International Journal of Mathematical Archive-10(6), 2019, 4-9
Available online through www.ijma.info ISSN 2229 – 5046

ON THE p-DOMINATION NUMBER AND p-REINFORCEMENT NUMBER OF THE JOIN OF SOME GRAPHS

EARL JUDE FRANCIS T. AWING*
Bayawan National High School,
Department of Education, Bayawan City Negros Oriental, Philippines.

MICHAEL P. BALDADO, Jr.
Mathematics Department,
Negros Oriental State University
Dumaguete City Negros Oriental, Philippines.

JOEL G. ADANZA
Mathematics Department,
Negros Oriental State University
Dumaguete City Negros Oriental, Philippines.

(Received On: 10-04-19; Revised & Accepted On: 18-05-19)

ABSTRACT

Let $G=(V,E)$ be a graph. A subset D of G is a p-dominating set of G if $\left|N_G(x)\cap D\right|\geq p$ for all $x\in V\setminus D$, where $N_G(x)$ is the set of all vertices which are adjacent to x. The p-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of p-dominating sets of G. The p-reinforcement number of G, denoted by $r_p(G)$, is the minimum number of edges in G^C that has to be added to G in order to reduce the p-domination number of the resulting graphs. In this study, we gave a tight upperbound for the p-domination number of the join of graphs, the p-domination number of a complete and any graph, the 2-domination number and 3-domination number of fans, and the 2-reinforcement number and 3-reinforcement number of fans.

Mathematics Subject Classification:

Keywords: *p-dominating set, p-domination number, p-reinforcement number, join, fans.*

I. INTRODUCTION

The concept p-domination in graphs was introduced by Fink *et al.* in [5]. Since then, many researchers studied the concept. Caro *et al.* [3], Blidia *et al.* [2], Lu *et al.* [8], Rautenbach *et al.* [13], and De La Viña *et al.* [4] gave bounds for the p-domination number of graphs. Lu and Xu [8] gave the p-domination number of complete multipartite graphs. Fujisawa *et al.* [6] gave the 2-domination number of the corona of some graphs. Thakkar *et al.* [14] and Mohan *et al.* [12] gave the 2-domination number of the Cartesian product of paths. Bakhshesh *et al.* [1] gave the 2-domination number of generalized Petersen graphs.

A study on the p-reinforcement number of graphs is found in [9]. Lu [8, 9, 10, 11] and other researchers worked on this concept and published a couple articles. Lu *et al.* [9] gave the p-reinforcement number of some graphs such as paths, cycles and complete t-partite graphs, and established some upper bounds. Lu and Xu [10] characterized all trees attaining the said upper bound for $p\geq 3$. Lu *et al.* [11] characterized trees with *2*-reinforcement number equal to 3. In particular, they showed that $r_2(T)=3$ if and only if there is a 2-dominating set *S* of *T* such that *T* contains neither an *S*-vulnerable vertices nor an *S*-vulnerable paths.

Corresponding Author: Earl Jude Francis T. Awing*
Bayawan National High School, Department of Education,
Bayawan City Negros Oriental, Philippines.

A *graph* or a *network* G is an ordered pair $G=(V,E)$, where V or $V(G)$ is a nonempty finite set whose elements are called *vertices,* and E or $E(G)$ is a set of 2-element subsets of V called *edges*. The *order* of G, denoted by $|V|$, is the number of vertices of G. The *size* of G, denoted by $|E|$, is the number of edges of G. The *degree* of a vertex v of a graph G, denoted by $\deg_G(v)$, is the number of edges incident with v. The *minimum degree* $\delta(G)$ and the *maximum degree* $\Delta(G)$ of G is given by $\delta(G)=\min\{\deg_G(x): x\in V\}$ and $\Delta(G)=\max\{\deg_G(x): x\in G\}$, respectively. A graph H is called a *subgraph* of G if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$.

The *complement* of a graph G, denoted by $\overline{G}$, is a graph with the same vertex set as G and where two distinct vertices are adjacent if and only if they are not adjacent in G. The *path* $P_n=(v_1,v_2,...,v_n)$ is the graph with distinct vertices $v_1,v_2,...,v_n$ and edges $v_1v_2,v_2v_3,...,v_{n-1}v_n$. A *complete graph* of order n, denoted by K_n, is the graph in which every pair of distinct vertices are adjacent. The *join* of two graphs G and H, denoted by $G+H$, is the graph with vertex set $V(G+H)=V(G)\dot{\cup}V(H)$ and edge set $E(G+H)=E(G)\dot{\cup}E(H)\dot{\cup}\{uv: u\in V(G), v\in V(H)\}$. The *fan* F_n is the graph of order $n+1$, obtained from P_n by adding a new vertex, say x_0, and joining x_0 by an edge to each of the n vertices of P_n, that is, $F_n=K_1+P_n$.

Let $G=(V,E)$ be a graph and $x\in V$. The *neighborhood* of x is the set consisting of all vertices y which are adjacent to x, that is, $N(x)=\{y\in V: xy\in E\}$. The elements of $N(x)$ are called *neighbors* of x. Let $S\subseteq V$. The *neighborhood* of S in G is the set $N_G(S)=\{v\in V(G): uv\in E(G)$ for some $v\in S\}=\bigcup_{v\in S} N_G(v)$. The *closed neighborhood* of S in G is the set $N_G[S]=S\cup N_G(S)$.

The *Pigeonhole Principle* implies that if there are n pigeons to enter into k pigeonholes with $n<k$, then there exists at least 1 pigeonhole that is empty.

Let $G=(V,E)$ be a graph and p a positive integer. A subset D of G is a p-dominating set of G if $|N_G(x)\cap D|\geq p$ for all $x\in V\setminus D$, where $N_G(x)$ is the set of all vertices which are adjacent to x. The p-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of p-dominating sets of G. The p-reinforcement number of G, denoted by $r_p(G)$, is the minimum number of edges in G^C that has to be added to G in order to reduce the p-domination number of the resulting graph.

II. RESULTS

A. *p*-Domination Number of the Join of Graphs

In this section, we gave a sharp upperbound of the p-domination number of the join of graphs. We also gave the p-domination number of the join of a complete graph of order $n\geq p$ and any graph.

Theorem 2.1: Let G and H be any graphs and p be a positive integer. If $|V(G)|\geq p$ and $|V(H)|\geq p$, then $\gamma_p(G+H)\leq 2p$.

Proof: Let G and H be any graphs with $|V(G)|\geq p$ and $|V(H)|\geq p$. Let $u_1,u_2,...,u_p\in V(G)$ and $v_1,v_2,...,v_p\in V(H)$. Consider $D=\{u_1,u_2,...,u_p,v_1,v_2,...,v_p\}$. Let $w\in V(G+H)\setminus D$, say $w\in V(H)\setminus D$. Then $|N(w)\cap D|\geq p$. This shows that D is a p-dominating set in $G+H$. Hence, $\gamma_p(G+H)\leq 2p$.

The next Corollary shows that the bound in Theorem 2.1 is sharp.

Corollary 2.2: Let P_m and P_n be paths of order m and n, respectively. If $m,n \geq 9$, then $\gamma_2(P_m + P_n) = 4$.

Proof: By **Theorem 2.1**, $\gamma_2(P_m + P_n) \leq 4$. Suppose $\gamma_2(P_m + P_n) < 4$, say without loss of generality $\gamma_2(P_m + P_n) = 3$. Let D be a 2-dominating set of $P_m + P_n$ with $|D| = 3$ and consider the following cases:

Case-1: $|V(P_n) \cap D| = 3$

Let $P_n = (1,2,3,...,n)$ and consider the partition

$$\alpha = \begin{cases} \{\{1,2,3\},\{4,5,6\},...,\{n-2,n-1,n\}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-3,n-2,n-1\},\{n\}\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-4,n-3,n-2\},\{n-1,n\}\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

of $V(P_n)$. Since $|V(P_n)| \geq 9$, $|\alpha| \geq 3$. By the *Pigeonhole principle,* there exists $A = \{u_i, u_{i+1(\text{mod}\, n)}, u_{i+2(\text{mod}\, n)}\} \in \alpha$ such that $|A \cap D| = 1$. Let $v \in u_{i+1(\text{mod}\, n)}$. Then $|N(v) \cap D| = 1$. This is a contradiction.

Case-2: $|V(P_n) \cap D| = 2$

Let $P_n = (1,2,3,...,n)$ and consider the partition

$$\alpha = \begin{cases} \{\{1,2,3\},\{4,5,6\},...,\{n-2,n-1,n\}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-3,n-2,n-1\},\{n\}\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{\{1,2,3\},\{4,5,6\},...,\{n-4,n-3,n-2\},\{n-1,n\}\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

of $V(P_n)$. Since $|V(P_n)| \geq 9$, $|\alpha| \geq 3$. By the *Pigeonhole principle,* there exists $A = \{u_i, u_{i+1(\text{mod}\, n)}, u_{i+2(\text{mod}\, n)}\} \in \alpha$ such that $|A \cap D| = 0$. Let $v \in u_{i+1(\text{mod}\, n)}$. Then $|N(v) \cap D| = 1$. This is a contradiction.

Therefore, $\gamma_2(P_m + P_n) = 4$.

The next Theorem gave the *p*-domination number of the join of a complete graph of order $n \geq p$ and any graph.

Theorem 2.3: Let G be a graph of order m and K_n be a complete graph of order *n*. If $p \leq n$, then $\gamma_p(K_n + G) = p$.

Proof: Let *G* be a graph of order *m* and $K_n = (\{u_1, u_2, ..., u_n\}, E(K_n))$ be a complete graph of order *n*. Let $D = \{u_1, u_2, ..., u_p\}$. Then clearly D is a *p*-dominating set in $K_n + G$. Thus, $\gamma_p(K_n + G) \leq p$. Since $p \leq \gamma_p(K_n + G)$, we must have $\gamma_p(K_n + G) = p$.

B. *2*-Domination and *3*-Domination Number of Fans

In this section, we gave the 2-domination number and 3-domination number of fans. Lemma 2.4 is found in [16] and Lemma 2.5 is an observation in [14].

Lemma 2.4: $\gamma(P_n) = \lceil n/3 \rceil$.

Lemma 2.5: $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.

Observation 2.6: Let P_n be a path of order *n*. Then $\gamma(P_n) < \gamma_2(P_n)$, that is, if *S* is a minimum dominating set then it cannot be a 2-dominating set.

Theorem 2.7: *D* is a minimum 2-dominating set in F_n ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

Proof: Let *D* be a minimum 2-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$) and $D \neq D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

Then there exist a subgraph $P = (u_i, u_{i+1(\text{mod}\, n)}, u_{i+2(\text{mod}\, n)})$ or $(u_{n-1}, u_n,)$ of $(u_1, u_2, \ldots, u_n)$ such that $V(P) \cap D = \varnothing$. If $P = (u_i, u_{i+1(\text{mod}\, n)}, u_{i+2(\text{mod}\, n)})$, then we let $v = u_{i+1(\text{mod}\, n)}$. While, if $P = (u_{n-1}, u_n,)$, then we let $v = u_n$. Thus, $|N(v) \cap D| = 1$. This is a contradiction.

Conversely, suppose that $D^* = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3 \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

and *D** is not a minimum 2-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$). Clearly, *D** is a 2-dominating set. Let *D* be a minimum 2-dominating set. Then $|D| < |D^*|$. Consider the following cases:

Case-1: $u \in D$
If $u \in D$, then $D \setminus \{u\}$ is not a dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, there exists $v \in \{u_1, u_2, \ldots, u_n\}$ such that $N(v) = \{u\}$. Thus, $|N(v) \cap D| = 1$. This is a contradiction.

Case-2: $u \notin D$
If $u \notin D$, then we note that *D'* is a minimum dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, by **Observation 2.6** *D'* cannot be a 2-dominating set of $(u_1, u_2, \ldots, u_n)$, and so is *D*. Since $u \notin D$, *D* cannot be a 2-dominating set of F_n. This is a contradiction.

Corollary 2.8: Let F_n ($n \geq 3$) be a fan of order $n+1$. Then $\gamma_2(F_n) = \lceil n/3 \rceil + 1$.

Observation 2.9: Let P_n be a path of order *n*. Then $\gamma_2(P_n) < \gamma_3(P_n)$, that is, if *S* is a minimum 2-dominating set of P_n then it cannot be a 3-dominating set.

Theorem 2.10: *D* is a minimum 3-dominating set in F_n ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

Proof: Let *D* be a minimum 3-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \geq 3$) and suppose that $D \neq D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

Then there exist a subset $A = \{u_i, u_{i+1(\text{mod}\, n)}\}$ or $\{u_n\}$ or $\{u_1\}$ of $\{u_1, u_2, \ldots, u_n\}$ such that $A \cap D = \varnothing$. Let $v = u_{i+1(\text{mod}\, n)}$ or u_1 or u_n. Then, $|N(v) \cap D| < 3$. This is a contradiction.

Conversely, suppose that $D^* = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2). \end{cases}$$

and D^* is not a minimum 3-dominating set in $F_n = (\{u\}, \varnothing) + (u_1, u_2, \ldots, u_n)$ ($n \ge 3$). Clearly, D^* is a 3-dominating set. Let D be a minimum 3-dominating set. Then $|D| < |D^*|$. Consider the following cases:

Case-1: $u \in D$

If $u \in D$, then $D \setminus \{u\}$ is not a 2-dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, there exists $v \in \{u_1, u_2, \ldots, u_n\}$ such that $|N(v)| < 2$. Thus, $|N(v) \cap D| < 3$. This is a contradiction.

Case-2: $u \notin D$

If $u \notin D$, then we note that *D'* is a minimum 2-dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, by **Observation 2.9** *D'* cannot be a 3-dominating set of $(u_1, u_2, \ldots, u_n)$, and so is *D*. Since $u \notin D$, *D* cannot be a 3-dominating set of F_n. This is a contradiction.

Corollary 2.11: Let F_n ($n \ge 3$) be a fan of order $n+1$. Then $\gamma_3(F_n) = \lceil n/2 \rceil + 1$.

C. *2*-Reinforcement and *3*-Reinforcement Number of Fans

In this section, we present the 2-reinforcement number and 3-reinforcement number of fans. Remark 2.12 is implied in an observation in [15].

Remark 2.12: Let P_n be a path of order *n*. Then

$$r(P_n) = \begin{cases} 1 & , \text{ if } n \equiv 1 \pmod 3) \\ 2 & , \text{ if } n \equiv 2 \pmod 3) \\ 3 & , \text{ if } n \equiv 0 \pmod 3). \end{cases}$$

Observation 2.13: Let P_n be a path of order *n*. Then

$$r_2(P_n) = \begin{cases} 1 & , \text{ if } n \equiv 0 \pmod 2) \\ 3 & , \text{ if } n \equiv 1 \pmod 2). \end{cases}$$

Theorem 2.14: Let F_n ($n \ge 4$) be a fan of order $n+1$. Then $r_2(F_n) = r(P_n)$.

Proof: Let $F_n = (\{u\}, \varnothing) + (u_1, u_2, ..., u_n)$ be a fan graph of order $n+1$. By **Theorem 2.7**, *D* is a minimum 2-dominating set in F_n ($n \ge 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_2, u_5, \ldots, u_{n-4}, u_{n-1}\} & , \text{ if } n \equiv 0 \pmod 3) \\ \{u_2, u_5, \ldots, u_{n-5}, u_{n-2}, u_{n-1} \text{ or } u_n\} \text{ or } \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 1 \pmod 3) \\ \{u_2, u_5, \ldots, u_{n-6}, u_{n-3}, u_{n-1} \text{ or } u_n\} & , \text{ if } n \equiv 2 \pmod 3) \end{cases}$$

If $n \equiv 1 (\bmod 3)$, then $D^* = D \setminus \{u_n\}$ is a 2-dominating set in $F_n + u_2 u_n$. If $n \equiv 2 (\bmod 3)$, then $D^* = D \setminus \{u_n\}$ is a 2-dominating set in $F_n + u_2 u_{n-1} + u_2 u_n$. If $n \equiv 0 (\bmod 3)$, then $D^* = D \setminus \{u_n\}$ is a 2-dominating set in $F_n + u_2 u_{n-2} + u_2 u_{n-1} + u_2 u_n$. Hence, by Remark 2.12 $r_2(F_n) \le r(P_n)$. Suppose $r_2(F_n) < r(P_n)$ for $n \equiv 0 \text{ or } 2 (\bmod 3)$. Then $D \setminus \{u\}$ must be a 2-dominating set in $F_n^{'}$ (where $F_n^{'}$ is the graph obtained from F_n by adding edges), that is, $D \setminus \{u\}$ must be a 2-dominating set in $P_n^{'}$ (where $P_n^{'}$ is the graph obtained from P_n by adding edges) – which is not possible.

Theorem 2.15: Let F_n ($n \geq 3$) be a fan of order $n+1$. Then $r_3(F_n) = r_2(P_n)$.

Proof: Let $F_n = (\{u\}, \emptyset) + (u_1, u_2, ..., u_n)$ be a fan graph of order $n+1$. By **Theorem 2.10**, D is a minimum 3-dominating set in F_n ($n \geq 3$) if and only if $D = D' \cup \{u\}$ where

$$D' = \begin{cases} \{u_1, u_3, \ldots, u_{n-3}, u_{n-1}, u_n\} & , \text{ if } n \equiv 0 \pmod 2 \\ \{u_1, u_3, \ldots, u_{n-2}, u_n\} & , \text{ if } n \equiv 1 \pmod 2. \end{cases}$$

If $n \equiv 0 (\mathrm{mod}\, 2)$, then $D^* = D \setminus \{u_n\}$ is a 3-dominating set in $F_n + u_1u_n$. If $n \equiv 1 (\mathrm{mod}\, 2)$, then $D^* = D \setminus \{u_n\}$ is a 3-dominating set in $F_n + u_1u_n + u_1u_{n-1} + u_3u_n$. Hence, by Observation 2.13 $r_3(F_n) \leq r_2(P_n)$. Suppose $r_3(F_n) < r_2(P_n)$ for $n \equiv 1 (\mathrm{mod}\, 3)$. Then $D \setminus \{u\}$ must be a 3-dominating set in $F_n^{'}$ (where $F_n^{'}$ is the graph obtained from F_n by adding edges), that is, $D \setminus \{u\}$ must be a 3-dominating set in $P_n^{'}$ (where $P_n^{'}$ is the graph obtained from P_n by adding edges) – which is not possible.

REFERENCES

1. D. Bakhshesh, M. Farchi, M. R. Hooshmandasl, 2-Domination Number of generalized Petersen graphs. Proc. Indian Acad. Sci.(math.Sci.), 128:17 (2018) 1-12.
2. M. Blidia, M. Chellali, L. Volkmann, Some Bounds on the *p*-Domination Number in Trees. Discrete Mathematics 306 (2006) 2031-2037.
3. Y. Caro, Y. Roditty, A note on the *k*-Domination Number of a Graph. International Journal of Mathematics and Mathematical Sciences vol.13 no.1 (1990), 205-206.
4. E. De La Viña, W. Goddard, M. A. Henning, R. Pepper, E. R. Vaughan, Bounds on the *k*-Domination Number of a Graph. Applied Mathematics Letters 24 (2011) 996-998.
5. J. Fink, M. S. Jacobson, *n*-Domination in Graphs: Graph Theory with Applications to Algorithms and Computer Science. New York: John Wiley and Sons (1985).
6. J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita, L. Volkmann, Independence and 2-Domination in Bipartite Graphs. Australasian Journal of Combinatorics 40 (2008) 265-268.
7. J. L. Gross, J. Yellen, Handbook of Graph Theory. Corporate Blvd, Boca Raton, Florida: CRC Press LLC, N. W. (2000).
8. Y. Lu, J. M. Xu, The *p*-Domination Number of Complete Multipartite Graphs. Fundamental Research Fund of NPU (2000).
9. Y. Lu, F. Hu, J. Xu, On the *p*-reinforcement and the complexity. Journal of Combinatorial Optimization 29:2 (2015) 389-405.
10. Y. Lu, J. Xu, Trees with maximum *p*-reinforcement number. Discrete Applied Mathematics 175 (2014) 43-54.
11. Y. Lu, W. Song, H. Yang, Trees with 2-reinforcement number three. Bulletin of the Malaysian Mathematical Sciences Society 39:2 (2016) 821-838.
12. J. Mohan, I. Kelkar, Restrained 2-Domination Number of Complete Grid Graphs. International Journal of Applied Mathematics and Computation 4(4) (2012) 352-358.
13. D. Rautenbach, L. Volkmann, New Bounds on the *k*-Domination Number and the *k*-tuple Domination Number. Applied Mathematics Letters 20 (2007) 98-102.
14. D. K. Thakkar, D. D. Pandya, 2-Domination Number and 2-Bondage Number of Complete Grid Graph. International Journal of Advanced Engineering Research and Studies 1:1 (2011) 24-29.
15. J.R.S. Blair, W. Goddard, S.T. Hedetnieme, S. Horton, P. Jones and G. Kubicki, On the domination and reinforcement numbers in trees. Discrete Math., 308 (2008): 1165-1175.
16. G. Chartrand, P. Zhang, Introduction to Graph Theory, Mc Graw Hill, Higher Education, 2005.

Source of support: Nil, Conflict of interest: None Declared.