International Journal of Mathematical Archive-2(10), 2011, Page: 1776-1793
IJMA Available online through www.ijma.info ISSN 2229 – 5046

WAVELETS AND THE EVALUATION OF FILTER COEFFICIENTS

[1]Sabina and [2]Vinod Mishra*

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148 106 Punjab, India

[1]sabinajindal8@gmail.com, [2]vinodmishra.2011@rediffmail.com

(Received on: 09-09-11; Accepted on: 05-10-11)

ABSTRACT

Recent years have received much attention to wavelets because of its comprehensive mathematical power and good application potential in many interesting physical phenomena. We introduce definition and brief historical development of wavelets, the fundamentals of Hilbert-space, Fourier transform and general theorems needed. In this paper we also briefly review some of the available methods and propose a method [Section9] based on Fourier transform, multiple shift orthogonality and normalization to obtain filter coefficients to certain restricted cases. We also propose convergence analysis.

Keywords: *Riesz Basis, Frame, Wavelet Transform, Multiresolution Analysis, Moments, Scalar and Filter Coefficients.*

1. WAVELET, HISTORY AND APPLICATIONS:

A wavelet is a wave pattern of small size, that is, its graph oscillates only over the short distance or damps very fast; it means value over the whole domain equates to zero. A wavelet is localizable both in time (position) and frequency (scale).

Wavelet: An oscillatory function $\psi(x) \in L^2(R)$ with zero mean is a wavelet if it has the desirable properties:

1. ***Smoothness*:** $\psi(x)$ is n times differentiable and that their derivatives are continuous.

2. ***Localization*:** $\psi(x)$ is well localized both in time and frequency domains, i.e. $\psi(x)$ and its derivatives must decay very rapidly. For frequency localization $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \to \infty$ and that $\hat{\psi}(\omega)$ becomes flat in the neighborhood of $\omega = 0$. The flatness is associated with number of vanishing moments of $\psi(x)$, i.e.

$$\int_{-\infty}^{\infty} x^k \psi(x) dk = 0 \quad \text{or equivalently} \quad \frac{d^k \hat{\psi}(\omega)}{d\omega^k} = 0 \ \text{for } k = 0, 1, \ldots\ldots\ldots, n.$$

in the sense that larger the number of vanishing moments more is the flatness when ω is small.

3. ***The admissibility condition***

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

suggests that $\left|\hat{\psi}(\omega)\right|^2$ decays at least as $|\omega|^{-1}$ or $|x|^{\epsilon-1}$ for $\in > 0$.

Let $\psi_{a,b}(x),\ a \in R, b \in R$ be a family of functions generated from mother wavelet $\psi(x)$ by scaling (a) and translation (b) and defined by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right), \quad \|\psi_{a,b}(x)\| = \|\psi\|_2 .$$

Corresponding author: [2]*Vinod Mishra*, ****E-mail:*** *vinodmishra.2011@rediffmail.com*

that a is a measure of degree of compression and b signifies that $\psi_{a,b}$ is centred (localized) around b.{ $\psi_{a,b}(x)$ } is an orthonormal basis of $L^2(R)$..

Dyadic Wavelet: Let $a = 2^{-j}$ and $b = k2^{-j}$.The function $\psi_{j,k}$ stands for the dyadic wavelet shrunk by a factor of 2^j if j is positive (magnified by a factor 2^{-j} if j is negative) and shifted by $k2^{-j}$ units.

Examples of Wavelets [10, pp. 288-289]

1. Gaussian wavelet: $\psi(x) = cxe^{-\pi x^2}$.
2. Mexican Hat or Maar's Wavelet: $\psi(x) = \dfrac{d}{dx}\left[\dfrac{cxe^{-\pi x^2}}{2\pi}\right] = c\left(\dfrac{1}{2\pi} - x^2\right)e^{-\pi x^2}$
3. Haar Wavelet (1910): $$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & otherwise \end{cases}$$
4. Poisson Wavelet: $$\psi(x) = -\left(1 + \frac{d}{dx}\right)\frac{1}{\pi}.\frac{1}{1+x^2}$$
5. Morlet wavelet: $\psi(x) = \exp\left(i\omega x - \dfrac{x^2}{2}\right)$,

The word, wavelets or ondelets was first introduced by J. Morlet, a French Geophysicist working in an oil company, Elf. Acquitaine, at the beginning of 1980's. The wavelet, which started attracting the scientific community of early eighties, is a synthesis of ideas which originated from various specialties including Mathematics (Harmonic analysis: Calderon-Zygmund Operator, Little Wood-Paley Theory(1937), Franklin basis, and atomic decomposition of function), Physics (coherent states formalism in quantum mechanics and renormalization group) and Engineering (quadratic mirror filters side bent coding in signal processing and pyramidal algorithm in image processing).

In fact the first orthonormal wavelet basis was discovered by Alfred Haar (1909) and was refined by P. Franklin in 1927. J.O. Stromberge (1981) was of course the first to be credited for constructing orthonormal basis of $L^2(R) : \{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), j,k \in Z\}$, where *j, k* represent scale and translation parameters respectively. For positive *j* the graphical display of $\psi_{j,k}$ is wider and flatter, whereas for negative *j*, the same is narrower and sharper.

In 1982, Morlet introduced the idea of transform and in 1984 Grossman and Morlet succeeded in establishing the inversion formula. In 1985, Y. Meyer, a pure mathematician, used Littlewood Paley methods of 1930's and Calderon's method of 1960's to formalize the notion of a wavelet. In 1986 Mallat realized that coarse features in an image are large objects, whereas fine scale feature should be studied much more locally. Subsequently, Daubechies, Grossman, Mallat, Meyer and Strang have developed the theory of wavelets to a considerable extent [11, p. 11]. The first application of a wavelet is due to Morlet (1983) and is $\psi(x) = \pi^{-1/4}\left\{e^{ikx} - e^{-2k^2}e^{-x^2/2}\right\}$ at $k = \pi(2\log 2)^{\frac{1}{2}}$.

Wavelet analysis is probably the most recent solution to overcome the shortcoming of Fourier transform. In the case of wavelet, we normally do not speak about time-frequency representation but about time–scale representation, scale being in a way the opposite of frequency, because the term frequency is reserved for the Fourier transform, since from Literature it is not always clear what is meant by small and large scales. It is defined as follows: The large scale is the big picture, while the small scale shows the details. Thus going from large to small scale is in this context equal to zooming in.

Applications of Wavelets

Electronics (signal compression and denoising; image and speech analysis), Computer (computer graphics, neural network), Mathematics (approximation theory, matrix theory, numerical analysis of ODEs and PDEs, operator theory, inverse problems), Mathematical Statistics (sampling theory, regression, density and function estimation, factor analysis modeling and forecasting in time series analysis, spatial statistics, pattern recognition), Meteorology (structure of the clouds), Universe (structure of galaxies and universe), Biomedical (bio-acoustics, electro-cardiography (ECG), electroencephalography (EEG)), Biomedical Imaging (biomedical image processing, i.e. noise reduction, image enhancement and detection of micro calcification in manimograms, computer assisted magnetic resonance imaging (MRI), functional image analysis), Fluid (turbulence) and many more.

2. PRELIMINARIES:

Definition: Let H be a Hillbert space with inner product $\langle , \rangle$. A set of vectors $\{x_n\}$ is an orthogonal system if $\langle x_n , x_m \rangle = \delta_{mn}$.

Lemma: 1 [10, p.355]. A set of vectors $\{x_n\}$ is orthonormal iff for every finite set of complex numbers $\{a_n\}$, we have $\|\sum_n a_n\ x_n\|^2 = \sum |a_n|^2$.

Definition: Let H be a Hilbert space. A set of vectors $\{x_n\}$ is a Riesz system, if there exist constants $0 \le c \le C < \infty$ such that for any finite set of complex numbers $\{a_n\}$

$$c\sum |a_n|^2 \le \left\| \sum_n a_n x_n \right\|^2 \le C\sum_n |a_n|^2 .$$

Definition: Let $L^2(R)$ is a vector space of square integrable function, i.e.

$L^2(R) : \left\{ f : R \to C : \int_R |f(x)|^2\, dx < \infty \right\}$ For $f, g \in L^2(R),$ define inner product $\langle f , g \rangle = \int_R f(x)\, \overline{g(x)}\, dx$.

In particular $\|f\| = \|f\|_2 = \left(\int_R |f(x)|^2\, dx \right)^{1/2}$, and we say that f is square integrable .

Definition: Let $L^1(R) : \left\{ f : R \to C : \int_R |f(x)|\ \ dx < \infty \right\}$. For $f, g \in L^1(R),$

let $\|f\|_1 = \int_R |f(x)| dx$. We say that f is integrable.

Lemma: 2 If $f \in L^1(R),$ then $\left| \int_R f(x) dx \right| \le \int_R |f(x)| dx = \|f\|_1$.

Definition: [4, p. 367]: Let $f : R \to C$ be a function. Then support of f , denoted by $supp\ f$, is the closure of the set $\{x \in R : f(x) \neq 0\}$. We say f has compact support if $supp\ f$ is a compact set.

In other words, f has compact support if there exists $r < \infty$ such that supp $f \subseteq [-r, r]$, that is, such that $f(x) = 0$ for all x satisfying $|x| > r$.

Definition: A sequence $\{\varphi_n\}$ of orthonormal basis in a Hilbert space H is called a frame if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \le B\|f\|^2 \quad \forall \ \ f \epsilon H$$

The constants A and B are called frame bounds. If $A = B$, then frame is called tight.

Cauchy-Schwartz Inequality: For $f, g \in L^2(R)$,

$$\left| \int_R f(x)\, \overline{g(x)}\, dx \right| \le \left(\int_R |f(x)|^2 dx \right)^{1/2} \left(\int_R |g(x)|^2 dx \right)^{1/2}, \text{ i.e.}$$

$$|\langle f , g \rangle| \le \|f\| \|g\| .$$

Triangle Inequality: For $f, g \in L^2(R)$,

$$\left(\int_R |f(x)+g(x)|^2 dx\right)^{1/2} \le \left(\int_R |f(x)|^2 dx\right)^{1/2} + \left(\int_R |g(x)|^2 dx\right)^{1/2}, \text{ i.e. } \|f+g\| \le \|f\| + \|g\|.$$

Lemma: 3 Suppose $f, g \in L^2(R)$, satisfies a Lipschitz condition of order $\alpha \in (0,1]$, which means that there exists a constant $c < \infty$ such that for all $x, y \in R$

$$|f(x) - f(y)| \le c|x-y|^{\alpha}.$$

For details refer to Frazier [4, pp. 349-351] and Pinsky [10, pp. 304-305].

Fourier Transform and Properties

Fourier Transform. Fourier transform [FT] is a well known mathematical tool to transform time domain signal to frequency domain for efficient extraction of information and vice- versa.

Definition (Fourier transform): For $f \in L^1(R)$ or $L^2(R)$ and $\omega \in R$, define

$$F(f) = \hat{f}(\omega) = \int_R f(x) e^{-ix\omega} dx,$$

where $\hat{f}$ is called the Fourier transform of f and the mapping $\wedge$ is called the Fourier transform.

Inverse Fourier Transform: For g$\in L^1(R)$ and $x \in R$, we define g^v, the inverse Fourier transform of *g* by

$$g^v(x) = \int_R \hat{g}(\omega) e^{ix\omega} d\omega.$$

The mapping ^{v} is the inverse Fourier Transform.

Characteristics of Fourier Transform: [16, pp. 30-31]

Boundedness: $\hat{f} \in L^{\infty}(R), \|\hat{f}\|_{\infty} \le \|f\|_1$

Uniform Continuity: $\hat{f}(\omega)$ is uniformly continuous on $-\infty < \omega < \infty$.

Decay: For $f \in L^1(R)$, $\hat{f}(\omega) \to 0$ when $|\omega| \to \infty$ [Reimann Lebesgue Lemma].

Linearity: $F[\alpha f(x) + \beta g(x)] = \alpha F[f(x)] + \beta F[g(x)]$.

Derivative: $F\left[f^{(n)}(x)\right] = (i\omega)^n \hat{f}(\omega)$.

Plancherel's Identity: $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$. If $g = f$, then the above identity reduces to

$$\|f\|^2 = \|\hat{f}\|^2.$$

The function $|\hat{f}(\omega)|^2$ is called the energy spectrum. Analogously, the area below the curve $|\hat{f}(\omega)|^2$ is equal to $\int |f(x)|^2 dx$ - the energy content of the signal.

Shifting: $Ff(x - x_0) = e^{-i\omega x_0} \hat{f}(\omega)$.

Scaling: $Ff(ax) = \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$.

Symmetry: $F[F[f(x)]] = f(-x)$.

Convolution: The convolution of f and g is defined as

$$f * g(x) = \int f(x-t) g(t) dt. \quad F\left[f * g(x) \right] = \hat{f}(\omega).\hat{g}(\omega).$$

Modulation Theorem: $f(x)g(x) = F(\omega) * G(\omega)$ [by symmetry property].

Moment Theorem: $\int_R x^n f(x) dx = (i)^n \dfrac{d^n \hat{f}(\omega)}{d\omega^n}$

Discrete Signal: $f_j = F\left({}^{j}/_{M}\right), \quad j = 0,1, \ldots, M-1.$

Fourier coefficients $\widetilde{f_n} = \frac{1}{M} \sum f_j e^{-2in\pi j}/_{M}, \quad n = \frac{-M}{2} + 1, \ldots, 0, \ldots, -\frac{M}{2}.$

Uncertainty Principle:

Definition: [2, pp. 123-124]. Let $f \in L^2(R)$. The dispersion of f about the point $a \in$ R is the quantity

$$\Delta_a f = \frac{\int_{-\infty}^{\infty} (t-a)^2 |f(t)|^2 dt}{\int_{-\infty}^{\infty} \int |f(t)|^2 dt}.$$

The dispersion about a point a is the measure of deviation or spread of its graph from $t = a$. This dispersion will be small if the graph of f is concentrated near $t = a$ and is spread out away from $t = a$.
In frequency domain,

$$\Delta_\alpha \hat{f} = \frac{\int_{-\infty}^{\infty} (\omega - \alpha)^2 \left|\hat{f}(\omega)\right|^2 d\omega}{\int_{-\infty}^{\infty} \int \left|\hat{f}(\omega)\right|^2 d\omega}.$$

Theorem: 1 (**Uncertainty Principle**) [2, pp.125-127]. Suppose f is a function in $L^2(R)$ which vanish at $+\infty$ and $-\infty$. Then

$$\Delta_a f .\ \Delta_\alpha \hat{f} \geq \frac{1}{4}$$

for all points $a, \alpha \in$ R.

The statement implies that $\Delta_a f$ and $\Delta_\alpha \hat{f}$ cannot simultaneously be small. In other words, when the time-frequency cell is narrow in time it is wider in frequency and vice-versa. In case of Gaussian function $f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}$ equality is achieved.

3. MATHEMATICAL THEORY OF WAVELET:

Continuous Wavelet Transform [CWT]

The wavelet transform or wavelet analysis is probably the most recent solution to overcome the shortcomings of the Fourier transform.

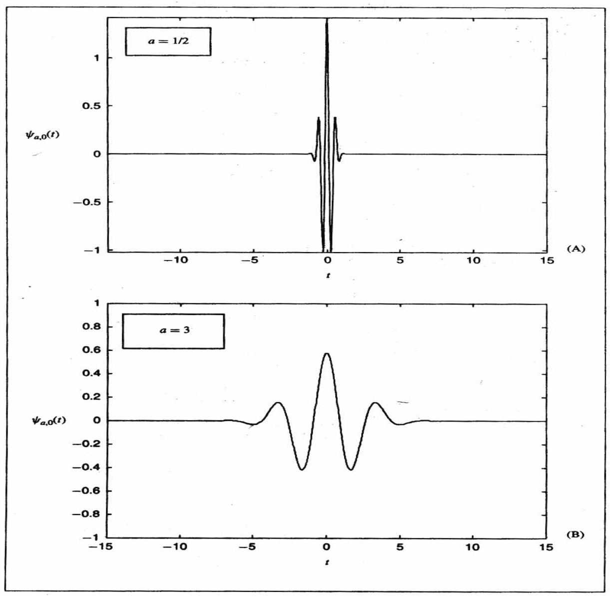
Wavelet constitutes a family of functions derived from one single function and indexed by two labels, one for position and other for frequency. That is, the wavelet transform of a one dimensional function is two dimensional; the wavelet transform of a two dimensional function is four dimensional. One imposes some additional condition on the wavelet function in order to make the wavelet transform decrease quickly with decreasing scale. These are the regularity conditions and state that the wavelet function should have some smoothness and concentration in both time and frequency domains.

The CWT of a function $f(x) \in L^2(R)$ at a scale a and position b with respect to $\psi(x) \in L^2(R)$ is given by

$$W_\psi[f](a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(x) \overline{\psi}_{a,b}(x) dx = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x) \overline{\psi}\left(\frac{x-b}{a}\right) dx.$$

In $\psi_{a,b}$ the parameter b gives the position of the wavelet, while the dilation parameter a governs frequency “for smaller values of $a(> 0)$, the wavelet is contracted in the time domain and the wavelet transform gives information about the finer details of the signal. For large values of a, the wavelet expands and the wavelet transform gives a global view of signal” (cf. [12, p.17]). Fig.1 [11, p.19] shows two dialation of the Morlet wavelet. If $a > 1$ there is a stretching of $\psi(x)$ along the time axis whereas if $0 < a < 1$ there is a contraction of $\psi(x)$.

Fig.1. A Morlet wavelet dilated by factor of $a = 1/2$ and $a = 3$.



The CWT as an Operator: The CWT takes a member of the set of square integrable function of one real variable in $L^2(R)$ and transforms it to a member of the set of functions of two real variables. Thus, it can been seen as a mapping operator from $L^2(R)$ to the latter set.

Define $W_\psi[f(x)] \equiv W(a,b)$. Then $W_\psi[f]$ is to be read CWT with respect to $\psi(x)$ of f. The notation for the operator use ψ as a subscript to remind us of the fact that the transform depends not only on the function $f(x)$ but also on the mother wavelet.

We now enumerate various properties of CWT using the operator notation:

Linearity:

$$W_\psi[\alpha f(x) + \beta g(x)] = \alpha W_\psi[f(x)] + \beta W_\psi[g(x)]$$

for scalar α, β and function $f(x), g(x) \in L^2(R)$.

Translation:

$$W_\psi[f(x-\tau)] = W[a, b-\tau]$$

Scaling:

$$W_\psi\left[\frac{1}{\sqrt{\alpha}} f(\frac{1}{\alpha})\right] = W\left[\frac{a}{\alpha}, \frac{b}{\alpha}\right] \quad \text{for } \alpha > 0$$

Wavelet Shifting: Let $\hat{\psi}(x) = \psi(x-\tau)$. Then

$$W_{\hat{\psi}}[f(x)] = W(a, b + a\tau) .$$

Observe that the CWT obtained by shifting the wavelet is different from the one obtained by shifting the signal.

Energy Conservation:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{c} \int_{-\infty}^{\infty} \int_0^{\infty} \left| \langle f, \psi_{a,b} \rangle \right|^2 \frac{dadb}{a^2}.$$

Localization: Let $f(x) = f(x - x_0)$ be the Dirac pulse at the point x_0, then $W_\psi[f](a,b) = \frac{1}{\sqrt{a}} \psi\left(\frac{x_0 - b}{a}\right)$.

Wavelet Series: A function $\psi \in L^2(R)$ is said to be orthonormal wavelet if the family $\{\psi_{j,k}\}_{j,k\in Z}$, where

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k), \|\psi_{j,k}\| = \|\psi\|_2$$

satisfies the conditions

$$<\psi_{j,k}, \psi_{l,m}> = O_{j,i}.....\delta_{k,m}; j,k,l,m \in Z$$

Wavelet series expansion of $f \in L^2(R)$ is defined by

$$f(x) = \sum_{j,k=-\infty}^{\infty} \beta_{j,k}\psi_{j,k}(x),$$

where the wavelet coefficients

$$\beta_{j,k} = <f, \psi_{j,k}> = \int_{-\infty}^{\infty} f(x)\overline{\psi}_{j,k}(x)dx.$$

4. MULTIRESOLUTION ANALYSIS:

The purpose of multiresolution analysis is to write a function $f \in L^2(R)$ as a collection (sequence) of its successive approximations, each of which is a smoothed version of the previous one.

Definition: A multiresolution analysis [MRA] of $L^2(R)$ is a sequence $\{V_n\}_{n\in Z}$ of the closed subspaces of functions $f \in L^2(R)$ satisfying the following properties:

(i) (Monotonicity) is $V_n \subset V_{n+1}\ \forall\ n \in Z \qquad (V_n \subset V_{n-1}\ \forall\ n \in Z)$

(ii) $\bigcap_{n=-\infty}^{\infty} V_n = \{0\}$, $\bigcup_{n=-\infty}^{\infty} V_n$ is dense in $L^2(R)$ i.e. $\overline{\bigcup_{n=-\infty}^{\infty} V_n} = L^2(R)$

(iii) (Dailation) $f(x) \in V_0$ iff $f(2^n x) \in V_n\ \forall\ n \in Z$, i.e. all the spaces are scaled versions of the central space V_0.

(iv) (Existence of scaling) There exists a scaling function $\varphi(x) \in V_0$ when integer translates space V_0, i.e. for each $f(x) \in L^2(R)$

$$V_0 = \left\{ f(x) = \varphi_{0,k}(x) = \sum_{k=-\infty}^{\infty} c_k \varphi(x-k) \right. \text{ and } \{\varphi(x-k), k \in Z\} \text{ is an orthogonal basis for } V_0.$$

Suppose $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k), j,k \in Z$. Since $\varphi_{0,k}(x) \in V_0 \forall k \in Z$ due to (iv). Further, if $j \in Z$ condition (iii) implies that the family $\{\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k), j,k \in Z\}_{k=-\infty}^{\infty}$ is an orthogonal basis for V_j. By definition part (iv) means that for any $f \in L^2(R)$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ such that each $f_n \in \cup_{j\in Z} V_j$ and $\{f_n\}_{n=1}^{\infty}$ converges to f in $L^2(R)$, that is $\|f_n - f\| \to 0$ as $n \to \infty$.

The functions, consisting of translations and dilations of wavelet function $\psi(2^j x - k)$, form a complete and orthonormal basis of $L^2(R)$.

The relation between two functions is expressed as

$$V_{j+1} = V_j \oplus W_j \qquad W_j \perp W_k \text{ for } j \neq k$$

where subspaces $V_j = 2^{j/2}\varphi(2^j x - k); \quad k = ..., -1, 0, 1, ...$

$$W_j = 2^{j/2}\psi(2^j x - k); \quad k = ..., -1, 0, 1, ...$$

Thus for $k > 0$, $V_{j+1} = V_0 \oplus (\overset{j}{\underset{j=0}{\oplus}} W_j)$, i.e. V_{j+1} can be expressed as a linear combination of functions in orthogonal spaces V_0 and W_j, $j = 0, 1, \ldots, j$ and analysed separately at different scales.

Since $\bigcup_{-\infty}^{\infty} V_j = L^2(R)$. For $j \to \infty$, $V_0 + (\overset{\infty}{\underset{j=0}{\oplus}} W_j) = L^2(R)$.

Similarly $V_0 = V_{-1} \oplus W_{-1} = V_{-j} \overset{1}{\underset{j}{\oplus}} W_{-j}$. But $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ implies $V_{-j} \to \{0\}$ as $j \to \infty$. We get $(\overset{\infty}{\underset{j=-\infty}{\oplus}} W_j) = L^2(R)$. Therefore, W_j is a decomposition of $L^2(R)$ into mutually orthogonal subspaces. Thus, $\psi(x) \in W_0$ such that $\psi_{j,k}$ is a complete orthonormal basis of W_j, i.e. $\{\psi_{j,k}(x)\}$ is an orthonormal basis of $L^2(R)$.

Note that for certain values of *j* and *N*,

$$supp\ \varphi_{j,k} = \left[\tfrac{k}{2^j}, \tfrac{N+k-1}{2^j}\right]$$

$$supp\ \psi_{j,k} = \left[\tfrac{k+1-N/2}{2^j}, \tfrac{k+N/2}{2^j}\right]$$

A scaling function can be used to expand a general function: Projection $P_j: L^2(R) \xrightarrow{onto} V_j$,

$$f(x) = P_j f = \sum_{k \in Z} 2^{j/2} c_k \varphi(2^j x - k).$$

It satisfies the following convergence property

$$\left\| f - \sum_k c_k \varphi(2^j x - k) \right\| \le C 2^{-jp} \left\| f^{(p)} \right\| \text{ for large } j,$$

where $c_k = \int f(x)\varphi(2^j x - k)dx$; C, p are constants.

Moment = 0 implies scaling bases can be represented as polynomials of degree $\left(\frac{N}{2} - 1\right)$.

Theorem: 2 [2, p. 205]. Let V_j, $j \in Z$ be a given MRA with scaling function φ and $P_j f$, projection of $f \in L^2(R)$ onto V_j such that

$$P_j f = \sum_k 2^{j/2} c_k \varphi(2^j x - k), \text{ where } c_k = \int f(x)\, \overline{\varphi(2^j x - k)} dx$$

then for *j* sufficiently large

$$c_k \cong m f(k 2^{-j}) \text{ with } m = \int \overline{\varphi(x)}\ dx.$$

Remarks: Let for $r > 0$, has compact support, that is, $\psi(x) = 0\ \forall x\ s.t\ |x| > r$. This means $\psi(2^j x)$ has compact support $\left[\frac{-r}{2^j}, \frac{r}{2^j}\right]$ since $\psi(2^j x)$=0 whenever $|2^j x| > 0$, i.e. when $|x| > \frac{r}{2^j}$.

The graph of $\psi(2^j x - k) = \psi(2^j(x - 2^{-j}k))$ is obtained by translating the graph of $\psi(2^j x)$ by $2^{-j}x$ along x-axis (to the right if $k > 0$) and to the left if $k < 0$. Hence if support of ψ is in $[r, r]$, then $\psi(2^j x - k)$ has support inside $[2^{-j}k - 2^{-j}r, 2^{-j}k + 2^{-j}r]$. Finally, graph of $\psi_{j,k}$ is obtained from the graph of $\psi(2^j x - k)$ by after multiplication by 2^j, which stretches the graph in *y* direction by this factor. For r very small $\psi_{j,k}$ is centered near the point $2^{-j}k$ and has a scale of about 2^{-j}.

5. GENERAL THEOREMS:

If $\{\varphi_n\}$ is an orthonormal basis then it is a tight frame, since

$$\sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 = \|f\|^2.$$

Sufficient condition for $\{\psi_{mn}\}$ to constitute a frame in $L^2(R)$

Theorem: 3 Let ψ and a_0 such that

i) $inf_{1\le|\omega|\le a_0} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\,(a_0^m\omega)\right|^2 > 0$

ii) $sup_{1\le|\omega|\le a_0} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\,(a_0^m\omega)\right|^2 > 0$

iii) $\left|\hat{\psi}\,(a_0^m\omega + x)\right| \le c(1+|x|)^{-(1+\epsilon)}$ for some $\epsilon > 0$ i.e. decays at least as fast as $(1+|x|)^{-(1+\epsilon)}$ for some $\epsilon > 0$, then there exists $\tilde{b} > 0$ such that

$\psi_{mn}(x) = a_0^{m/2}\,\psi\,(a_0^m x - nb_0)$ form a frame for any $b_0 < \tilde{b}_0$, i.e. for any $b_0 \in (0, \tilde{b})$.

Theorem: 4 For the scaling function it holds $\int_R \varphi(x)dx = 1$ or equivalently $\hat{\varphi}(0) = 1$, where $\hat{\varphi}(\omega)$ is the Fourier transform of φ.

Theorem: 5 For a given multiresolution analysis, there exists an orthonormal wavelet basis for $L^2(R)$. Let W_j be the orthogonal complement of V_j in V_{j+1} Then $L^2(R) = V_0 + (\bigoplus_{j=0}^{\infty} W_j)$. In particular, each $f \in L^2(R)$ can be uniquely expressed as a sum $\sum_k w_k$ with $w_k \in W_k$, where w_k's are mutually orthogonal. Equivalently, the set of all wavelets $\{\psi_{j,k}\}_{j,k\in Z}$ is an orthogonal basis for $L^2(R)$.

Theorem: 6 [5]. For any $n \in N$ there exists Daubechies MRA with function φ and ψ that have compact support of length $2N-1$. Moreover, Daubechies wavelet has N vanishing moments, i.e.

$$\langle x^k, \psi \rangle_{L^2(R)} = 0, \quad k = 0, N-1.$$

Theorem: 7 Let ψ be an admissible mother wavelet satisfying

$$\int (1+|x|)\left|\psi\,(x)\right| dx < \infty.$$

i) If f is bounded function that satisfies Lipschitz condition of order α, $0 < \alpha \le 1$, then $\left|W_\psi f(a,b)\right| \le c|a|^{\alpha+1/2}$ for some constant $c > 0$.

ii) If f is bounded and continuous at x_0 with $0 < \alpha \le 1$, i.e. $|f(x+\lambda) - f(x_0)| \le \alpha|\lambda|^\alpha$ for some $\alpha > 0$.

Then $\left|W_\psi f(a,b)\right| \le c|a|^{1/2}\left(|a|^\lambda + |b|^\lambda\right)$, for some constant $c > 0$.

Lemma: 4 Suppose $\varphi \in L^1(R) \cap L^2(R)$ satisfies $\int_{-\infty}^{\infty} \varphi(x)dx = 1$ and $\int |x|^\alpha\,|\varphi(x)|dx = c_2 < \infty$.

Then $\left|2^{j/2}\langle f, \varphi_{jk}\rangle - f(2^{-j}k)\right| \le c_1c_2 2^{-m\alpha}$.

Or equivalently, $\left|\langle f, \varphi_{jk}\rangle - 2^{-j/2} f(2^{-j}k)\right| \le c_1c_2 2^{-j(\alpha+1/2)}$.

Theorem: 8 [5] Let φ be continuous function with compact support that satisfies:

i) $\{\varphi(x-k)\}_{k\in z}$ is an orthonormal system

ii) $\int_R \varphi(x)dx = 1$

iii) Only finite number of the coefficients a_k in $\varphi(x) = \sum_k a_k \varphi(2x-k)$ are non-zero. Then φ is a scaling function, i.e. φ can be used in construction of MRA.

Theorem: 9 [5] Suppose that the polynomial $P(z) = \frac{1}{2}\sum a_k z^k$ satisfies:
i) $P(1) = 1$
ii) $|P(z)| \neq 0$ for any z with $|z| = 1$
iii) $|P(z)|^2 + |P(-z)|^2 = 1$ for any z with $|z| = 1$
Then the iteration $\varphi_0 = \chi_{[0,1]}$, $\varphi_n(x) = \sum_k a_k \varphi_{n-1}(2x-k)$ converges pointwise and in $L^2(R)$ to a scaling function φ.

Lemma: 5 Let $\varphi, \psi \in L^2(R)$, then
(i) The set $\{\varphi(x-n)\}_{n=-\infty}^{\infty}$ is orthonormal iff $\sum|\hat{\varphi}(\omega + 2\pi k)|^2 = 1$
(ii) The set $\{\varphi(x-n)\}_{n=-\infty}^{\infty}$ and $\left\{\psi(x-m)\right\}_{m=-\infty}^{\infty}$ are biorthogonal, i.e.

$$\langle \varphi_n, \psi_m \rangle = 0 \quad \forall m, n \text{ iff } \sum \hat{\varphi}(\omega + 2\pi k)\, \hat{\psi}(\omega + 2\pi k) = 0,$$

where $\varphi_n(x) = \varphi(x-n)$ and $\psi_m(x) = \psi(x-m)$.

Lemma: 6 The scaling function φ satisfies the following conditions:
i) $\sum|\hat{\varphi}(\omega + 2\pi k)|^2 = 1$
ii) $\hat{\varphi}(\omega) = H\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$

where $H(\omega)$ is a 2π periodic function that belongs to $L^2[0,2\pi]$ and satisfies $|H(\omega)|^2 + |H(\omega + 2\pi)|^2 = 1$.

Lemma: 7 The set $\left\{\psi(x-m)\right\}_{m=-\infty}^{\infty}$ are orthonormal iff $\sum\left|\hat{\psi}(\omega + 2\pi k)\right|^2 = 1$ and $\hat{\psi}(\omega) = M\left(\frac{\omega}{2}\right)\hat{\psi}\left(\frac{\omega}{2}\right)$, where $M(\omega)$ is a 2π periodic function that belongs to $L^2[0,2\pi]$ and satisfies $|M(\omega)|^2 + |M(\omega + \pi)|^2 = 1$.

Lemma: 8 The Fourier transform of any function $f \in W_0$ can be written in the form $\hat{f}(\omega) = \lambda_f(\omega)\hat{\psi}(\omega)$, where $\lambda_f(\omega)$ is a periodic function with period 2π and $\hat{\psi}$ is a independent of f. Moreover, $\lambda_f \in L^2[0,2\pi]$ and $\|f\|_{L^2(R)} = \|\lambda_f\|_{L^2[0,2\pi]}$.

Frequency Domain Characterization of Filter Coefficients:

Fourier Transform of $\{h(k)\}$ is

$$H(\omega) = \frac{1}{\sqrt{2}}\sum_k h(k)e^{-ik\omega}.$$

Fourier Transform of $\{g(k)\}$ is

$$M(\omega) = \frac{1}{\sqrt{2}}\sum_k g(k)e^{-ik\omega}.$$

Notice that $H(\pi) = 0$ and $H(0) = 1$.

$$\hat{\varphi}(\omega) = H\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) = \hat{\varphi}(0)\prod_{j=1}^{\infty} H\left(\frac{\omega}{2^j}\right) \qquad [\varphi(t) = \sum_{k=-\infty}^{\infty} h_k\sqrt{2}\varphi(2t-k)]$$

$$\hat{\psi}(\omega) = M\left(\frac{\omega}{2}\right)\hat{\psi}\left(\frac{\omega}{2}\right) = M\left(\frac{\omega}{2}\right)\prod_{j=2}^{\infty} M\left(\frac{\omega}{2^j}\right) \qquad [\psi(t) = \sum_{k=-\infty}^{\infty} g_k\sqrt{2}\varphi(2t-k)]$$

Lemma: 9 Suppose $M: R \to C$ satisfies $M(0) = 1$, $|M(\omega)| \leq 1$ for all $\omega \in R$ and there exist $\alpha > 0$ and $c < \infty$ such that

$$|M(\omega)| - |M(0)| \leq c|\omega|^{\alpha} \quad \forall \omega \in R$$

For $n \in$N, let

$$M_n(\omega) = \prod_{j=1}^{n} M\left(\frac{\omega}{2^j}\right).$$

Then $M_n(\omega)$ converges as $n \to \infty$ uniformly on every bounded subset of R, hence pointwise at every point $\omega \in R$.

Theorem:10 [9] Let $p = \frac{N}{2}$ be the number of vanishing moments for a wavelet ψ_{jk} and let $f \in C^p(R)$.Then the wavelet coefficients decay as following:

$$|d_{jk}| \leq C_p 2^{-j\left(p+\frac{1}{2}\right)} \max_{\omega \in I_{j,k}} |f^p(\omega)|$$

where C_p is a constant independent of j, k and f and $I_{j,k} = Supp\left\{\psi_{jk}\right\} = \left[\frac{k}{2^j}, \frac{k+N-1}{2^j}\right]$.

Notice that

$$d_{jk} = \int_{I_{j,k}} f(x)\psi_{jk}(x)dx,$$

$$f(x) = \sum_{p=0}^{p-1} f^p\left({}^{k}/_{2^j}\right)\frac{\left(x-\frac{k}{2^j}\right)^p}{p!} + f^p(\omega)\frac{\left(x-\frac{k}{2^j}\right)^p}{p!}, \omega \in \left[\frac{k}{2^j}, x\right]$$

Theorem: 11 [9] If ψ has p vanishing moments, then

(a) $H(0) = 1$.

(b) $\frac{d^p}{d\omega^p}H(\omega)|_{\omega=\pi} = 0, \quad p = 0,1, \dots, p-1.$

Corollary: [9]

$$H(n\pi) = \begin{cases} 1 & n\ even \\ 0 & n\ odd \end{cases}$$

Lemma: 10 [9]

$$\varphi(2\pi n) = \delta_{0,n}, n \in Z.$$

6. GENERALIZED MOMENTS:

Probability Densities: The functions $\varphi^2(x)$ and $\psi^2(x)$ of orthonormal scaling function and wavelet are the probabilities densities.

Generalized Moments: The generalized moments of $\varphi(x)$ and $\psi(x)$ are defined as:

$$\mu_{k,t} = \int_R x^k \varphi(x)\varphi(x-t)dt$$

$$\xi_{k,t} = \int_R x^k \psi(x)\psi(x-t)dt$$

$\mu_{1,0}, \xi_{1,0}$ are the first moments (the means) of $\varphi(x)$ and $\psi(x)$ respectively.

$$\varphi(x) = \sum_n h_n\sqrt{2}\varphi(2x-n), \quad \psi(x) = \sum_n g_n\sqrt{2}\varphi(2x-n)$$

Theorem:12 [13] Let $T = 2N - 2$. The vector $\mu_k = \{\mu_{k,t}\}$, $t \leq T$, is a solution of the system

$$\left(I - \frac{1}{2^k}A\right)\mu_k = b_k,$$

where $A_{ij} = \sum h_n h_{n+i-2j}, \quad -T \leq i,j \leq T$ is the translation matrix (or Lawton matrix). The vector b_k has components

$$b_{k,t} = \frac{1}{2^k}\sum_l \sum_n h_l h_n \sum_{j=1}^{k} \binom{k}{j} n^j M_{k-j,1-n+2t}, \ -T \leq i,j \leq T.$$

Remark: $\mu_{1,t} = \mu_{1,-t}$.

Note: $\mu_{0,t} = \delta(t)$.

Theorem: 13 [15] $\mu_{1,0} = \frac{1}{2}\sum_i p_i\mu_{1,i} + \frac{1}{2}\sum_i ih_i^2$,

where $p_n = \sum_i h_i h_{n+i}$ satisfy $p_{2k} = \delta(k)$ and $p_n = p_{-n}$.

Theorem: 14 [15] The mean $\xi_{1,0} = \int x\psi^2(x)dx$ is at the center of support of $\psi(x)$, i.e. equal to $\frac{2N-1}{2}$.

7. SCALING FUNCTION AND WAVELET:

The scaling function $\varphi(x)$ is the solution of dilation equation of a particular type

$$\varphi(x) = \sum_{k=-\infty}^{\infty} a_k \varphi(2x-k),\ a_k = \sqrt{2}\,h_k \tag{1}$$

The constants a_k are called filter coefficients. The associated wavelet $\psi(x)$, is orthogonal to scaling function, defined by ψ

$$\psi(x) = \sum_{k=2-N}^{1} (-1)^k a_{1-k}\varphi(2x-k)\,. \tag{2}$$

Daubechies Wavelet:

Daubechies wavelets are compactly supported functions. This means that they have non zero values within a finite interval and have a zero value everywhere else. That's why it is useful for representing the solution of differential equation. In 1988, Ingrid Daubechies defined scaling function as

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k)$$

where N denotes the genus of the Daubechies wavelet. The functions generated with these coefficients will have supp $(\varphi) = [0, N-1]$ and $(N/2-1)$ vanishing wavelet moments.

Derivation of Filter Coefficients:

The N coefficients a_k are uniquely derived under the following conditions:

1. $a_k = 0$ for $k \notin \{0, 1, 2, \ldots.., N-1\}$
2. Area under the scaling function is normalized to unity, i.e. $\int \varphi dx = 1$ implies $\sum_{k=0}^{N-1} a_k = 2.$
3. The translates of scaling function φ are required to be orthonormal, i.e. $\int \varphi(x-k)\varphi(x-l)dx = \delta_{k,l}$

   yield $\sum_{k=0}^{N-1} a_k a_{k-2l} = 2\delta_{0l},\ 1-\frac{N}{2} \le l \le \frac{N}{2}-1.$
4. The first $p = \frac{N}{2}$ moments $\int x^l \psi(x)dx = 0$, i.e. $\sum_{k=0}^{N-1} (-1)^k k^l a_k = 0,\ l = 0, 1, 2, \ldots.., \frac{N}{2}-1$

The functions, consisting of translations and dilations of wavelet function $\psi(2^j x - k)$, form a complete and orthonormal basis of $L^2(R)$. The last condition is derived when a function $f(x) = \sum_{k=-\infty}^{\infty} \alpha_l x^l$ is exactly represented by the expansion f of the form $f(x) = \sum_{k=-\infty}^{\infty} a_k \varphi(2x-k)$ and that $< f(x), \psi(x) >= 0$.

Construction of Scaling Function:

There are no explicit formulae for the basic scaling function and wavelet. Cascade algorithm, successive approximation, Daubechies-Lagarias algorithm and subdivision scheme are the methods which involve direct evaluation of scaling function and wavelet at dyadic rational points.

Cascade Algorithm: The dilation equation is expressible in the matrix form as $M\varphi = \varphi$ using the fact that $\varphi(i) = 0$ for all $i < 0, i > N-1$. The solution $[M-I]\varphi = 0$ is the set of Eigen vectors corresponding to Eigen value 1, is not unique. Normalization condition is imposed, i.e. $\sum_{i=-\infty}^{\infty} \varphi(i) = 1 \quad \forall\, i = 0,1,2,\ldots\ldots, N-1$. Once φ is known for the integral values of x, $\varphi(i/2)$ can be found. The process can be repeated to get $\varphi\left(\frac{i}{2^j}\right), i, j \in Z$.

For details refer to Chen et al. [3], Maninder [7], Sabina[12], Soman [14] and Mishra-Sabina [17].

Fig.1. 2D graph of scaling (phi) and wavelet (psi) functions of Daub6

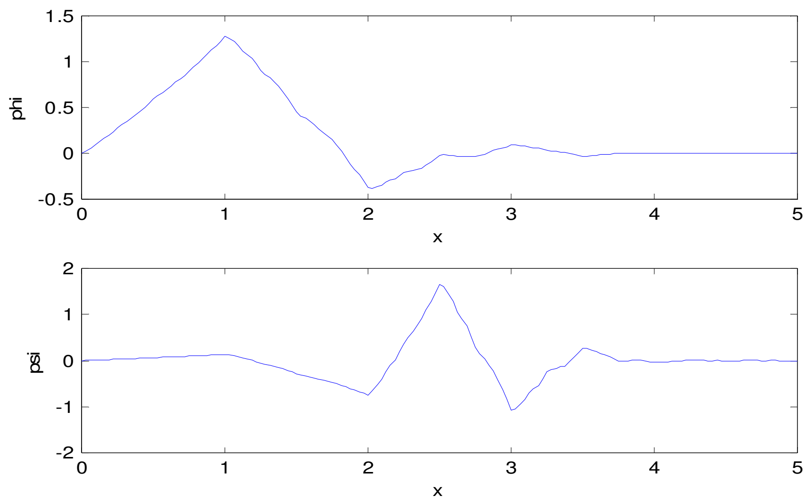
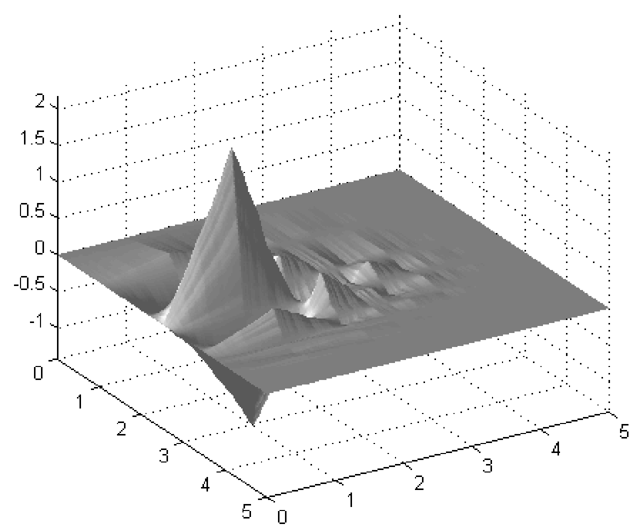


Fig.2. 3D graph of scaling function of Daub6



8. EVALUATION OF FILTER COEFFICIENTS:

To find $a_k s$ for Daub4

Taking Fourier transform to two scale relation

$$\varphi(x) = \sum_k a_k \varphi(2x - k),$$

We find

$$\hat{\varphi}(\omega) = P(e^{-i\omega/2})\hat{\varphi}(\omega/2),$$

where $P(z)$ is given by

$$P(z) = \frac{1}{2}\sum_k a_k z^k$$

Similarly taking Fourier transform to associated wavelet relation, we obtain

$$\psi(\omega) = Q(e^{-i\omega/2})\hat{\varphi}(\omega/2),$$

where $Q(z) = \frac{1}{2}\sum_k (-1)^k a_{1-k} z^k = -zP(-z^{-1})$.

Remarks: $P(1) = 1$ implies $P(-1) = 0$, i.e. $z = -1$ is a repeated root of $P(z)$. Thus for finite number of non zero $a_k s$, $P(z)$ is a polynomial with $z = -1$ a root, is expressible as

$$P(z) = (1+z)^p \tilde{P}(z),\ \tilde{P}(-1) \neq 0,\ p = \frac{N}{2},$$

where $\tilde{P}(z)$ is the product of the remaining factors of P after dividing out $z+1$ an appropriate number of times.
For N=4, let $P(z) = (1+z)^2\,(\alpha + \beta z)$.

$$P(1) = 1 \Longrightarrow \alpha = \frac{1}{4} - \beta.$$

$$[P(i)]^2 + [P(-i)]^2 = 1 \Longrightarrow 16\beta^2 - 4\beta + \frac{1}{2} = 0, \text{ i.e. } \beta = \frac{1 \pm \sqrt{3}}{8}.$$

Selecting –ve sign, $\beta = \frac{1-\sqrt{3}}{8}$. So

$$P(z) = \frac{1}{8}(1+z)^2\left[\left(1+\sqrt{3}\right) + \left(1-\sqrt{3}\right)z\right]$$

$$= \frac{1}{2}[a_0 + a_1 z + a_2 z^2 + a_3 z^3],$$

where $a_0 = \frac{1+\sqrt{3}}{4}, a_1 = \frac{3+\sqrt{3}}{4}, a_2 = \frac{3-\sqrt{3}}{4}, a_3 = \frac{1-\sqrt{3}}{4}$.

If we choose $\beta = \frac{1+\sqrt{3}}{8}$, the coefficient will be in reverse order. Narcowich[8] points out that the above method does not work for $p = 3$.

Bezout Technique [6]

Consider wavelets with p vanishing moments.
To find $H(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^p H_0(\omega), \quad H_0(\pi) \neq 0$

which satisfies the orthogonal relation

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1. \tag{3}$$

Notice $\frac{1+e^{-i\omega}}{2} = \frac{1+cos\omega}{2} - \frac{i}{2}sin\omega = e^{-i\frac{\omega}{2}}cos\left(\frac{\omega}{2}\right)$.

$$|H(\omega)|^2 = H(\omega)\overline{H(\omega)} = \left[cos^2\left(\frac{\omega}{2}\right)\right]^p |H_0(\omega)|^2.$$

$$|H(\omega+\pi)|^2 = H(\omega+\pi)\overline{H(\omega+\pi)} = \left[sin^2\left(\frac{\omega}{2}\right)\right]^p |H_0(\omega+\pi)|^2.$$

(3) gives $\left[cos^2\left(\frac{\omega}{2}\right)\right]^p |H_0(\omega)|^2 + \left[sin^2\left(\frac{\omega}{2}\right)\right]^p |H_0(\omega+\pi)|^2 = 1$ (4)

As $H_0(\omega)$ has real coefficients; this implies that $|H_0(\omega)|^2$ is an even function. So we can write in terms of cosines. Since

$$cos\omega = 1 - 2sin^2\left(\frac{\omega}{2}\right).$$

Letting $y = sin^2\left(\frac{\omega}{2}\right)$ and $|H_0(\omega)|^2 = q(y)$, (5)

(4) implies Bezout equation: $[1-y]^p q(y) + y^p q(1-y) = 1$.

Daubechies proved that $q(y) = \sum \binom{p+k-1}{k} y^k$.

Theorem: 15 [Bezout] If p_1 and p_2 are polynomials of degree n_1 and n_2 respectively, with no common zeros, then there exist unique polynomials q_1 and q_2 of degree $n_2 - 1$ and $n_1 - 1$ respectively, so that

$$p_1(x)q_1(x) + p_2(x)q_2(x) = 1.$$

Lemma: 11 The general solution of the Bezout equation, where r is the arbitrary polynomial
$$(1-x)^n q_1(x) \pm x^n q_2(1-x) = 1,$$

where $q_1(y) = \sum_{k=0}^{p-1} \binom{n+k-1}{k} y^k + y^n r(y)$ and

$$q_2(y) = \pm \sum_{k=0}^{p-1} \binom{n+k-1}{k} y^k - y^n r(1-y).$$

Since $q_1 = q_2 = q$ implies additional constraint
$$r(x) + r(1-x) = 0\ .$$

Therefore, r must have odd symmetry about $x = \frac{1}{2}$ so that

$$q(y) = \sum_{k=0}^{p-1} \binom{p+k-1}{k} y^k + y^p r\left(\frac{1}{2} - y\right). \tag{6}$$

Further $y = sin^2\left(\frac{\omega}{2}\right) = \frac{1}{2} - \frac{cos\omega}{2} = \frac{-z}{4} + \frac{1}{2} - \frac{1}{4z},\ \ z = e^{-i\omega}$.

Example: For $p = 2$, (6) implies $q(y) = 1 + 2y$.

Using (5),
$$2z|H_0(z)|^2 = 4z - z^2 - 1 = -(z^2 - 4z + 1).$$

The roots are $z = 2 \pm \sqrt{3}$. If we take $= 2 + \sqrt{3}$,
$$H_0(z) = \frac{\sqrt{3}-1}{2}\left(z - \left(2 + \sqrt{3}\right)\right).$$

$H(\omega) = \left(\frac{1+z}{2}\right)^p H_0(\omega)$ leads to the filter coefficients $a_0,\ a_1\ , a_2, a_3$. Coefficients for $p = 3, 4, 5, 6$ are shown in tabular form in Maleknejad et al.[6].With standard substitution $z = e^{-i\omega}$, $|H_0(\omega)|^2$ turns into a Laurent polynomial with power ranging from $-n$ to n, where n is the degree of q. If we multiply $|H_0(z)|^2$ by z^n it turns to a regular polynomial of degree $2n$. If the roots are real or lie on the unit circle, then the roots of this polynomial come in the form $z_k, \frac{1}{z_k}$. For simplicity if we select $z_k,\ k = 1,2, \ldots\ldots, n$ be the choosen roots, then
$$H_0(z) = \frac{(z-z_1)(z-z_2)\ldots\ldots\ldots(z-z_n)}{(1-z_1)(1-z_2)\ldots\ldots\ldots(1-z_n)}$$

will be a solution of equation $|H_0(\omega)|^2 = q(y)$.

9. FILTER DESIGN AND CONVERGENCE:

Here we propose a convenient method to find filter coefficients to certain restricted cases with input from [14].

We calculate the filter coefficients by using multiple double shift orthogonality property, normalization and Theorem9(i).

Following Remark (Section8), consider the polynomial
$$P(z) = (1+z)^p \left[\sum_{i=0}^{p-1} a_i z^i\right], \mathrm{p} = \frac{\mathrm{N}}{2}.$$

Using $P(1) = 1$(Theorem9) and $l = \frac{1}{2^p}$,
$$l = \sum_{i=0}^{p-1} a_{i,}\ \ \text{i.e. } a_{p-1} = l - \sum_{i=0}^{p-2} a_{i.}$$

Thus
$$P(z) = (1+z)^p \left[\sum_{i=0}^{p-2} a_i z^i + a_{p-1} z^{p-1}\right]$$

$$= (1+z)^p \left[\sum_{i=0}^{p-2} a_i z^i + \left(l - \sum_{i=0}^{p-2} a_i \right) z^{p-1} \right]. \qquad (7)$$

I. For $p = 2$

$$P(z) = (1+z)^2[a + (l-a)z]\,,\ l = 1/4.$$

Convolution of (1,2.1) with $(a, l-a)$ gives $a, l+a, 2l-a, l-a$.

Applying double shift orthogonality, we obtain

$$2a^2 - 2la - l^2 = 0\ .$$

This gives $a = \frac{1+\sqrt{3}}{8}$ taking +ve sign.

Using of the fact that $P(z) = \frac{1}{2}\sum_{k=0}^{3} a_k z^k$, we find that

$$a_0 = \frac{1+\sqrt{3}}{4}, a_1 = \frac{3+\sqrt{3}}{4}, a_2 = \frac{3-\sqrt{3}}{4}, a_3 = \frac{1-\sqrt{3}}{4}$$

Here $\sum_{k=0}^{3} a_k = 2$ is satisfied.

The corresponding Daub 4 filter coefficients h_k's are as given in the Table1 by using $\sum_{k=0}^{3} h_k = \sqrt{2}$, since $a_k = \sqrt{2} h_k$.

Taking –ve sign in *a* will reverse the order of coefficients.

II. For $p = 3$

$$P(z) = (1+z)^3[a + bz + (l-a-b)z^2]\,,\ l = 1/8.$$

Convolving $(1,4,6,4,1)$ with $(a, b, (l-a-b))$ give filter $a, 3a+b, 2a+2b+l, 3l-2a, 3l-3a-2b, l-a-b$.
Applying first and second double shift orthgonality, we obtain

$$4(-a^2 + al - ab) + 2(bl - b^2) + 3l^2 = 0$$
$$6(-a^2 + al - ab) + (bl - b^2) = 0$$

These imply

$$4b^2 - 4lb - 9l^2 = 0.$$

That is, $b = \frac{1-\sqrt{10}}{16}$ taking -ve sign.

Also

$$8a^2 - 8a(l-b) - 3l^2 = 0, \text{ i.e.}$$
$$8a^2 - \frac{(1+\sqrt{10})}{2}a - \frac{3}{64} = 0.$$

Thereby

$$a = \frac{1+\sqrt{10}+\sqrt{(1+\sqrt{10})^2+6}}{32} \text{ taking +ve sgn.}$$

Solving in the way as in (**I**) above and using $P(z) = \frac{1}{2}\sum_{k=0}^{5} a_k z^k$ will give Daub6 filter coefficients as shown in Table1.

Taking +ve sign in *b* will reverse the order of coefficients.

III. Now let $p = 4$ and

$$P(z) = (1+z)^4[a + bz + cz^2 + (l-a-b-c)z^3],\ l = 1/16.$$

On convolving (1, 4, 6, 4, 1) with $(a, b, c, (l-a-b-c))$, we obtain filter as $a, 4a+b, 6a+4b+c, l+3a+5b+3c, 4l-3a+2c, 6l-6a-5b-2c, 4l-4a-4b-3c, l-a-b-c$.

Applying three double shift orthogonality,

$$15b(l-b-c)+8c(l-c)-7ab+7ac+28l^2=0 \quad (8)$$

$$4a(l-a)+26b(l-b-c)-34ac+6c(l-c)-74ab+l^2=0 \quad (9)$$

and

$$a(8l-8a-9b-7c)+b(l-b-c)=0 \quad (10)$$

By solving (8), (9) and (10), we obtain

$$a=0.32580030356, b=-0.29226519423, c=0.10699752983.$$

Solving in the way as in (**I**) above and using $P(z)=\frac{1}{2}\sum_{k=0}^{7}a_k z^k$ will give Daub6 filter coefficients as shown in Table1.

Beyond $p=4$, the theory is hard to apply.

Table of Filter Coefficients

Table1: Filter coefficients for Daub $N=4,\ 6,\ 8$

k	$N=4$	$N=6$	$N=8$
0	0.4829629131445341	0.3326705529500825	0.2303778133088964
1	0.8365163037378077	0.8068915093110924	0.7148465705529154
2	0.2241438680420134	0.4598775021184914	0.6308807679298587
3	-0.1294095225512603	-0.1350110200102546	-0.0279837694168599
4		-0.0854412738820267	-0.1870348117190931
5		0.0352262918857095	0.0308413818355607
6			0.0328830116668852
7			-0.0105974017850690

Complete data for N=2,4,6,8,10,12,14,16,18,20, based on some other techniques, are available in Altaisky [1, p. 75] and Vidakovic [15].

Now we propose and prove the following Lemma:

Lemma: 12. Let $P(z)$ be defined by (7). For $p\to\infty$, $P(z)$ is convergent if $\sum_{i=0}^{p-2}a_i > l$ and $|z|<|k|, k=\frac{a_i}{a_{i+1}}$.

Proof: Let $\sum_{i=0}^{p-2}a_i > l$. Then $l-\sum_{i=0}^{p-2}a_i<0$ so that

$$P(z)<(1+z)^p\sum_{i=0}^{p-2}a_i z^i$$

$$=\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}\sum_{i=0}^{p-2}a_i z^{p+i-j.}$$

$P(z)$ will be convergent if $\sum_{i=0}^{p-2}a_i\, z^{p+i-j}$ is convergent.

D'Alembert Ratio Test implies that $|z|<|k|$, where $k=\frac{a_i}{a_{i+1}}$.

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