\right)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}\,n\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{A}\,n\,\mathcal{A}\left|n\,\mathcal{S}\,\overset{\frown}{1}\,\mathcal{S}\,n\,\mathcal{A}\,n\right|\right) =$$
$$= \left(n\,\mathcal{S}\,\overset{\frown}{1}\right)\mathcal{D}\left(\left|n\,\mathcal{S}\,\overset{\frown}{1}\right|\right) = \overset{\frown}{(-1)} = -\overset{\frown}{1}\,,$$

so,

(3.11) $\qquad n - \overset{\frown}{1} = -\overset{\frown}{1}\left(= \overset{\frown}{-1}\right) \quad , \quad n \in \mathbb{N}.$

We mention that (3.11) is given by (2.2) with $u = n(\in \mathbb{N})\ and\ v = \overset{\frown}{(-1)}$, too.

It is an exciting possibility the extension of division for $\overset{\frown}{1}$ by (2.16).

First of all we remark that we have no possibiliy for "$\frac{\overset{\frown}{1}}{\overset{\frown}{1}}$" because extending (2.16) for $u = \overset{\frown}{1}$ and $v = \overset{\frown}{1}$, $u\,\mathcal{S}\,(u\,\mathcal{M}\,v) = 0$ and $(u\,\mathcal{A}\,v)\,\mathcal{S}\left(\overset{\frown}{2}\,\mathcal{M}\,u\,\mathcal{M}\,v\right) = 0$,so,

(3.12) $\qquad \frac{\overset{\frown}{1}}{\overset{\frown}{1}}$ is undetermined.

On the other hand assuming that $n = 1,2,3\ldots$ and extending (2.16) for $u = \overset{\frown}{1}$ we can write

$$\frac{\overset{\frown}{1}}{n} = \left(\overset{\frown}{1}\,\mathcal{S}\left(\overset{\frown}{1}\,\mathcal{M}\,n\right)\right)\mathcal{D}\left(\overset{\frown}{2}\,\mathcal{M}\,\overset{\frown}{1}\,\mathcal{M}\,n\right) = \left(\overset{\frown}{1}\,\mathcal{S}\,n\right)\mathcal{D}\left(\overset{\frown}{1}\,\mathcal{S}\,n\right) = \overset{\frown}{1}\,,$$

so,

(3.13) $\qquad \frac{\overset{\frown}{1}}{n} = \overset{\frown}{1} \quad , \ n = 1,2,3, \ldots$

is obtained. Moreover, extending (2.16) for $v = \overset{\frown}{1}$ we can write

$$\frac{n}{\overset{\frown}{1}} = \left(n\,\mathcal{S}\left(n\,\mathcal{M}\,\overset{\frown}{1}\right)\right)\mathcal{D}\left(\left(n\,\mathcal{A}\,\overset{\frown}{1}\right)\mathcal{S}\left(\overset{\frown}{2}\,\mathcal{M}\,n\right)\right) = 0$$

because $\left(n\,\mathcal{A}\,\overset{\frown}{1}\right)\mathcal{S}\left(\overset{\frown}{2}\,\mathcal{M}\,n\right) = \overset{\frown}{\left(\frac{1}{n+1}\right)} \neq 0$.So,

(3.14) $\qquad \frac{n}{\overset{\frown}{1}} = 0 \quad , n \in \mathbb{N},$

is obtained.

***Remark* 3.15:** Of course, by (3.14) we have no „ $n = 0 \cdot \overset{\frown}{1}$" because this multiplication is undetermined. (See, (3.6)*.)

4. ALTEREGOS OF NATURAL NUMBERS

The explosion formula (1.1) gives that the exploded form of natural numbers $n \in \mathbb{N}$

$$\overset{\frown}{\left(\frac{n}{n+1}\right)} = (n, 0) = n.$$

In the set $\mathbb{R}$ the sequence of natural numbers $\{n\}_{n=0}^{\infty}$ is unbounded from above but in the $\overset{\frown\frown}{\mathbb{R}}$ it has upper bound $\overset{\frown\frown}{1}$. Moreover $\overset{\frown\frown}{1}$ is the least upper bound because supposing that there exists a (may be exploded number) $\boldsymbol{b}$ such that for any natural number $n \leq \boldsymbol{b} < \overset{\frown\frown}{1}$ holds, we get contradiction. Really if we suppose that „for any natural number $n \leq \boldsymbol{b} < \overset{\frown\frown}{1}$" , then applying Theorem 1.17 we have that $\underset{\smile\smile}{n} \leq \underset{\smile\smile}{\boldsymbol{b}} < 1$. So, by (1.10) we have that for any natural number $\frac{n}{n+1} \leq \underset{\smile\smile}{\boldsymbol{b}} < 1$. Hence, $\lim_{n\to\infty} \frac{n}{n+1} = 1$ a is contradiction. So, the assumption „for any natural number $n \leq \boldsymbol{b} < \overset{\frown\frown}{1}$" is false.
In next $\mathbb{N}$ is called as $0-$ th *level* and denoted by as follows

(4.1) $$_0\mathbb{N} = \{0\,, 1, \ldots, n, n+1, \ldots\}.$$

***Definition* 4.2:** Let μ be an positive arbitrary natural number. The subset of $\overset{\frown\frown}{\mathbb{R}}$

(4.2) $$_\mu\mathbb{N} = \{\overset{\frown\frown}{\mu}\,\mathcal{A}\,0\,, \overset{\frown\frown}{\mu}\,\mathcal{A}\,1\,, \ldots, \overset{\frown\frown}{\mu}\,\mathcal{A}\,n\,,\ \overset{\frown\frown}{\mu}\ \mathcal{A}\,(n+1), \ldots\}$$

is called μ – th *level* .
The explosion formula (1.1) with (1.13) and (1.10) gives

(4.3) $$\overset{\frown\frown}{\mu}\,\mathcal{A}\,n = \overbrace{\mu + \frac{n}{n+1}} = (n, \mu)\ ,\ \ n = 0,1,3, \ldots.$$

***Theorem* 4.4:** Considering the set $_\mu\mathbb{N}$, ($\mu = 1,2,3, \ldots$) and the basic concepts

a) "natural number" is the exploded number (n, μ) , $n = 0,1,2, \ldots$,
b) "$zero$" $= (0, \mu)$
c) „ successor" *of* (n, μ) *is* $(n+1\,, \mu)$, $n = 0,1,2, \ldots$, the Peano axioms are satisfied.

Proof: We refer to the Peano axioms of natural numbers using the definition of equality of exploded numbers and Theorem 1.2.

1. By the point a) the exploded number $(0, \mu)$ is „natural number" .
2. For any natural number *n* we have that $(n, \mu) = (n, \mu)$ because $n = n$ *and* $\mu = \mu$.
3. For any natural numbers *n* and *m* if $(n, \mu) = (m, \mu)$ then $(m, \mu) = (n, \mu)$ because if $n = m$ then $m = n$ and $\mu = \mu$.
4. For any natural numbers *n* , *m* and *k* if $(n, \mu) = (m, \mu)$ and $(m, \mu) = (k, \mu)$ then $(n, \mu) = (k, \mu)$ because if $n = m$ and $m = k$ then $n = k$, moreover $\mu = \mu$.
5. For any, natural number" (n, μ) its „successor" is „natural number".
6. Really, the „successor" of (n, μ) is $(n+1, \mu)$ because the successor *n* is *n*+1, moreover $\mu = \mu$ and $(n+1, \mu)$ is also „natural number.
7. For all „natural numbers" $(n, \mu) = (m, \mu)$ if and only if $(n+1, \mu) = (m+1, \mu)$.
8. Really, $(n, \mu) = (m, \mu) \Leftrightarrow (n+1 = m+1 \text{ and } \mu = \mu) \Leftrightarrow (n = m \text{ and } \mu = \mu)$
9. For every $n = 1,2,3, \ldots$ the equality $(n, \mu) = (0, \mu)$ is false.
10. Really, by (4.3) we write $(n, \mu) = (0, \mu) \Leftrightarrow \overset{\frown\frown}{\mu}\,\mathcal{A}\,n = \overset{\frown\frown}{\mu} \Leftrightarrow \overset{\frown\frown}{\mu}\,\mathcal{A}\,n\,\mathcal{S}\,\overset{\frown\frown}{\mu} \Leftrightarrow \overset{\frown\frown}{\mu}\ \mathcal{S}\ \overset{\frown\frown}{\mu} \Leftrightarrow n = 0$ is a contradiction.

In the next we define the μ-th *level – addition* and μ-the *level –multiplication* denoted by $+^{(\mu)}$ and $\cdot^{(\mu)}$, respectively

(4.5) $$(n, \mu) +^{(\mu)} (m, \mu) = (n+m, \mu) \quad , n, m \in \mathbb{N}$$

and

(4.6) $$(n, \mu) \cdot^{(\mu)} (m, \mu) = (n \cdot m, \mu) \quad , n, m \in \mathbb{N}.$$

***Remark* 4.7:** Between $\mathbb{N}$ and $_\mu\mathbb{N}$, ($\mu = 1,2,3, \ldots$) the mapping $n \leftrightarrow (n, \mu)$ where $n \in \mathbb{N}$ is isomorphism.
Let $\mu = 1,2,3, \ldots$ *and* $\nu = 0,1,2, \ldots$ be given such that $\mu > \nu$.

***Definition* 4.8:** We say that $_\mu\mathbb{N}\ \Xi\ _\nu\mathbb{N}$ is the μ-th *level- jumping with rescpect* $_\nu\mathbb{N}$ if for each element of $_\mu\mathbb{N}$ the connection

(4.9) $$(n, \mu) = (n, \nu)\mathcal{A}\,\overbrace{\mu - \nu} \quad , \quad n = 0,1,2, \ldots$$

is fulfilled.

So, $_1\mathbb{N}\ \Xi\ \mathbb{N}$ is the first level – jumping with respect for the set of natural numbers. (See (4.1).)
Using (4.2) and (4.3) and applying Definition 1.15 we have

$$0 = (0,0) < 1(= (0,1)) < 2(= (0,2)) < \cdots < \overset{\frown\frown}{1},$$
$$\overset{\frown\frown}{1} = (0,1) < (1,1) < (2,1) < \cdots < (n,1) < \cdots \overset{\frown\frown}{2},$$
$$\vdots$$
$$\overset{\frown\frown}{\mu} = (0, \mu) < (1, \mu) < (2, \mu) < \cdots < (n, \mu) < \cdots \overbrace{\mu + 1},$$
$$\vdots$$

and the new question is: Is there any exploded number which is comparable to the cardinal number of the set of real numbers?

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