International Journal of Mathematical Archive-10(12), 2019, 12-15
Available online through www.ijma.info ISSN 2229 – 5046

NUMERICAL SOLUTION OF NON-LINEAR BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS USING THE FINITE DIFFERENCE TECHNIQUE

J. S. MAREMWA[1], N. SANG[2*]

**Department of Mathematics and Computer Science,
University of Eldoret P.O. Box 1125-30100, Eldoret, Kenya.**

(Received On: 02-10-19; Revised & Accepted On: 25-11-19)

ABSTRACT

Ordinary Differential Equations (ODEs) of the Initial Value Problem (IVP) or Boundary Value Problem (BVP) type can model phenomena in wide range of fields including science, engineering, economics, social science, biology, business, health care among others. Often, systems described by differential equations are so complex that purely analytical solutions of the equations are not traceable. Therefore techniques for solving differential equations based on numerical approximations take centre stage. In this paper, we review the finite difference technique as a method of solution to both linear and non-linear BVPs.

Keywords: *Linear Boundary Value Problem, Non-Linear Boundary Value Problem, finite difference technique.*

1. NON LINEAR BVP OF ORDINARY DIFFERENTIAL EQUATIONS

1.1 Introduction

Virtually all systems that undergo change can be described by differential equations. Differential equation model phenomena in wide range of fields including science, engineering, economics, social science, biology, business, health care among others. Often, systems described by differential equations are so complex that purely analytical solutions of the equations are not traceable. Therefore techniques for solving differential equations based on numerical approximations take centre stage.

1.2 Initial and Boundary Value Problems

A differential equation defines a relationship between unknown function and one or more of its derivatives. The derivatives are of the dependent variable with respect to independent variable(s). If the independent variable is single, the differential equation is called an ordinary differential equation (ODE), otherwise it is a partial differential equation (PDE). An Initial Value Problem of an ODE usually abbreviated as IVP is an ordinary differential equation whose solution is specified at only one point in the domain of the equation. This condition is often called an initial condition. An ordinary differential equation whose solution in an interval domain say $[a,b]$ where $a,b \in \Re$ is specified at more than one point is called a Boundary Value Problem (BVP). The conditions are then called boundary conditions. Some specified conditions of ODEs are: In engineering, the dissolution of contaminant into ground water is governed by a first order differential equation. In science, the rate of cooling of a beverage is proportional to the difference in temperature between the beverage and the surrounding air. This is governed by first order ODE.

Corresponding Author: N. Sang[2*]
Department of Mathematics and Computer Science,
University of Eldoret P.O. Box 1125-30100, Eldoret, Kenya.

An analytical solution of a differential equation (partial or ordinary) is called ‘closed form solution’. At best, there are only a few differential equations that can be solved analytically by closed form. There exist many different methods in the literature for the analytical solutions of both IVPs and BVPs [1]. These include the methods for first order ordinary differential equations such as linear equations solved by use of integrating factor, exact equations, homogenous equations, ODEs in which variables are separable and Bernoulli type equations. For second and higher order ODEs techniques such as use of complementary functions and particular integrals and variation of parameters method are available. There are only fairly few kinds of equations for which the solution is in terms of standard elementary mathematical functions such as cosines, sine logarithms, exponentials etc. Some simple second order linear differential equations can be solved using various special functions such as Legendre, Bessel’s etc. Beyond second order, the kind of functions needed to solve even fairly simple linear differential equations becomes extremely complicated. Solutions of most practical problems involving differential equations require the use of numerical methods. Numerical solutions of IVPs of ODEs are classified into two major groups namely, one step-method and multi-step methods. The one step method include among others, Taylor’s methods, Euler’s method, Heun’s method and Runge-Kutta methods. Linear multi-step methods include implicit Euler’s method, Trapezium rule method, Adams-Bashforth method, Adams-Moulton method and predictor-Corrector methods. For BVPs of ODEs there exist some methods such as shooting methods and finite difference method for both linear and non-linear BVPs. The finite difference method is the subject of this paper.

2. THE FINITE DIFFERENCE METHOD

Consider $y'' = f(x, y)$ (1)

We assume that the differential equation does not contain y' explicitly with the boundary conditions,

$$y(a) = a_3 \tag{2}$$

$$y(b) = b_3 \tag{3}$$

That is $a_1 = b_1 = 1$ and $a_2 = b_2 = 0$

We divide the interval (a, b) into n smaller intervals of width h

Where $h = \dfrac{b-a}{n}$ (4)

Then define $x_r = a + rh$ (5)

So that $x_0 = a$ and $x_n = b$ and

$$y(x_r) = y_r \tag{6}$$

The boundary conditions (2) and (3) can be written as $y_0 = a_3$, $y_n = b_3$

$$y_0 = a_3,\ y_n = b_3 \tag{7}$$

We express the second derivative $y''(x_r) = y_r''$ in terms of values of y.

Using Taylor’s series,

$$y(x_r + t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \dots \tag{8}$$

Where $c_m = \dfrac{y_r^{(m)}}{m!}$ (9)

Also $y_r^{(m)}$ denotes the m^{th} derivative of y evaluated at $x = x_r$

Thus $y_r'' = y''(x_r) = 2c_2$ (10)

From equation (8),

$$y_{r-1} = y(x_r - h) = c_0 - c_1 h + c_2 h^2 - c_3 h^3 + c_4 h^4 - \dots \tag{11}$$

$$y_r = y(x_r) = c_0 \tag{12}$$

$$y_{r+1} = y(x_r + h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + c_4 h^4 + \dots \tag{13}$$

We eliminate c_1 from these last three equations by adding (11) and (13). Thus

$$y_{r-1} + y_{r+1} = 2c_0 + 2c_2h^2 + 2c_4h^4 + ...$$

We can eliminate c_0 by subtracting twice equation (12) to give

$$y_{r-1} - 2y_r + y_{r+1} = 2c_2h^2 + 2c_4h^4 + ... \tag{14}$$

From equation (10) and (14) we have,

$$h^2 y_r^{"} = y_{r-1} - 2y_r + y_{r+1} \tag{15}$$

With error approximately

$$2c_4h^4 \text{ or } \frac{1}{12}h^4 y_r^{(iv)} \tag{16}$$

Equation (15) is called a finite difference approximation since it enables us to evaluate $y_r^{"}$ by using the finite difference between y_{r-1}, y_r, y_{r+1} rather than the infinitesimal differences used in calculus. Finite differences were extensively used in the days of hand computation but nowadays their use is largely restricted to a few formulae like equation (15).

We have (n+1) values of y_r entering into our present problem namely $y_0, y_1, y_2, ..., y_n$. Two of these (y_0 and y_n) are known from boundary condition (2)

Using equation (15) and the differential equation (1) we can write down the approximate equations

$$y_{r-1} - 2y_r + y_{r+1} = h^2 f(x_r, y_r),$$
$$r = 1,2,3,....,(n-1) \tag{17}$$

Given that y_0 and y_n are known, equation (17) form (n-1) equations for (n-1) unknowns $y_1, y_2, y_3, ..., y_{n-1}$, if n=5 equations are

$$-y_1 + y_2 = h^2(f(x_1, y_1) - a_3$$
$$y_1 - 2y_2 + y_3 = h^2 f(x_2, y_2)$$
$$y_2 - 2y_3 + y_4 = h^2 f(x_3, y_3)$$
$$y_3 - 2y_4 = h^2 f(x_{4,} y_4) - b_3$$

Since the non-linear term is small, with multiple h^2 we can solve above equations using an iterative method of solution.

We start with some initial approximation $y_r^{(0)}$ to y_r and obtain successive approximations $y_r^{(k)}$ by solving for each value of k, a set of simultaneous equations.

In case n=5 equations are

$$-2y_1^{(k)} + y_2^{(k)} = h^2 f(x_1, y_1^{(k-1)}) - a_3$$
$$y_1^{(k)} - 2y_2^{(k)} + y_3^{(k)} = h^2 f(x_2, y_2^{(k-1)})$$
$$y_2^{(k)} - 2y_3^{(k)} + y_4^{(k)} = h^2 f(x_3, y_3^{(k-1)})$$
$$y_3^{(k)} - 2y_4^{(k)} = h^2 f(x_4, y_4^{(k-1)}) - b_3$$

We then solve equations for $k = 1,2,3$ until no further changes in values of y_r is seen.

Generally,

$$y_{r-1}^{(k)} - 2y_r^{(k)} + y_{r+1}^{(k)} = h^2 f(x_r, y_r^{(k-1)}),$$
$$\text{r=1, 2, 3,…(n-1)} \tag{18}$$

Together with $y_0^{(k)} = a_3$, $y_n^{(k)} = b_3$

This reduces to

$$-(r+1)y_r^{(k)} + ry_{r+1}^{(k)} = U_r^{(k)}, \text{ r=1, 2, 3……..,(n-1)} \tag{19}$$

Where $U_1^{(k)} = h^2 f(x_1, y_1^{(k-1)}) - a_3$

And $U_r^{(k)} = rh^2 f(x_r, y_r^{(k-1)}) + U_{r-1}^{(k)}$,

$$\text{r=1, 2, 3……..,(n-1)} \tag{20}$$

Using back substitution from equation (19)

$$y_r^{(k)} = \frac{1}{r+1}\left\{ry_{r+1}^{(k)} - U_r^{(k)}\right\},$$

r = (n-1), (n-2),……., 1

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