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SPHERICALLY SYMMETRIC STRING COSMOLOGICAL MODEL IN ROSEN'S THEORIES OF GRAVITATION

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ABSTRACT

In this paper the solutions of spherical symmetric cosmological model are obtained in the context of Rosen's bimetric theory of Relativity(BR) and bimetric general theory of relativity (BGR).In both the theories under the assumption that of space-time admits one parameter group of conformal motion, it is observed that the model satisfied energy conditions. Also it is noticed that the arbitrary function Ψ *(i.e. conformal factor) in the Lie derivative of the metric tensor is remain constant in BR.*

Keywords: *Spherical symmetric cosmological model, Bimetric theory of relativity, conformal motion.*

1. INTRODUCTION

There were several attempts to modify the general relativity but with little success. Rosen [1-14] proposed the bimetric theory of relativity where two metric tensors at each point of the space-time are defined: the Riemannian metric g_{ij} and the background flat space-time metric γ_{ij}. The tensor g_{ij} describes the geometry of a curved space-time and the γ_{ij} refers to inertial forces. According to the BR, the field equations of the theory are

$$N_{ij} - \frac{1}{2} g_{ij} N = -8\pi\kappa\, T_{ij} \tag{1}$$

where $N^i_j = \frac{1}{2}\gamma^{ab}(g^{hi} g_{hj|a})_{|b}$

$$N = N^i_j,\ \kappa = \sqrt{g/\gamma},\ \ g = \det(g_{ij}),\ \gamma = \det(\gamma_{ij})$$

Here T_{ij} is usual energy momentum tensor of the matter or other non-gravitational fields satisfying the conservation law

$$T^{ij}_{;j} = 0$$

where (;) denotes covariant differentiation with respect to g_{ij} and a vertical bar (|) stands for covariant differentiation with respect to γ_{ij}.

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According to the BGR (Rosen [10-12]), in addition to g_{ij} (the physical, or gravitational metric), there is a second metric γ_{ij} (the background metric), describing the geometry of space-time which one would have if there were no matter in the universe. In this respect the theory is similar to BR (Rosen [11, 12]).However the field equations of BGR are different than those of BR. The field equations in BGR are taken to the same as in general relativity but with an additional term on right hand side. They are taken to be

$$K_{ij} - \frac{1}{2} g_{ij} K = -8\pi\kappa\, T_{ij}, \qquad K_{ij} = R_{ij} - P_{ij} \tag{2}$$

where K_{ij} has the same form as the Ricci tensor R_{ij} formed from g_{ij}, except the that ordinary derivatives are replaced by covariant derivatives with respect to γ_{ij}, P_{ij} is the Ricci tensor formed from γ_{ij}, $K = g^{\alpha\beta} K_{\alpha\beta}$.

The equation (2) takes the form

$$G_{ij} = S_{ij} - 8\pi T_{ij} \tag{3}$$

where G_{ij} is the Einstein's tensor $R_{ij} - \frac{1}{2} g_{ij} R$, T_{ij} is the energy stress tensor and

$$S_{ij} = \frac{3}{a^2}\left(\gamma_{ij} - \frac{1}{2} g_{ij}\; g^{\alpha\beta}\gamma_{\alpha\beta}\right) \tag{4}$$

From equation (4) it can be seen that, if one is dealing with a physical system that is small compared to the size of the universe, e.g., the solar system, then S_{ij} is negligible in the field equations (3) and the present theory gives agreement with the Einstein general theory of relativity.

It is a challenging problem to determine the exact physical solution at very early stages of the formation of our universe. In the recent years there has been considerable interest in string cosmology. The study of cosmic string is of considering importance because it may generate density perturbation theory of the galaxy formation (Zeldovich [15]; Vilekin, [16]). It has been noted that the presence of string in the early universe does not contract present day observation of the universe (Kibble, [17]). Banergee *et al*. [18], Chakraborty [19] extensively investigated several aspects of cosmic strings in the context of general relativity. Camci [20] have studied conformal collineation and Ricci in heritance collineation defined by $L_\xi R_{ab} = 2\alpha\; R_{ab}$ for a string cloud and string fluid in general relativity.

In this paper the comparatively study is taken of cosmic strings in case of spherically symmetric space-time in BR and BGR. It is observed that by making use of Letelier [21] form of energy momentum tensor for a cloud string dust under the assumption that space-time admit one parameter group of conformal motion the resulting space-times are different in nature.

The paper is organized as: In section (2) field equations in BR with cosmic string for spherically symmetric space-time are considered and their solutions are deduced in section (3) with an assumption that the space-time admits a one parameter group of conformal motion. In the section (4) the solution of field equations in BGR is investigated for spherically symmetric space-time. The comparatively discussions are given in section (5).

2. FIELD EQUATIONS DUE TO BR

For a static and spherically symmetric system the physical metric is considered as

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \tag{5}$$

where λ and ν are functions of r alone.

The energy momentum tensor T_{ij} for a system of cosmic string is given by

$$T_{ij} = \rho u_i u_j - \wedge\, \chi_i\; \chi_j \tag{6}$$

where ρ is the energy density of the string , $\wedge$ is the tension density of the string . As pointed out by the Letelier [21], $\wedge$ may be positive or negative, u^i describes the system of four- velocity and χ^i represents a direction of strings.

In the commoving system, choose

$$u^i = (u^0, 0, 0, 0)\,; \quad \chi^i = (0, \chi^1, 0, 0) \tag{7}$$

From $u^i\ u_i = -\chi^i\ \chi_i = 1$, it obtain

$$u^0 = e^{-\nu/2}, \chi^1 = e^{-\lambda/2} \tag{8}$$

The non- vanishing components of energy momentum tensors T_j^i are

$$T_0^0 = \rho, \quad T_1^1 = \lambda, \quad T_2^2 = T_3^3 = 0 \tag{9}$$

The background flat space-time corresponding to (5) is

$$d\sigma^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{10}$$

Thus the field equation (1) for the metric (5) and (10) with energy momentum tensor (6) can be written as

$$-\frac{\nu''}{4} - \frac{\nu'}{2r} + \frac{\lambda''}{4} + \frac{\lambda'}{2r} = -8\pi\kappa\rho \tag{11}$$

$$\frac{\nu''}{4} + \frac{\nu'}{2r} - \frac{\lambda''}{4} - \frac{\lambda'}{2r} + \frac{e^{\lambda} - e^{-\lambda}}{r^2} = -8\pi\kappa \wedge \tag{12}$$

$$\frac{\nu'}{4} + \frac{\nu'}{2r} + \frac{\lambda''}{4} + \frac{\lambda'}{2r} + \frac{e^{-\lambda} - e^{\lambda}}{2r^2} = 0 \tag{13}$$

The three equations (11) – (13) connect four unknown variables ν, λ, ρ and $\wedge$. Thus one more relation connecting these variables is needed to solve these equations. Instead of considering a new relation, it is assumed that the physical metric (5) admits a one parameter group of conformal motion and the solutions of the field equations are obtained in the next section.

3. CONFORMAL MOTION AND SOLUTION OF FIELD EQUATIONS

A space-time admits a one parameter group of conformal motions generated by the vector field ξ^μ if

$$\mathrm{L}\, g_{\mu\nu} = g_{\mu\nu,\sigma}\xi^\sigma + g_{a\nu}\xi^a_{,\mu} + g_{\mu a}\xi^a_{,\nu} = \psi g_{\mu\nu} \tag{14}$$

where a comma denotes the covariant differentiation with respect to $\gamma_{\mu\nu}$ and ψ is an arbitrary function of coordinates. If the vector field is collinear with χ^μ then by virtue of spherical symmetry and independence of the metric tensor on the time-like coordinates the most general form of ξ^μ is

$$\xi^\mu = F(r)\,\chi^\mu \tag{15}$$

From equations (5), (14) and (15), it is obtained

$$\psi = F\nu' e^{-\lambda/2} = 2F' e^{-\lambda/2} = \frac{2Fe^{-\lambda/2}}{r} \tag{16}$$

A straight forward calculation gives

$$e^{-\lambda} = \frac{\psi^2}{4C_2^2}, \quad e^{\nu} = C_1 r^2, \quad F = C_2 \tag{17}$$

where C_1 and C_2 are constant of integration. Without-loss generality one can take $C_1 = 1$.

Using equation (17) in equations (11) - (13), it is obtained

$$\frac{\psi}{4C_2 r^3} + \frac{\psi''}{4C_2 r} - \frac{\psi'^2}{4C_2 r\psi} + \frac{\psi'}{2r^2 C_2} = 8\pi\rho \tag{18}$$

$$\frac{-\psi}{4C_2r^3}-\frac{\psi''}{4C_2r}+\frac{\psi'^2}{4C_2r\psi}-\frac{\psi'}{2r^2C_2}-\frac{2C_2}{r^3\psi}+\frac{\psi^3}{8C_2^3r^3}=8\pi\wedge \tag{19}$$

$$\frac{\psi}{4C_2r^3}-\frac{\psi''}{4C_2r}+\frac{\psi'^2}{4C_2r\psi}-\frac{\psi'}{2r^2C_2}-\frac{C_2}{r^3\psi}+\frac{\psi^3}{16C_2^3r^3}=0 \tag{20}$$

where $\kappa=e^{\frac{\nu+\lambda}{2}}=r\frac{2C_2}{\psi}$

with the help of equation (20), equations (18) and (19) reduces to

$$8\pi\rho=\frac{\psi}{2C_2r^3}-\frac{C_2}{r^3\psi}+\frac{\psi^3}{16C_2^3r^3} \tag{21}$$

$$8\pi\wedge=-\frac{\psi}{2C_2r^3}-\frac{C_2}{r^3\psi}+\frac{\psi^3}{16C_2^3r^3} \tag{22}$$

It is noted that the two independent equations (21) and (22) connect three unknown variables $\rho, \wedge$ and ψ. Thus, one more relation connecting these variables is needed to solve these equations. Consider Takabayasi's [22] equation of state

$$\rho=b\wedge \tag{23}$$

Using equation (23) into equations (21) and (22), it is obtained

$$8\pi\rho=\frac{2b}{(b-1)^{3/2}r^3}[(b+1)+\sqrt{2}\sqrt{b^2+1}]^{1/2} \tag{25}$$

$$8\pi\wedge=\frac{2}{(b-1)^{3/2}r^3}[(b+1)+\sqrt{2}\sqrt{b^2+1}]^{1/2} \tag{26}$$

From equations (17) and (24), the metric coefficients are given by

$$e^{-\lambda}=\frac{(b+1)+\sqrt{2}\sqrt{b^2+1}}{b-1}$$

$$e^{\nu}=r^2 \text{ (as } C_1=1)$$

Thus, the metric (5), in this case becomes

$$ds^2=r^2dt^2-Adr^2-r^2(d\theta^2+r^2\sin^2\theta d\phi^2) \tag{27}$$

where $A=\frac{(b+1)+\sqrt{2}\sqrt{b^2+1}}{b-1}$

In the next section, the solutions of field equations for the metric (1) in BGR are obtained under the assumption that a space-time admits a one-parameter group of conformal motion.

4. FIELD EQUATIONS DUE TO BGR

Consider a metric given by (5) and the energy momentum T_{ij} for a system of cosmic string given by (6).

The background metric γ_{ij} is taken in a static de-sitter form so that the line element is given by

$$d\sigma^2=(1-\frac{r^2}{a^2})dt^2-(1-\frac{r^2}{a^2})^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

For $r<<a$ this line element on the flat space has the form

$$d\sigma^2=dt^2-dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2) \tag{28}$$

Let us consider a region with $r << a$ and neglect the term which are small throughout the region, the non-vanishing components S_i^j

$$S_0^0 = -S_1^1 = -S_2^2 = -S_3^3 = \frac{3}{2a^2} e^{-\nu} \tag{29}$$

Following the procedure of Rosen (1980), field equations (3) for $r << a$ are

$$e^{-\lambda}\left[\frac{-\lambda'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = \frac{3}{2a^2} e^{-\nu} - 8\pi\rho \tag{30}$$

$$e^{-\lambda}\left[\frac{\lambda'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = \frac{3}{2a^2} e^{-\nu} - 8\pi \wedge \tag{31}$$

$$e^{-\lambda}\left[\frac{\lambda''}{r} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu'}{2r} - \frac{\lambda'}{2r}\right] = -\frac{3}{2a^2} e^{-\nu} \tag{32}$$

where a prime ($'$) denotes a derivative with respect to r .

Due to the non linearity of the field equations (30) – (32), it is rather difficult to obtain their solution and therefore one has to make certain simplifying assumption to desire the useful results. It is assumed that the physical metric (5) admits a one parameter group of conformal motion and the solutions of the field equations (30) – (32) are obtained.

Substituting the values from the equation (17) into equation (30) – (32), to obtain

$$\frac{-\psi\psi'}{2c_2^2 r^2} - \frac{\psi^2}{4c_2^2 r^2} + \frac{1}{r^2} + \frac{3}{2a^2}\frac{1}{r^2} = 8\pi\rho \tag{33}$$

$$-\frac{3\psi^2}{4c_2^2 r^2} + \frac{1}{r^2} - \frac{3}{2a^2}\frac{1}{r^2} = 8\pi \wedge \tag{34}$$

$$\frac{\psi\psi'}{2c_2^2 r} + \frac{\psi^2}{4C_2^2 r^2} + \frac{3}{2a^2}\frac{1}{r^2} = 0 \tag{35}$$

Equation (35) with substitution $\psi^2 = t$ gives

$$\psi^2 = -\frac{6C_2^2}{a^2} + \frac{C}{r} \tag{36}$$

where C is a constant of integration.

Using equation (36) into equation (33) and (34), to obtain

$$8\pi\rho = \frac{1}{r^2} + \frac{3}{2a^2}\frac{1}{r^2} \tag{37}$$

$$8\pi\wedge = \frac{1}{r^2} + \frac{3}{2a^2}\frac{1}{r^2} - \frac{3C}{4C_2^2 r^3} \tag{38}$$

If one choose the constant of integration $C = 0$ to get the geometric string

$$\rho = \wedge \tag{39}$$

From equations (37) and (38), to obtain

$$\rho - \wedge = \frac{3C}{32\pi C_2^2 r^3} \geq 0, \qquad (C \geq 0) \tag{40}$$

From equations (17) and (36) the metric coefficients are obtained as:

$$e^{\lambda} = 4\left[-\frac{6}{a^2} + \frac{C}{r}\right]^{-1} \text{ and } e^{\nu} = r^2$$

Thus the metric (5) can be written as

$$ds^2 = r^2 dt^2 - 4\left(-\frac{6}{a^2}+\frac{C}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{41}$$

Comparative observations of theories BR and BGR for a spherically symmetric line element:

Variables	Observations due to Rosen Theories	
	BR	BGR
Metric coefficients e^λ, e^ν	$e^\lambda = \dfrac{b-1}{(b+1)+\sqrt{2}\sqrt{b^2+1}}$ = constant , $e^\nu = r^2$	$e^\lambda = 4[-\dfrac{6}{a^2}+\dfrac{C}{r}]^{-1} = f(r)$, $e^\nu = r^2$
Conformal factor, ψ	$\psi^2 = \dfrac{4[(b+1)+\sqrt{2}\sqrt{b^2+1}]C_2^2}{b-1}$ = constant	$\psi^2 = -\dfrac{6C_2^2}{a^2}+\dfrac{C}{r} = f(r)$
String density, ρ	$8\pi\rho = \dfrac{2b}{(b-1)^{3/2}r^3}[(b+1)+\sqrt{2}\sqrt{b^2+1}]^{1/2}$ $\rho \geq 0$, $\rho \to 0$ as $r \to \infty$, $\rho \to \infty$ as $r \to 0, b \neq 1$	$8\pi\rho = \dfrac{1}{r^2}+\dfrac{3}{2a^2}\dfrac{1}{r^2}$ $\rho \geq 0$, $\rho \to 0$ as $r \to \infty$, $\rho \to \infty$ as $r \to 0$
Tension density, λ	$8\pi\wedge = \dfrac{2}{(b-1)^{3/2}r^3}[(b+1)+\sqrt{2}\sqrt{b^2+1}]^{1/2}$ $\wedge \geq 0$, $\wedge \to 0$, as $r \to \infty$ $\wedge \to \infty$, as $r \to 0, b \neq 1$	$8\pi\wedge = \dfrac{1}{r^2}+\dfrac{3}{2a^2}\dfrac{1}{r^2}-\dfrac{3C}{4C_2^2 r^3}$ $\wedge \geq 0$, $\wedge \to 0$, as $r \to \infty$ $\wedge \to \infty$, as $r \to 0$
Resulting space-time	$ds^2 = r^2dt^2 - Adr^2 - r^2(d\theta^2 + r^2\sin^2\theta d\phi^2)$, where A is constant.	$ds^2 = r^2dt^2 - 4\left(-\dfrac{6}{a^2}+\dfrac{C}{r}\right)^{-1}dr^2$ $- r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

5. DISCUSSIONS

In this paper the solutions of spherically symmetric string cosmological model are obtained in BR and BGR. In both the theories under the assumption that of space-time admits one parameter group of conformal motion, it is observed that the model satisfies energy conditions $\rho \geq 0, \wedge \geq 0$, $\rho - \wedge \geq 0$ and $\rho, \wedge$ diverges as $r \to 0$ and vanishes at $r \to \infty$. In BR, the model (27) is singular at $b = 1$ and when $b \neq 1$ energy density (ρ), string tension $(\wedge)$ is positive and both diverge at $r \to 0$ and vanish at $r \to \infty$. Hence the geometric strings are absent in BR. Thus, for a relativistic model one should not take $b = 1$. Also the arbitrary function ψ and the metric potential e^λ appear to be constant. Whereas in BGR both $\rho, \wedge$ diverges at $r \to 0$ and vanishes at $r \to \infty$ and for $C = 0$ we get the geometric string $\rho = \wedge$. In this case ψ and metric potential e^λ are functions of radial coordinate r and for a physical system which is small compare to the size of universe ,the term $\dfrac{3}{2a^2}e^{-\nu}$ in the field equations (30) – (32) is negligible and hence the present result gives agreement with Einstein general relativity.

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