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STABILITY AND PHASE STUDY OF A HARVESTED PREDATOR-PREY SYSTEM WITH HOLLING –IV TYPE PREDATION

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ABSTRACT

In this paper we have considered a prey-predator model with Holing type IV of predation and independent harvesting in either species. The purpose of the work is to offer mathematical analysis of the model and to discuss some significant qualitative results that are expected to arise from the interplay of biological forms. Our study shows that using the harvesting efforts as controls, it is possible to break the cycle behaviour of the system and drive it to a required state. Also it is possible to introduce globally stable limit cycle in the system using the above controls.

Keywords: *Prey-predator system, harvesting, limit cycle, controllability.*

INTRODUCTION

In population dynamics, group defense is a term used to describe the phenomenon whereby predator is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough. An example of this phenomenon is described by Tener [11]. Lone musk ox can be successfully attacked by wolves. Small herds of musk ox (2 to 6 animals) are attacked but with rare success. No successful attacks have been observed in larger herds. A second example described by Holmes and Bethel [12] involves certain insect populations. Apparently, large swarms of insects make individual identification difficult for their predators. The third example was observed by Davidowicz, Gliwicz, and Gulati [18]. Filamentous algae are often qualified as inedible by herbivorous zooplankton. However, experiments show that Daphnia can consume them at low concentrations, while they jam their filtering apparatus in high concentrations.

In a famous paper Rosenzweig [17] warns that "man must be careful in attempting to enrich ecosystems in-order to increase its food yields. There is a real chance that such activity may result in a decimation of the food species that are wanted in greater abundance".

Later on several other mathematicians like H.I Freedman, G.S.K. Wolkowicz [9] Franz Rothe and Donglar S. Shafer [22], Shigai Ruan and Dongmei Xiao [25], etc., worked on several predator prey systems where there is group defence on the part of the prey and illustrated the warning of Rosenzweig. In this work we consider predator prey system, which is prone to the warning of Rosenzweig, and study the effect of harvesting on the biological interplay of predators and the prey population. We find strategies under which we can avoid the danger of predator extinction. Hence we assume that both predator and prey species are harvested independently and the system dynamics are observed treating the harvesting efforts as parameters of the system. From this study, we can always deduce the results for the sub cases where only one of the species is harvested.

In this work we assume that the coupled equations governing the dynamics of predator prey interaction are

$$\frac{dx}{dt} = \gamma x\left(1 - \frac{x}{K}\right) - \frac{xy}{a + x^2} \quad (1)$$

$$\frac{dy}{dt} = y\left[\frac{\mu x}{a + x^2} - D\right] \quad (2)$$

where x, y denote the prey and predator population respectively. $\gamma > 0$ represents the intrinsic growth rate of the prey, K is the carrying capacity of the prey in the absence of the predator and harvesting in the environment.

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The term $\frac{x}{a+x^2}$ denotes the functional response of the predator. This response function is termed as Holling type IV function. $\mu > 0$ is the conversion factor denoting the number of newly predators for each captured prey. D>0 is the natural death rate of the predator and $\sqrt{a}$ is the half saturation value.

The dynamics of the system (1) and (2) is studied in Shigui Ruan and Dongmei Xiao [25]. If we assume that both predator and prey species are harvested which corresponding harvesting effors $E_1 \geq 0$, $E_2 \geq 0$ respectively, then the system (1) and (2) gets modified and takes the following form.

$$\frac{dx}{dt} = (\gamma - E_1)x - \frac{\gamma x^2}{K} - \frac{xy}{a+x^2} \tag{3}$$

$$\frac{dy}{dt} = y\left[\frac{\mu x}{a+x^2} - (D+E_2)\right] \tag{4}$$

where the terms E_1x and E_2y represent the catch of the respective species per unit time. Actually it is more appropriate to take these terms as q_1E_1x and q_2E_2y where q_1 and q_2 represent the catchability coefficients of the prey and predator respectively. But, for the sake of mathematical simplicity, we have taken these terms to be unity. From (3) and (4), we see that the dynamics of the prey is governed by the logistic equation in the absence of predator and harvesting. For ecological information of the system (3) and (4), one can refer [4]-[8] and [15]-[16].

EXISTENCE OF EQUILIBRIUM POINTS

From the dynamics of the system (3) and (4), it is easy to observe that if $E_1 \geq \gamma$ then the system admits only one equilibrium point given by $S_1 = (0, 0)$ which is trivial equilibrium. Moreover, in this case, we have $\frac{dx}{dt} < 0 \, \forall t$ which implies that x approaches zero as $t \to \infty$. As a result, $\frac{dy}{dt}$ becomes negative for large t and hence y also goes to zero eventually. If $\gamma > E_1$ then there exists another equilibrium point on the boundary of the first quadrant given by $S_2 = \left(K\left(1 - \frac{E_1}{\gamma}\right), 0\right)$

Looking for an interior equilibrium point, we solve the following equations.

$$(\gamma - E_1)x - \frac{\gamma x^2}{K} - \frac{xy}{a+x^2} = 0 \tag{5}$$

$$y\left[\frac{\mu x}{a+x^2} - (D+E_2)\right] = 0 \tag{6}$$

and (6) implies

$$(D+E_2)x^2 - \mu x + a(D+E_2) = 0 \tag{7}$$

Note that the equation (7) is quadratic in x. Now let us consider the case where the system parameters satisfy the relation.

Case-(i): $\mu^2 - 4a(D+E_2)^2 = 0$

$$i.e., E_2 = \frac{\mu}{2\sqrt{a}} - D \tag{8}$$

In this case we obtain the interior equilibrium point

$$S_3 = (x^*, y^*) = \left(\sqrt{a}, \left(\gamma - E_1 - \frac{\gamma\sqrt{a}}{K}\right)2a\right)$$

$$if \; K\left(1 - \frac{E_1}{\gamma}\right) > \sqrt{a} \tag{9}$$

Therefore under the assumption that (8) and (9) hold good, the system admits three equilibrium points $S_1(0,0)$,

$$S_2 = \left(K\left(1 - \frac{E_1}{\gamma}\right), 0 \right) \text{ and}$$

$$S_3 = \left(\sqrt{a}, \left(\gamma - E_1 - \frac{\gamma\sqrt{a}}{K} \right) 2a \right)$$

Case-(ii): $\mu^2 - 4a(D + E_2)^2 < 0$

$$i.e., E_2 > \frac{\mu}{2\sqrt{a}} - D \tag{1.0}$$

In this case, there is no interior equilibrium point in the first quadrant. Therefore in this case the system admits only two equilibrium points given by $S_1(0,0)$ and $S_2 = \left(K\left(1 - \frac{E_1}{\gamma}\right), 0 \right)$ under the assumption $E_1 < \gamma$

Case-(iii): $\mu^2 - 4a(D + E_2)^2 > 0$

$$i.e., E_2 < \frac{\mu}{2\sqrt{a}} - D \tag{1.1}$$

In this case the system has at most four equilibrium points given by $S_1(0, 0)$, $S_2 = \left(K\left(1 - \frac{E_1}{\gamma}\right), 0 \right)$, $S_4(x_1,y_1)$ and $S_5(x_2,y_2)$
where

$$x_1 = \frac{\mu - \sqrt{\mu^2 - 4a(D + E_2)^2}}{2(D + E_2)} \text{ and}$$

$$x_2 = \frac{\mu + \sqrt{\mu^2 - 4a(D + E_2)^2}}{2(D + E_2)} \text{ ;}$$

$$y_1 = \left(\gamma - E_1 - \frac{\gamma x_1}{K} \right)(a + x_1)^2 \text{ and}$$

$$y_2 = \left(\gamma - E_1 - \frac{\gamma x_2}{K} \right)(a + x_2)^2$$

Clearly S_1, S_2 are boundary equilibrium points and S_4, S_5 are interior equilibrium points. More precisely, there are three possibilities.

(a) When both x_1 and x_2 are greater than $K\left(1 - \frac{E_1}{\gamma}\right)$ then the system has only two equilibrium points S_1 and S_2. Observe that $K\left(1 - \frac{E_1}{\gamma}\right)$ is nothing but the carrying capacity of the prey population in the absence of the predator in the environment. Therefore, in the absence of the predator in the environment, the prey population x(t) will be less than or equal to $K\left(1 - \frac{E_1}{\gamma}\right)$. Therefore $x_1, x_2 > K\left(1 - \frac{E_1}{\gamma}\right)$ is biologically meaningless.

(b) When $x_1 < K\left(1 - \frac{E_1}{\gamma}\right) < x_2$ then the system has three equilibrium points $S_1(0, 0)$, $S_2 = \left(K\left(1 - \frac{E_1}{\gamma}\right), 0 \right)$ and the interior equilibrium point $S_4(x_1,y_1)$. The interior equilibrium point $S_5(x_2, y_2)$ is biologically meaningless for the reason given in (a).

(c) When $K\left(1-\frac{E_1}{\gamma}\right) > x_2$, the system has four equilibrium points $S_1(0, 0)$, $S_4(x_1,y_1)$ and $S_5\,(x_2,y_2)$ where S_1, S_2 are boundary equilibrium points and S_4, S_5 are interior equilibrium points.

Finally we observe that S_3, S_4 are mutually exclusive. i.e., if the system admits S_3 as its interior equilibrium then S_4 does not appear in the picture and vice versa. The system cannot admit S_5 as its equilibrium point without having S_4 as its equilibrium point as the x coordinate of S_4 is always less than that of S_5.

NATURE OF THE EQUILIBRIA

Now we wish to study the nature of these equilibrium points and their dependence on the harvesting efforts. During the stability analysis of these equilibrium points, we come across the following important equations.

$$k^2\left(1-\frac{E_1}{\gamma}\right)^2 - 3a = 0$$

$$3\mu^2 = 16a(D+E_2)^2 \quad or \quad E_2 = \frac{\sqrt{3}\mu}{4\sqrt{a}} - D$$

$$x_1 = \frac{1}{3}K\left(1-\frac{E_1}{\gamma}\right) - \frac{1}{3}\sqrt{K^2\left(1-\frac{E_1}{\gamma}\right)^2 - 3a} = J_1$$

$$x_1 = \frac{1}{3}K\left(1-\frac{E_1}{\gamma}\right) + \frac{1}{3}\sqrt{K^2\left(1-\frac{E_1}{\gamma}\right)^2 - 3a} = J_2$$

$$x_1 = K\left(1-\frac{E_1}{\gamma}\right) \quad \text{and} \quad x_2 = K\left(1-\frac{E_1}{\gamma}\right)$$

All these curves are represented in Fig.1. We are aware that the interior equilibrium points are nothing but the intersection points between the predator isocline and the prey isocline depends on the values of the system parameters.

Here we denote $K\left(1-\frac{E_1}{\gamma}\right)$ *by M*

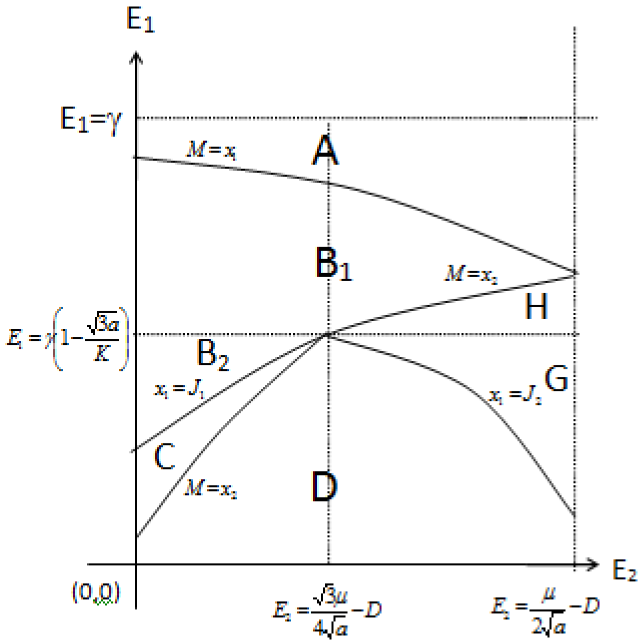

Figure-1

If $k^2\left(1-\frac{E_1}{\gamma}\right)^2 < 3a$ then the isocline will be a decreasing function w.r.t 'x' with x intercept as $K\left(1-\frac{E_1}{\gamma}\right)$. Note that $K\left(1-\frac{E_1}{\gamma}\right)$ represents the carrying capacity of the prey population in the absence of the predator in the ecosystem

If $k^2\left(1-\frac{E_1}{\gamma}\right)^2 > 3a$ then the prey isocline will have two extreme points J_1 and J_2 in the interval $\left(0,\left(1-\frac{E_1}{\gamma}\right)\right)$ as shown in the Fig.2.

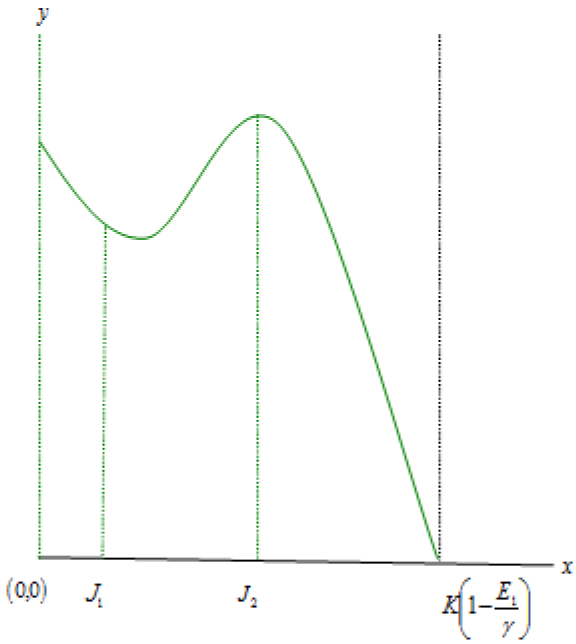

Figure-2

If $x_1 \in (J_1, J_2)$ then S_4 is unstable. If $x_1 \in (0, J_1] \cup [J_2, K\left(1-\frac{E_1}{\gamma}\right)$ then S_4 is stable locally. If the system admits two interior equilibrium points S_4 and S_5 the dynamics of the system under goes lot of changes. S_5 always enters the system as saddle with one of its unstable manifold branches connecting S_5 and S_2. Thus the presence of S_5 brings in several paths leading to S_2. That is possibility of extinction of predator species. The phase portraits of the paths of the system in different regions are shown in the following figures. In the region A the prey-predator isoclines and paths of the system (3) and (4) are shown in Fig.3 we can find S_2 is globally asymptotically stable whenever (E_2, E_1) is restricted to the region A. Clearly S_1 is saddle.

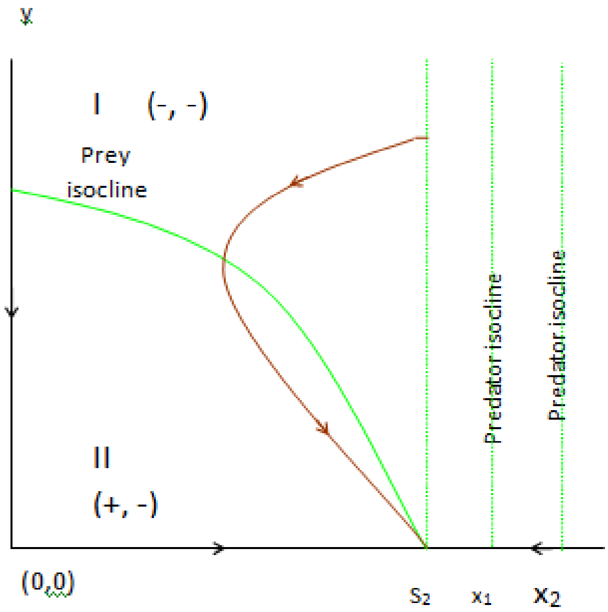

Figure-3

Now let us consider $(E_2, E_1) \in B_1$. The geometric configuration of the paths of (3) and (4) in B_1 is shown in Fig.4. In this region S_4 is globally asymptotically stable and S_1, S_2 are unstable. Now consider (E_2, E_1) belonging to B_2. In this region the geometric configuration of the system is

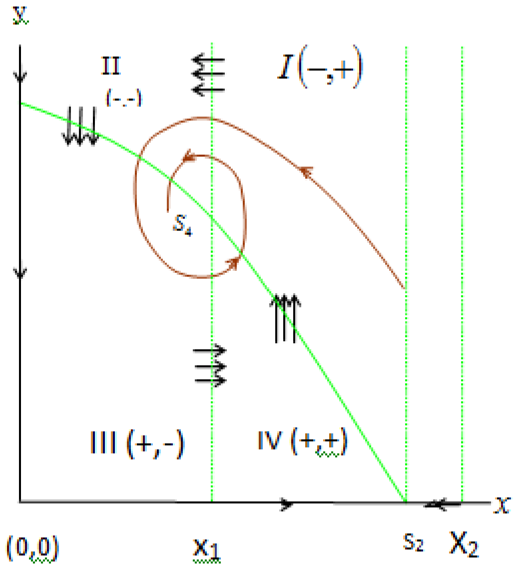

Figure-4

shown in Fig.5. In this case also all solutions initiating in the interior of the first quadrant will move as shown in Fig.5 and eventually approach S_4 thus making S_4 globally asymptotically stable in the interior of the first quadrant as earlier S_1 and S_2 are saddle

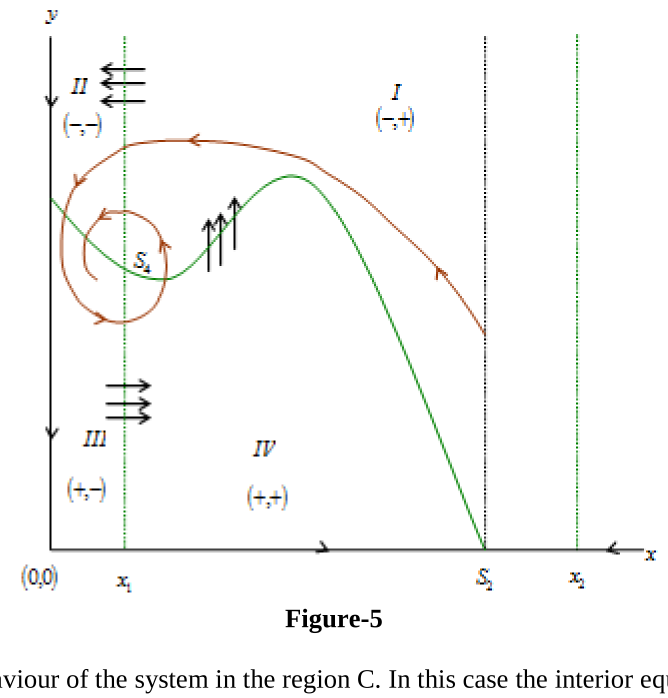

Figure-5

Now let us consider the behaviour of the system in the region C. In this case the interior equilibrium point S_4 is unstable and all solutions initiating in the interior of the first quadrant will approach a limit cycle surrounding S_4. The points S_1, S_2 are unstable(Refer Fig. 6).

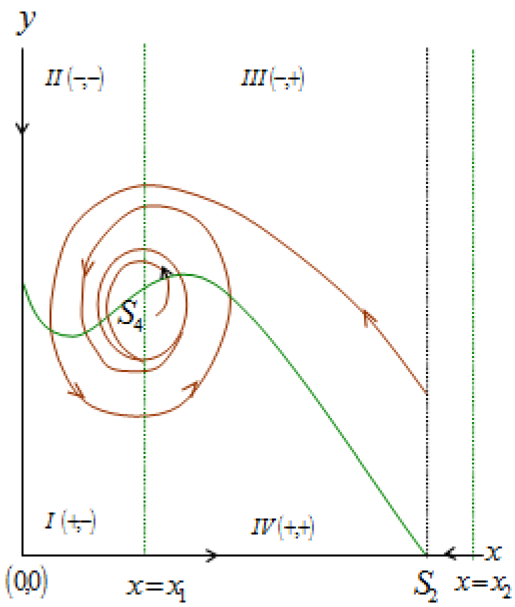

Figure-6

When$(E_2,E_1)\in$ D the paths initiating in the region I to V will move in any one of the ways shown in the figures 7,8 and 9.

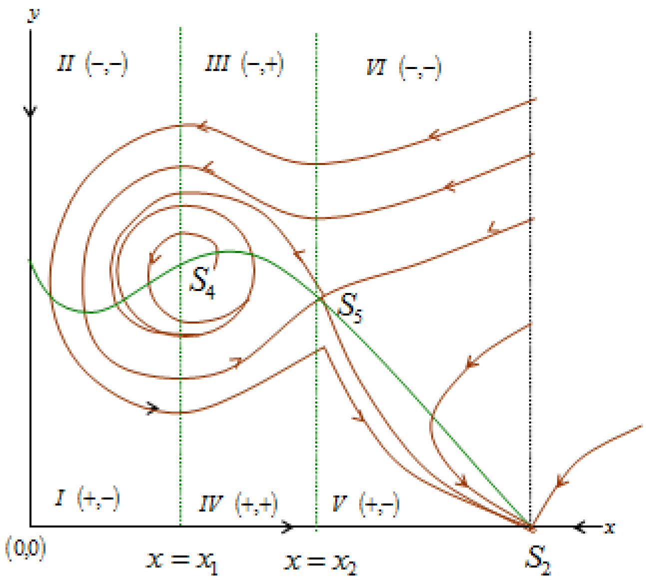

Figure-7

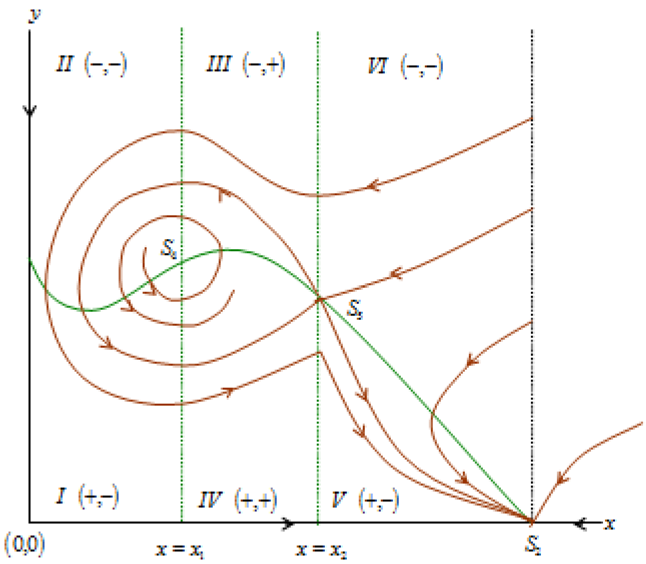

Figure-8

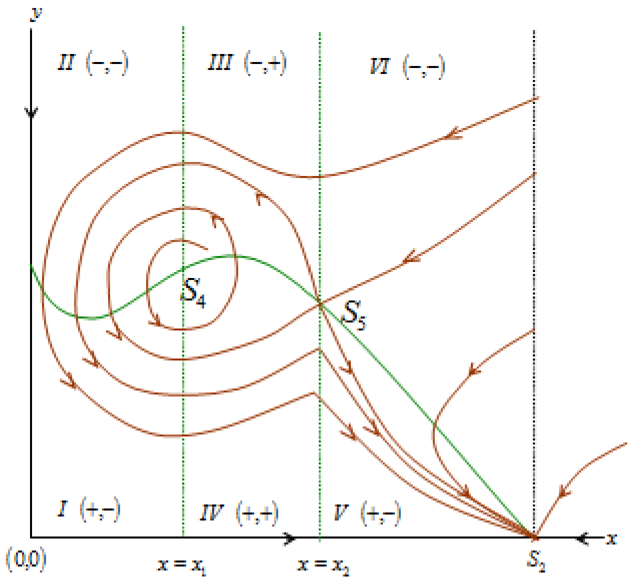

Figure-9

Now let us consider the case $(E_2, E_1) \in G$. In this case S_2, S_4 are stable S_1, S_5 are saddle. (Refer Fig. 10). In case when $(E_2, E_1) \in H$, Fig. 11 exhibit the stable nature of S_2, S_4 and S_1, S_5 are saddle.

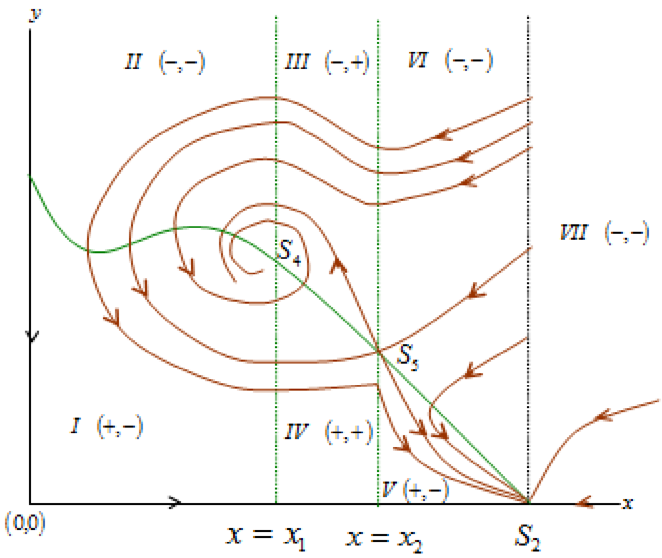

Figure-10

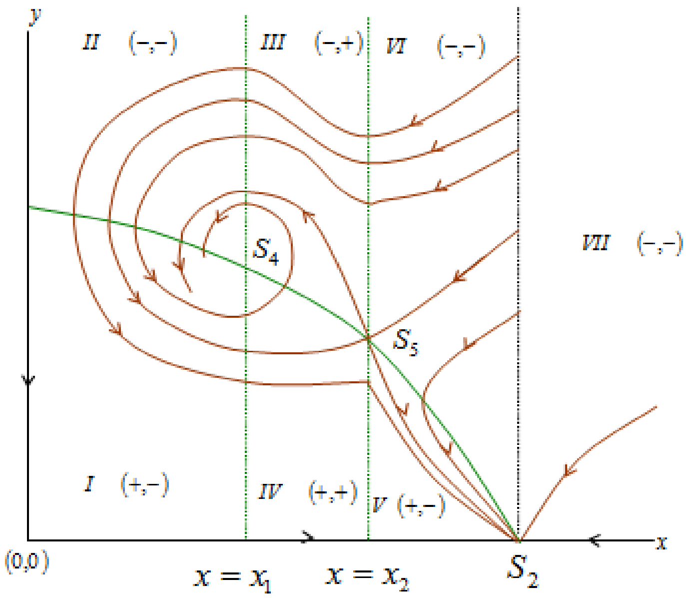

Figure-11

SUMMARY AND CONCLUSIONS

In this work, we have considered a predator-prey system where there is group defence on the part of the prey discrete harvesting in both the species. We associate the group defences to the considered system by assuming the predator response function to be Holling type IV. This work presents the Predator-prey model. This analysis also illustrates methods to control the predator-prey system.

The considered system admits at most four equilibrium points, of which two are interior equilibrium points and the other two are boundary equilibrium points. The interior equilibrium points are represented by $S_3(x^*,y^*)$, $S_4(x_1,y_1)$, $S_5(x_2,y_2)$ and the boundary equilibrium points are represented by $S_1(0,0)$, $S_2\left[K\left(1-\frac{E_1}{\gamma}\right),0\right]$. We observe that the equilibrium point S_3 and the equilibrium point S_4 are mutually exclusive in the sense that if the system admits S_3 as its interior equilibrium then S_4 does not appear in the picture and vice versa. Also the system may admit S_4 as the only interior equilibrium or it may be admit both S_4&S_5 as its interior equilibrium points. But S_5 alone cannot be admitted as the systems interior equilibrium point without S_4 being one as the x-coordinate of S_4 is always less then that of S_5. As long as the system admits S_4 as its sole interior equilibrium, it may either be globally stable or the system would admit a limit cycle. Thus in this case any solution initiating in $\{(x,y)/\,x>0, y>0\}-\{S_4\}$will either approach S_4 or a limit cycle surrounding S_4 depending on whether the unique interior equilibrium point S_4 is stable or unstable respectively. If the system admits two interior equilibrium points namely S_4 &S_5, the dynamics of the system under goes lot of changes. S_5 always enters the system as saddle with one of its unstable manifold branches connecting S_5 and S_2. Thus, presence of S_5 brings in several paths leading to S_2. That is possibility of extinction of predator species. From this observation, we see that the extinction of predator species can be avoided if S_5 does not get in to the picture. i.,e., the system does not admit as its equilibrium point. This can be achieved by using the harvesting efforts E_1 &E_2 as controls. The predator extinction can be avoided if we can restrict the harvesting efforts (E_{2r},E_1) to the region $B_1 \cup B_2 \cup C$.In this case all the paths initiating in the positive quadrant will approach S_4 or a limit cycle surrounding S_4 which depend on the other parameters of the system. In either case, we are assured of persistence of both the species prey and predator. Thus the strategy for the persistence of both the species is to limit (E_2, E_1) to $B_1 \cup B_2 \cup C$. extinction of predator species. From this observation, we see that the extinction of predator species can be avoided if S_5 does not get in to the picture. i., e., the system does not admit as its equilibrium point. This can be achieved by using the harvesting efforts E_1 &E_2 as controls. The predator extinction can be avoided if we can restrict the harvesting efforts (E_{2r}, E_1) to the region $B_1 \cup B_2 \cup C$. In this case all the paths initiating in the positive quadrant will approach S_4 or a limit cycle surrounding S_4 which depend on the other parameters of the system. In either case, we are assured of persistence of both the species prey and predator. Thus the strategy for the persistence of both the species is to limit (E_2,E_1) to $B_1 \cup B_2 \cup C$.

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