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# ON THE $p$-DOMINATION NUMBER AND $p$-REINFORCEMENT NUMBER OF THE JOIN OF SOME GRAPHS

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***ABSTRACT***

*Let $G=(V,E)$ be a graph. A subset D of G is a p-dominating set of G if $|N_G(x)\cap D|\geq p$ for all $x\in V\setminus D$, where $N_G(x)$ is the set of all vertices which are adjacent to x. The p-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of p-dominating sets of G. The p-reinforcement number of G, denoted by $r_p(G)$, is the minimum number of edges in $G^C$ that has to be added to G in order to reduce the p-domination number of the resulting graphs. In this study, we gave a tight upperbound for the p-domination number of the join of graphs, the p-domination number of a complete and any graph, the 2-domination number and 3-domination number of fans, and the 2-reinforcement number and 3-reinforcement number of fans.*

***Mathematics Subject Classification:***

***Keywords:*** *p-dominating set, p-domination number, p-reinforcement number, join, fans.*

## I. INTRODUCTION

The concept $p$-domination in graphs was introduced by Fink *et al.* in [5]. Since then, many researchers studied the concept. Caro *et al.* [3], Blidia *et al.* [2], Lu *et al.* [8], Rautenbach *et al.* [13], and De La Viña *et al.* [4] gave bounds for the $p$-domination number of graphs. Lu and Xu [8] gave the $p$-domination number of complete multipartite graphs. Fujisawa *et al.* [6] gave the 2-domination number of the corona of some graphs. Thakkar *et al.* [14] and Mohan *et al.* [12] gave the 2-domination number of the Cartesian product of paths. Bakhshesh *et al.* [1] gave the 2-domination number of generalized Petersen graphs.

A study on the $p$-reinforcement number of graphs is found in [9]. Lu [8, 9, 10, 11] and other researchers worked on this concept and published a couple articles. Lu *et al.* [9] gave the $p$-reinforcement number of some graphs such as paths, cycles and complete $t$-partite graphs, and established some upper bounds. Lu and Xu [10] characterized all trees attaining the said upper bound for $p\geq 3$. Lu *et al.* [11] characterized trees with $2$-reinforcement number equal to 3. In particular, they showed that $r_2(T)=3$ if and only if there is a 2-dominating set $S$ of $T$ such that $T$ contains neither an $S$-vulnerable vertices nor an $S$-vulnerable paths.

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A *graph* or a *network* $G$ is an ordered pair $G=(V,E)$, where $V$ or $V(G)$ is a nonempty finite set whose elements are called *vertices,* and $E$ or $E(G)$ is a set of 2-element subsets of $V$ called *edges*. The *order* of $G$, denoted by $|V|$, is the number of vertices of $G$. The *size* of $G$, denoted by $|E|$, is the number of edges of $G$. The *degree* of a vertex $v$ of a graph $G$, denoted by $\deg_G(v)$, is the number of edges incident with $v$. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ of $G$ is given by $\delta(G)=\min\{\deg_G(x): x\in V\}$ and $\Delta(G)=\max\{\deg_G(x): x\in G\}$, respectively. A graph $H$ is called a *subgraph* of $G$ if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$.

The *complement* of a graph $G$, denoted by $\overline{G}$, is a graph with the same vertex set as $G$ and where two distinct vertices are adjacent if and only if they are not adjacent in $G$. The *path* $P_n=(v_1,v_2,...,v_n)$ is the graph with distinct vertices $v_1,v_2,...,v_n$ and edges $v_1v_2,v_2v_3,...,v_{n-1}v_n$. A *complete graph* of order $n$, denoted by $K_n$, is the graph in which every pair of distinct vertices are adjacent. The *join* of two graphs $G$ and $H$, denoted by $G+H$, is the graph with vertex set $V(G+H)=V(G)\dot\cup V(H)$ and edge set $E(G+H)=E(G)\dot\cup E(H)\dot\cup\{uv: u\in V(G), v\in V(H)\}$. The *fan* $F_n$ is the graph of order $n+1$, obtained from $P_n$ by adding a new vertex, say $x_0$, and joining $x_0$ by an edge to each of the $n$ vertices of $P_n$, that is, $F_n=K_1+P_n$.

Let $G=(V,E)$ be a graph and $x\in V$. The *neighborhood* of $x$ is the set consisting of all vertices $y$ which are adjacent to $x$, that is, $N(x)=\{y\in V: xy\in E\}$. The elements of $N(x)$ are called *neighbors* of $x$. Let $S\subseteq V$. The *neighborhood* of $S$ in $G$ is the set $N_G(S)=\{v\in V(G): uv\in E(G)$ for some $v\in S\}=\bigcup_{v\in S} N_G(v)$. The *closed neighborhood* of $S$ in $G$ is the set $N_G[S]=S\cup N_G(S)$.

The *Pigeonhole Principle* implies that if there are $n$ pigeons to enter into $k$ pigeonholes with $n<k$, then there exists at least 1 pigeonhole that is empty.

Let $G=(V,E)$ be a graph and $p$ a positive integer. A subset $D$ of $G$ is a $p$-dominating set of $G$ if $|N_G(x)\cap D|\geq p$ for all $x\in V\setminus D$, where $N_G(x)$ is the set of all vertices which are adjacent to $x$. The $p$-domination number of $G$, denoted by $\gamma_p(G)$, is the minimum cardinality of $p$-dominating sets of $G$. The $p$-reinforcement number of $G$, denoted by $r_p(G)$, is the minimum number of edges in $G^C$ that has to be added to $G$ in order to reduce the $p$-domination number of the resulting graph.

## II. RESULTS

### A. *p*-Domination Number of the Join of Graphs

In this section, we gave a sharp upperbound of the $p$-domination number of the join of graphs. We also gave the $p$-domination number of the join of a complete graph of order $n\geq p$ and any graph.

**Theorem 2.1:** Let $G$ and $H$ be any graphs and $p$ be a positive integer. If $|V(G)|\geq p$ and $|V(H)|\geq p$, then $\gamma_p(G+H)\leq 2p$.

***Proof:*** Let $G$ and $H$ be any graphs with $|V(G)|\geq p$ and $|V(H)|\geq p$. Let $u_1,u_2,...,u_p\in V(G)$ and $v_1,v_2,...,v_p\in V(H)$. Consider $D=\{u_1,u_2,...,u_p,v_1,v_2,...,v_p\}$. Let $w\in V(G+H)\setminus D$, say $w\in V(H)\setminus D$. Then $|N(w)\cap D|\geq p$. This shows that $D$ is a $p$-dominating set in $G+H$. Hence, $\gamma_p(G+H)\leq 2p$.

The next Corollary shows that the bound in Theorem 2.1 is sharp.

**Corollary 2.2:** Let $P_m$ and $P_n$ be paths of order $m$ and $n$, respectively. If $m,n \geq 9$, then $\gamma_2(P_m + P_n) = 4$.

***Proof:*** By **Theorem 2.1**, $\gamma_2(P_m + P_n) \leq 4$. Suppose $\gamma_2(P_m + P_n) < 4$, say without loss of generality $\gamma_2(P_m + P_n) = 3$. Let $D$ be a 2-dominating set of $P_m + P_n$ with $|D| = 3$ and consider the following cases:

**Case-1:** $|V(P_n) \cap D| = 3$

Let $P_n = (1,2,3,...,n)$ and consider the partition

$$\alpha = \begin{cases} \{\{1,2,3\},\{4,5,6\},...,\{n-2,n-1,n\}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-3,n-2,n-1\},\{n\}\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-4,n-3,n-2\},\{n-1,n\}\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

of $V(P_n)$. Since $|V(P_n)| \geq 9$, $|\alpha| \geq 3$. By the *Pigeonhole principle,* there exists $A = \{u_i, u_{i+1(\bmod n)}, u_{i+2(\bmod n)}\} \in \alpha$ such that $|A \cap D| = 1$. Let $v \in u_{i+1(\bmod n)}$. Then $|N(v) \cap D| = 1$. This is a contradiction.

**Case-2:** $|V(P_n) \cap D| = 2$

Let $P_n = (1,2,3,...,n)$ and consider the partition

$$\alpha = \begin{cases} \{\{1,2,3\},\{4,5,6\},...,\{n-2,n-1,n\}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-3,n-2,n-1\},\{n\}\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-4,n-3,n-2\},\{n-1,n\}\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

of $V(P_n)$. Since $|V(P_n)| \geq 9$, $|\alpha| \geq 3$. By the *Pigeonhole principle,* there exists $A = \{u_i, u_{i+1(\bmod n)}, u_{i+2(\bmod n)}\} \in \alpha$ such that $|A \cap D| = 0$. Let $v \in u_{i+1(\bmod n)}$. Then $|N(v) \cap D| = 1$. This is a contradiction.

Therefore, $\gamma_2(P_m + P_n) = 4$.

The next Theorem gave the *p*-domination number of the join of a complete graph of order $n \geq p$ and any graph.

**Theorem 2.3:** Let $G$ be a graph of order $m$ and $K_n$ be a complete graph of order *n*. If $p \leq n$, then $\gamma_p(K_n + G) = p$.

***Proof:*** Let *G* be a graph of order *m* and $K_n = (\{u_1, u_2, ..., u_n\}, E(K_n))$ be a complete graph of order *n*. Let $D = \{u_1, u_2, ..., u_p\}$. Then clearly $D$ is a *p*-dominating set in $K_n + G$. Thus, $\gamma_p(K_n + G) \leq p$. Since $p \leq \gamma_p(K_n + G)$, we must have $\gamma_p(K_n + G) = p$.

### B. *2*-Domination and *3*-Domination Number of Fans

In this section, we gave the 2-domination number and 3-domination number of fans. Lemma 2.4 is found in [16] and Lemma 2.5 is an observation in [14].

**Lemma 2.4:** $\gamma(P_n) = \lceil n/3 \rceil$.

**Lemma 2.5:** $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.

**Observation 2.6:** Let $P_n$ be a path of order *n*. Then $\gamma(P_n) < \gamma_2(P_n)$, that is, if *S* is a minimum dominating set then it cannot be a 2-dominating set.

 

**Theorem 2.7:** *D* is a minimum 2-dominating set in $F_n$ ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

***Proof:*** Let *D* be a minimum 2-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$) and $D \neq D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

Then there exist a subgraph $P = (u_i, u_{i+1(\text{mod } n)}, u_{i+2(\text{mod } n)})$ or $(u_{n-1}, u_n,)$ of $(u_1, u_2, \ldots, u_n)$ such that $V(P) \cap D = \varnothing$. If $P = (u_i, u_{i+1(\text{mod } n)}, u_{i+2(\text{mod } n)})$, then we let $v = u_{i+1(\text{mod } n)}$. While, if $P = (u_{n-1}, u_n,)$, then we let $v = u_n$. Thus, $|N(v) \cap D| = 1$. This is a contradiction.

Conversely, suppose that $D^* = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

and *D** is not a minimum 2-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$). Clearly, *D** is a 2-dominating set. Let *D* be a minimum 2-dominating set. Then $|D| < |D^*|$. Consider the following cases:

**Case-1:** $u \in D$
If $u \in D$, then $D \setminus \{u\}$ is not a dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, there exists $v \in \{u_1, u_2, \ldots, u_n\}$ such that $N(v) = \{u\}$. Thus, $|N(v) \cap D| = 1$. This is a contradiction.

**Case-2:** $u \notin D$
If $u \notin D$, then we note that *D'* is a minimum dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, by **Observation 2.6** *D'* cannot be a 2-dominating set of $(u_1, u_2, \ldots, u_n)$, and so is *D*. Since $u \notin D$, *D* cannot be a 2-dominating set of $F_n$. This is a contradiction.

**Corollary 2.8:** Let $F_n$ ($n \geq 3$) be a fan of order $n+1$. Then $\gamma_2(F_n) = \lceil n/3 \rceil + 1$.

**Observation 2.9:** Let $P_n$ be a path of order *n*. Then $\gamma_2(P_n) < \gamma_3(P_n)$, that is, if *S* is a minimum 2-dominating set of $P_n$ then it cannot be a 3-dominating set.

**Theorem 2.10:** *D* is a minimum 3-dominating set in $F_n$ ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

***Proof:*** Let *D* be a minimum 3-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$) and suppose that $D \neq D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

Then there exist a subset $A = \{u_i, u_{i+1(\text{mod } n)}\}$ or $\{u_n\}$ or $\{u_1\}$ of $\{u_1, u_2, \ldots, u_n\}$ such that $A \cap D = \varnothing$. Let $v = u_{i+1(\text{mod } n)}$ or $u_1$ or $u_n$. Then, $|N(v) \cap D| < 3$. This is a contradiction.

Conversely, suppose that $D^* = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

and $D^*$ is not a minimum 3-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$). Clearly, $D^*$ is a 3-dominating set. Let $D$ be a minimum 3-dominating set. Then $|D| < |D^*|$. Consider the following cases:

**Case-1:** $u \in D$

If $u \in D$, then $D \setminus \{u\}$ is not a 2-dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, there exists $v \in \{u_1, u_2, \ldots, u_n\}$ such that $|N(v)| < 2$. Thus, $|N(v) \cap D| < 3$. This is a contradiction.

**Case-2:** $u \notin D$

If $u \notin D$, then we note that $D'$ is a minimum 2-dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, by **Observation 2.9** $D'$ cannot be a 3-dominating set of $(u_1, u_2, \ldots, u_n)$, and so is $D$. Since $u \notin D$, $D$ cannot be a 3-dominating set of $F_n$. This is a contradiction.

**Corollary 2.11:** Let $F_n$ ($n \geq 3$) be a fan of order $n+1$. Then $\gamma_3(F_n) = \lceil n/2 \rceil + 1$.

**C. *2*-Reinforcement and *3*-Reinforcement Number of Fans**

In this section, we present the 2-reinforcement number and 3-reinforcement number of fans. Remark 2.12 is implied in an observation in [15].

**Remark 2.12:** Let $P_n$ be a path of order *n*. Then

$$r(P_n) = \begin{cases} 1 & , \text{ if } n \equiv 1 \pmod 3 \\ 2 & , \text{ if } n \equiv 2 \pmod 3 \\ 3 & , \text{ if } n \equiv 0 \pmod 3). \end{cases}$$

**Observation 2.13:** Let $P_n$ be a path of order *n*. Then

$$r_2(P_n) = \begin{cases} 1 & , \text{ if } n \equiv 0 \pmod 2 \\ 3 & , \text{ if } n \equiv 1 \pmod 2). \end{cases}$$

**Theorem 2.14:** Let $F_n$ ($n \geq 4$) be a fan of order $n+1$. Then $r_2(F_n) = r(P_n)$.

***Proof:*** Let $F_n = (\{u\}, \varnothing) + (u_1, u_2, ..., u_n)$ be a fan graph of order $n+1$. By **Theorem 2.7**, $D$ is a minimum 2-dominating set in $F_n$ ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

If $n \equiv 1 (\text{mod } 3)$, then $D^* = D \setminus \{u_n\}$ is a 2-dominating set in $F_n + u_2 u_n$. If $n \equiv 2 (\text{mod } 3)$, then $D^* = D \setminus \{u_n\}$ is a 2-dominating set in $F_n + u_2 u_{n-1} + u_2 u_n$. If $n \equiv 0 (\text{mod } 3)$, then $D^* = D \setminus \{u_n\}$ is a 2-dominating set in $F_n + u_2 u_{n-2} + u_2 u_{n-1} + u_2 u_n$. Hence, by Remark 2.12 $r_2(F_n) \leq r(P_n)$. Suppose $r_2(F_n) < r(P_n)$ for $n \equiv 0 \text{ or } 2 (\text{mod } 3)$. Then $D \setminus \{u\}$ must be a 2-dominating set in $F_n'$ (where $F_n'$ is the graph obtained from $F_n$ by adding edges), that is, $D \setminus \{u\}$ must be a 2-dominating set in $P_n'$ (where $P_n'$ is the graph obtained from $P_n$ by adding edges) – which is not possible.

**Theorem 2.15:** Let $F_n$ ($n \geq 3$) be a fan of order $n+1$. Then $r_3(F_n) = r_2(P_n)$.

***Proof:*** Let $F_n = (\{u\}, \emptyset) + (u_1, u_2, ..., u_n)$ be a fan graph of order $n+1$. By **Theorem 2.10**, $D$ is a minimum 3-dominating set in $F_n$ ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

If $n \equiv 0 (\bmod 2)$, then $D^* = D \setminus \{u_n\}$ is a 3-dominating set in $F_n + u_1 u_n$. If $n \equiv 1 (\bmod 2)$, then $D^* = D \setminus \{u_n\}$ is a 3-dominating set in $F_n + u_1 u_n + u_1 u_{n-1} + u_3 u_n$. Hence, by Observation 2.13 $r_3(F_n) \leq r_2(P_n)$. Suppose $r_3(F_n) < r_2(P_n)$ for $n \equiv 1 (\bmod 3)$. Then $D \setminus \{u\}$ must be a 3-dominating set in $F_n^{'}$ (where $F_n^{'}$ is the graph obtained from $F_n$ by adding edges), that is, $D \setminus \{u\}$ must be a 3-dominating set in $P_n^{'}$ (where $P_n^{'}$ is the graph obtained from $P_n$ by adding edges) – which is not possible.

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