International Journal of Mathematical Archive-10(12), 2019, 39-46
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A REVIEW ON HYERS-ULAM STABILITY OF FUNCTIONAL EQUATIONS IN SEVERAL VARIABLES

SUBHASHISH BISWAS*[1], SUSHMA[2] AND PRIYANKA RANI[3]

[1]Kalinga University, Raipur - 492101, India.

[2]Kanya Mahavidyalaya, Kharkhoda - 124001, India.

[3]Kalinga University, Raipur - 492101, India.

(Received On: 11-12-19; Revised & Accepted On: 02-12-19)

ABSTRACT

In this paper, we study the stability of functional equations that has originated a few decades ago. Its origin can be traced back to the problem posed by S. M. Ulam and D. H. Hyers, who laterprovided the solution to the same. In particular, during the last three decades, the notionof stability of functional equations has attracted the attention of many researchers to do work in this vast field. In this paper, we shall review an area of research on this topic.

Key Word: *Ulam-Hyers stability, homomorphism, functional inequality, additive mapping, Banach space.*

INTRODUCTION

Over the last few decades the development of stability problem has attracted considerable attention due to its numerous applications and applicability in both pure Mathematics and Physics. The notion of stability of functional equations has evolved into an area of continuing research from both pure and applied viewpoints. Different generalizations of the notion of stability of functional equations have been generalized according to the requirement and their applicability to solve a particular problem by using different conditions and spaces.

To quote S.M. Ulam[23], for very general functional equations, one can ask the following question: When is it true that the solution of an equation differing slightly from a given one must be necessarily close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near the solutions of the strict equation?

This paper is designed to provide an account to the solution of such problems by examining different researches that have been done during the last years since 1940 when Ulam posed the fundamental problem in the subject where he provided a comprehensive approach before the Mathematics Club of the University of Wisconsin. He discussed about a number of important and solved problems concerning a wide variety of questions including, the stability of homomorphisms.

These days, it is widely accepted that the stability problem is considered as the beginning of this field, which is as follows:

"Let G_1 be a group and G_2 be a metric group with the metric d(. , .). Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h: G_1 \to G_2$ satisfies the inequality $d\left(h(xy), h(x)h(y)\right) < \delta$ for all x, $y \in G_1$ then there exists a homomorphism $H: G_1 \to G_2$ with $d\left(h(x), H(x)\right) < \varepsilon$ for all $x \in G_1$? "

The first answer to Ulam's question came within a year when D. H. Hyers [13] has excellently answered this question of Ulam for the case where G_1, G_2 are Banach spaces:

Corresponding Author: Subhashish Biswas*[1],
[1]Kalinga University, Raipur, 492101, India.

Theorem 1: "Let $f : E_1 \to E_2$ be a mapping between Banach spaces such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \delta, \quad x, y \in E_1 \tag{1}$$

for some $\delta > 0.$ Then the limit

$$l(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{2}$$

exists for each $x \in E_1$ and $l : E_1 \to E_2$ is the unique additive mapping such that

$$\left\| f(x) - l(x) \right\| \le \delta, \quad x \in E_1 \tag{3}$$

Moreover, if f(tx) is continuous in t for each fixed $x \in E_1$, then the mapping l is linear."

Taking this famous result into consideration, the additive Cauchy equation

$$f(x+y) = f(x) + f(y), \tag{4}$$

is said to have the Hyers-Ulam stability on (G, E), where G and E are given spaces, if for every mapping f: $G \to E$ satisfy the inequality (1), for some $\delta \ge 0$ and all x, $y \in G,$ there exists an additive mapping $l : G \to E$ such that f – l is bounded on G.

The method in (2), given by Hyers, produces the additive mapping l which is called as Direct Method. It is imperative and potent tool for the study of the stability for various functional equations.

The Hyers's result was generalized by T. Aoki [1] in 1950, for additive mappings. Let E_1 and E_2 be Banach spaces and f(x) be a transformation from E_1 into E_2 which is "approximately linear."

In generalizing the definition of Hyers, Aoki called a transformation f(x) from E_1 into E_2 "approximately linear", when there exists K(≥ 0) and p $(0 \le p \le 1)$ such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \le K\left(\|x\|^b + \|y\|^b \right) \text{ for any x and y in } E_1.$$

Let f(x) and l(x) be transformations from E_1 into E_2. These are called "near", when there exist K(≥ 0) and p($0 \le p \le 1$) such that

$$\left\| f(x) - l(x) \right\| \le K \|x\|^p \quad \text{for any x in } E_1.$$

Theorem 2: "If f(x) is an approximately linear transformation from E_1 into E_2, then there is a linear transformation l(x) near f(x). And such l(x) is unique."

In 1978, an approach was made to weaken the condition for the bound of the norm of the Cauchy difference f(x+y) – f(x) – f(y) by Th. M. Rassias [20] and later proved to be considerable generalized result of Hyers by making a direct method.

Theorem 3: "Let $f : E_1 \to E_2$ be a mapping between Banach spaces. If f satisfies

$$\left\| f(x+y) - f(x0 - f(y) \right\| \le K\left(\|x\|^p + \|y\|^p \right), x, y \in E_1 \tag{5}$$

for some $K \ge 0$ and some p with $0 \le p < 1$, then there exists a unique additive mapping l : $E_1 \to E_2$ such that

$$\left\| f(x) - l(x) \right\| \le \frac{2\theta}{2 - 2^p} \|x\|^p, x \in E_1. \tag{6}$$

If, in addition, f(tx) is continuous in t for each fixed $x \in E_1$, then the mapping l is linear."

This result was a significant generalization of Hyers result and inspired many mathematicians to investigate the stability problem of several functional equations. By regarding a large influence of Theorem 3 on the study of stability problems of several functional equations, the Hyer-Ulam stability of such type is called the Hyers-Ulam-Rassias stability.

In 1991,Z. Gajda worked extensively on working result of Russias proved in [20] to provide a more comprehensive approach to extend its validity [7] by raising the point that was mentioned by Rassias so that it works for every p from the interval $(-\infty,1)$ and hence, making the theorem intact for all such p's. It can also be noted that the only purpose of assuming that all the transformations of the form $t \to f(tx)$ are continuous is to guarantee the real homogeneity of the mapping T. Without this assumption one can show that f is approximated by an additive mapping T which means that T satisfies the following equation

$$T(x+y) = T(x) + T(y),$$

for all x, $y \in E_1$. However, there is still one non-trivial question concerning a possible extension of the range of the validity of Theorem 3. Finally, it should be noticed that the completeness of the space E_1 may be removed from the assumption of the theorem 3. It turns out that, for p>1, modification of Rassias's proof is possible. So, Gajda modified Rassias's work in the following theorem.

Theorem 4: "Let E_1 and E_2 be two(real) normed linear spaces and assume that E_2 is complete. Let $f: E_1 \to E_2$ be a mapping for which there exist two constants $\varepsilon \in (0,\infty)$ and $p \in R \setminus \{1\}$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p),$$

for all x,y $\in E_1$. Then there exists a unique additive mapping l: $E_1 \to E_2$ such that

$$\|f(x) - l(x)\| \le \delta \|x\|^p \text{ for all } x \in E_1,$$

where

$$\delta = \begin{cases} \dfrac{2\varepsilon}{2-2^p} & \text{for } p < 1, \\ \dfrac{2\varepsilon}{2^p - 2} & \text{for } p > 1. \end{cases}$$

Moreover, for each $x \in E_1$, the transformation $t \to f(tx)$ is continuous, then the mapping l is linear."

In 1994, P. Gavruta [8] gave a further generalization by replacing the Cauchy differences by a control mapping ϕ, in the spirit of Rassais approach.

Theorem 5: "Let (G, +) be an abelian group, $(X, \|.\|)$ a Banach space and $\phi: G \times G \to [0,\infty)$ a mapping such that

$$\phi(x,y) = \sum_{k=0}^{\infty} 2^{-k} \phi(2^k x, 2^k y) < \infty, \text{ for all } x, y \in G. \tag{7}$$

Let $f: G \to X$ be such that

$$\|f(x+y) - f(x) - f(y)\| \le \phi(x,y), \text{ for all } x, y \in G. \tag{8}$$

Then there exists a unique mapping l: $G \to X$ such that

$$l(x+y) = l(x) + l(y), \text{ for all } x,y \in G. \tag{9}$$

And

$$\|f(x) - l(x)\| \le \frac{1}{2}\phi(x,x), \text{ for all } x \in G.\text{"} \tag{10}$$

In the same year, G. Issac and Th. M. Rassias [15] proved the stability of functional equations by introducing the notion of a ψ - additive mapping [14] and proved the following theorem.

Theorem 7: "Suppose that ψ verifies

(A1) $\lim_{t\to\infty} \dfrac{\psi(t)}{t} = 0$.

(A2) $\psi(ts) \le \psi(t)\psi(s)$, for all t, s ≥ 0.

(A3) $\psi(t+s) \le \psi(t) + \psi(s)$, for all t, s ≥ 0.

(A4) ψ is monotone increasing on R_+.

(A5) $\psi(t) < t$, for all t >1 .

Then $f: E_1 \to E_2$ is a ψ - additive mapping if and only if there exist a constant c > 0 and an additive mapping $l: E_1 \to E_2$ such that

$$\|f(x) - l(x)\| \le c\psi(\|x\|), \text{for all } x \in E_1.\text{"}$$

In 1995, Gavruta *et al.* [11] extended the Gavruta [7] work for providing an better generalization of the Hyers-Ulam stability mappings.

Theorem 8: "Let (G, +) be an abelian group, k an integer, $k \ge 2$, (X, $\|.\|$) a Banach space, $\phi: G \times G \to [0, \infty)$ a mapping such that

$$\phi_k(x, y) = \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \phi(k^n x, k^n y) < \infty, \tag{11}$$

for all x, y $\in G$ and $f: G \to X$ a mapping with the property

$$\|f(x+y) - f(x) - f(y)\| \le \phi(x, y), \text{ for all } x, y \in G.$$

Then there exists a unique additive mapping $l: G \to X$ such that

$$\|f(x) - l(x)\| \le \sum_{m=1}^{k-1} \phi_k(x, mx), \text{for all } x \in G.\text{"} \tag{12}$$

In 2001, P. Gavruta [9] gave an answerto a problem of G. Issac and Rassias concerning Hyers-Ulam-Rassias stability of the linear mappings and a new characterization of $\psi -$ additive mappings.

Theorem 9: "Let E_1 be a normed space and E_2 be a real Banach space. Suppose
(A5') There exists a $t_0 > 0$ such that $\psi(t_0) < t_0$.
Suppose that ψ verifies (A2), (A3), (A4) and (A5').
Then $f: E_1 \to E_2$ is a ψ - additive mapping if and only if there exists c > 0 and an additive mapping $l: E_1 \to E_2$ such that

$$\|f(x) - l(x)\| \le c\psi(\|x\|), \text{for all } x \in E_1.\text{"}$$

Condition (A5') is essential for this theorem. Now we give some examples which satisfy different condition from Theorem 7 and Theorem 9.

Example 1: There exists functions ψ that verify (A3), (A4), (A5) but do not verify (A2).

$$\psi(n) = \frac{n}{n+1}, n \ge 0.$$

Example 2: There exist functions ψ that verify (A2), (A3), (A5') which do not verify (A4),

$$\psi(n) = \begin{cases} \dfrac{1+n}{\sqrt{n}}, & \text{if } n \ge 0 \\ 0, & \text{if } n = 0. \end{cases}$$

For a number of generalizations of Hyers theorem for the stability of the additive mapping as well as Hyers-Rassias approach for the stability of the linear mapping, one can refer to [22], [2], [3], [4], [6], [16], [17], [18] and [19].

A lot of work has been done in finding a new method for the generalized Hyers-Ulam-Rassias stability. Eventually, in 2010, Gavruta *et.al* [10] proposed a new method called weighted method for the study of Hyers-Ulam-Rassias stability. They used this method for Volterra and Fredholm integral operators type functional equations.
For the stability of Volterra integral operator, they consider I - [a, b] and denoted by C(I), the space of all complex-valued continuous functions on I. Consider the functions $L: I \to [0, \infty)$ to be integrable, g belongs to C(I), $f: I \times C \to C$ and $\phi: I \to (0, \infty)$ continuous.

Theorem 10: "Suppose that there exist a unique $\alpha \in [0,1]$ so that

$$\left|\int_c^x L(t)\phi(t)dt\right| \le \alpha\phi(x), x \in I$$

$$\left|f(t,u(t)) - f(t,v(t))\right| \le L(t)\left|u(t)-v(t)\right|, t \in I, \text{for all } u, v \in C(I).$$

If y $\in$ C(I) so that $\left|y(x) - g(x) - \int_c^x f(t, y(t))dt\right| \le \phi(x), x \in I$

then there exist a unique $y_0 \in C(I)$:

$$y_0(x) = g(x) + \int_c^x f(t, y_0(t))d \ ,$$

and $$\left|y(x) - y_0(x)\right| \le \frac{\phi(x)}{1-\alpha}, x \in I."$$

For the stability of the Fredholm integral operator consider I = [a, b], $g \in C(I), \phi: I \to (0,\infty)$ continuous, L: $I \times I \to [0,\infty)$ integrable $K: I \times I \times C \to C$ continuous.

Theorem11: "Suppose that there exists $\beta > 0$,

$$\int_I L(x,t)\phi(t)dt \le \beta\phi(x), x \in I,$$

$$\left|K(x,t,u(t) - K(x,t,v(t)\right| \le L(x,t)\left|u(t)-v(t)\right|, u, v \in C(I).$$

Let $y \in C(I)$ be so that

$$\left|y(x) - g(x) - \lambda\int_I K(x,t,y(t)dt\right| \le \phi(x), x \in I.$$

If $\left|\lambda\right| < \frac{1}{\beta}$, then there exists a unique $y_0 \in C(I)$ such that

$$y_0(x) = g(x) + \lambda\int_I K(x,t,y_0(t))dt,$$

and

$$\left|y(x) - y_0(x)\right| \le \frac{\phi(x)}{1-\left|\lambda\right|\beta}, x \in I."$$

In 2012, Cadariu *et.al* [5] proved the stability of a non-linear functional equation, linear functional equation as well as a volterra integral equation, by using the weighted space method.

Firstly, we listed their result to obtain the generalized Hyers-Ulam stability for the nonlinear equation

$$y(x) = F\big(x, y(x), y(\eta(x))\big), \tag{13}$$

by using the weighted space method.

Theorem 12: "Consider a function y: S → X, is the unknown, S is a nonempty set, (X, d) is a complete metric space, F: S × X × X→ X and $\eta: S \to S$ are given functions. Suppose that there exists L∈ [0, 1) such that the mappings $\lambda, \mu: S \to [0,\infty)$ satisfy

$$\lambda(x)\phi(x) + \mu(x)\phi(\eta(x)) \le L\phi(x), \text{for all } x \in S \tag{14}$$

for some given function $\phi: S \to (0,\infty)$. Suppose also that the given mapping F: S × X × X→ X satisfies

$$d\Big(F\big(x,u(x),u(\eta(x))\big), F\big(x,v(x),v(\eta(x))\big)\Big)$$
$$\le \lambda(x)d\big(u(x),v(x)\big) + \mu(x)d\big(u(\eta(x)),v(\eta(x))\big),$$

for all x ∈ S and for all u, v ∈ X.

If y: S → X is a fixed mapping with the property

$$d\Big(y(x), F\big(x, y(x), y(\eta(x))\big)\Big) \le \phi(x), \text{for all } x \in S. \tag{15}$$

Then there exists a unique $y_0 : S \to X$ such that

$$y_0(x) = F\big(x, y_0(x), y_0(\eta(x))\big), \text{for all } x \in S$$

and the inequality

$$d\big(y(x), y_0(x)\big) \le \frac{\phi(x)}{1-L}, \tag{16}$$

holds for all x ∈ S.”

Next, they showed the Hyers-Ulam stability of a linear functional equation. In which they consider a nonempty set S, a real Banach space X, with the norm $\|.\|$ and the given functions g: S → R, η: S → S and h: S → X.

Theorem 13: “suppose that there exists L ∈ [0, 1) such that the functions $\lambda, \mu : S \to [0,\infty)$ satisfy

$$\lambda(x)\phi(x) + \mu(x)\phi(\eta(x)) \le L\phi(x), \text{for all } x \in S,$$

for some fixed mapping $\phi : S \to (0,\infty)$. Suppose also that the function F: S × X × X → X verifies

$$\big(|g(x)| - \mu(x)\big).\big\|u(\eta(x)) - v(\eta(x))\big\| \le \lambda(x)\|u(x) - v(x)\|,$$

for all x ∈ S and for all u, v from S into X.
If y: S → X has the property

$$\big\|y(x) - g(x)y(\eta(x)) - h(x)\big\| \le \phi(x), \text{for all } x \in S.$$

Then there exists a unique mapping $y_0 : S \to X$ defined by

$$y_0(x) = h(x) + \lim_{n\to\infty}\left(y(\eta^n(x)).\prod_{i=0}^{n-1} g(\eta^i(x)) + \sum_{j=0}^{n-2}\left(h(\eta^{j+1}(x)).\prod_{i=0}^{j} g(\eta^i(x))\right)\right),$$

for all x∈ S, which satisfies the equation

$$y(x) = g(x).y(\eta(x)) + h(x), \text{for all } x \in S$$

and the inequality

$$\|y(x) - y_0(x)\| \le \frac{\phi(x)}{1-L}, \text{for all } x \in S.\text{”}$$

At last, they obtained the generalized Hyers-Ulam stability for the nonlinear volterra integral equations in Banach spaces.

Theorem 14: “Suppose that there exists a Banach space X over the field K, a positive nonzero constant α, an interval I = [a, b] (a < b) and the continuous functions L: I × I $\to [0,\infty)$ and $\phi : I \to (0,\infty)$ such that

$$\int_a^x L(x,t)\phi(t)\,dt \le \alpha\phi(x), \text{for all } x \in I.$$

Suppose also that G: I × I × X→ X is a continuous function and denoted by C(I, X) = { f : I → X, f is continuous} and by $\|.\|$ the norm of Banach space X, which satisfies

$$\big\|G(x,t,u(t)) - G(x,t,v(t))\big\| \le L(x,t)\|u(t) - v(t)\|, \text{for all } x, t \in I, \text{for all } u, v \in C(I,X).$$

If the continuous mapping y: I → X has the property

$$\left\|y(x) - h(x) - \lambda\int_a^x G(x,t,y(t))\,dt\right\| \le \phi(x), \text{for all } x \in I$$

and if

$$|\lambda| < \frac{1}{\alpha}.$$

Then there exists a unique $y_0 \in C(I, X)$ such that

$$y_0(x) = h(x) + \lambda \int_a^x G(x, t, y_0(t))\,dt, \text{for all } x \in I$$

A and the inequality

$$\left\| y(x) - y_0(x) \right\| \le \frac{\phi(x)}{1 - |\lambda|\alpha}, \text{for all } x \in I$$

holds."

In 2015, Huang *et.al* [12] discussed the stability of some classes of linear functional differential equations by open mapping theorem method. We need some terminologies and definitions before displaying their result which completely explained in [21].

Theorem 15: "Given $g(t) \in C[0, b]$, then

$$y^{(n)}(t) = g(t)y(t - \tau) \tag{17}$$

has the Hyers-Ulam stability on $[-\tau, b]$."

Hyers-Ulam Stability problem is used in many dimensions of mathematics for solving many problems. A lot of work has been done using different techniques and methods in various spaces. Even now it is considered as a fundamental basis for understanding problems regarding Hyers-Ulam stability problem because of its wide applications in various fields.

REFERENCES

1. Aoki, T. (1950). On the stability of the linear transformation in Banach spaces. Journal of the Mathematical Society of Japan, Vol. 2, pp. 64-66.
2. Badea, C. (2005). The general linear equation is stable. Non-linear Funct. Anal. Appl., Vol. 10, No. 1, 155-164.
3. Belaid, B., Elhoucien, E. and Rassias, Th. M. (2008). On the Hyers-Ulam stability of approximately Pexider-mappings. Math. Ineq. Appl., 11(4), 805-818.
4. Bojor, F. (2009). Generalized additive Cauchy equations and their Ulam-Hyers stability. Creative Math. Inf. 18, No. 2, 129-135.
5. Cadariu, L., Gavruta, L. and Gavruta, P. (2012). Weighted space method for the stability of some nonlinear equations. Appl. Anal. Discrete Math, 126-139.
6. Czerwik, S. (2002). Functional equations and inequalities in several variables. World Scientific, New Jersey.
7. Gajda, Z. (1991). On the stability of additive mappings. International Journal of Mathematics and Mathematical Sciences, Vol. 14, No. 3, pp. 431-434.
8. Gavruta, P. (1994). A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. Journal of Mathematical analysis and applications, Vol. 184, No. 3, pp. 431-436.
9. Gavruta, P. (2001). On a problem of G. Isac and Th. M. Rassias concerning the stability of mapping. Journal of Mathematical analysis and application, Vol. 261, No. 2, pp. 543-553.
10. Gavruta, P. and Gavruta, L. (2010). A new method for the generalization Hyers-Ulam-Rassias stability. International Journal of Non-linear Analysis and Applications. Vol. 1, No. 2, pp. 11-18.
11. Gavruta, P., Hossu, M., Popescu, D. and Caprau, C. (1995). On the stability of mappings and an answer to a problem of Th. M. Rassias. Annales Mathematiques Blaise Pascal, Vol. 2, pp. 55-60.
12. Huang, J. and Li, Y. (2015). Hyers-Ulam stability of linear functional differential equations. J. Math. Anal. Appl., pp. 1192-1200.
13. Hyers, D. H. (1941). On the stability of the linear functional equation. Proceedings of the National Academy of Sciences of the United States of America, Vol. 27, pp. 222-224.
14. Isac, G. and Rassias, Th. M.(1993). On the Hyers-Ulam stability of ψ- additive mappings. Journal Approx. Theory 72, pp. 131-137.
15. Isac, G. and Rassias, Th. M.(1994). Functional inequalities for approximately additive mappings in "Stability of Mappings of Hyers-Ulam type". Hadronic Press, Palm Harbour, pp. 117-125.
16. Jung, S. M. And Rassias, Th. M. (2007). Ulam's problem for approximate homomorphisms in connection with Bernoulli's differential equation. Appl. Math. Comput., 187, pp. 223-227.
17. Kaiser, Z. and Pales, Z. (2005). An example of a stable functional equation when the Hyers method does not work. J. Ineq. Pure Appl. Math., 6(1), Article 14.
18. Kim, G. H.(2001). On the stability of the quadratic mapping in normed spaces. International Journal of Mathematics and Mathematical Sciences, 25(4), pp. 217-229.
19. Popa, D.(2005). Hyers-Ulam-Rassias stability of linear recurrence. Journal Math. Anal. Appl., 309(2), pp. 591-597.

20. Rassias, Th. M.(1978). On the stability of the linear mapping in Banach spaces. Proceedings of the American Mathematical Society. Vol. 72, No. 2, pp. 297-300.
21. Takagi, H., Miura, T. and Takahasi, S. E. (2003). Essential norms and stability constants of weighted composition operators on C(X). Bull. Korean Math. Sc. 40 (4), pp. 583-591.
22. Trif, T.(2002). On the stability of a functional equation deriving from an inequality of Popviciu for convex functions. Journal Math. Anal. Appl. 272(2), pp. 604-616.
23. Ulam, S. M.(1960). Problems in Modern Mathematics. Chapter 6, Science Editors, Wiley, New York, USA.

Source of support: Nil, Conflict of interest: None Declared.