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A NEW LOG-LOGISTIC DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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ABSTRACT

In this work, we derive a new two parameter continuous log-logistic distribution with a strong motivation and wide applications. The new model is constructed using the zero truncated Poisson distribution. Some of the fundamental properties of the new model are derived. Four applications to real data sets are provided for illustrating the wide applicability of the new model. The new two parameter model is the best among other ten competitive models which have two parameters or more. We estimated the unknown parameters via the method of maximum likelihood. The new log-logistic model provides the small values for AIC, BIC, CAIC and HQIC.

Keywords: *Burr XII Distribution; Log-Logistic Distribution; Maximum Likelihood; Generating Function; Moments; Zero Truncated Poisson.*

1. INTRODUCTION AND PHYSICAL MOTIVATION

Consider the well-known two parameter (a, b) lifetime Burr XII (BrXII) model with cumulative distribution function (CDF)

$$\mathbf{G}_{\text{BrXII}}(x;a,b) = 1-\left(1+x^{a}\right)^{-b},$$

when $b=1$ the BrXII reduces to the one parameter Log-Logistic (LL) model with CDF

$$\mathbf{G}_{\text{LL}}(x;a) = 1-\left(1+x^{a}\right)^{-1}, \tag{1}$$

where a dominate the shape of the model. The corresponding probability density function (PDF) of (1) is given by

$$\mathbf{g}_{\text{LL}}(x;a) = ax^{a-1}\left(1+x^{a}\right)^{-2}. \tag{2}$$

Following Yousof *et al.* (2016), we can define the Rayleigh Log-Logistic (RLL) with the CDF given by

$$H_{\text{RLL}}(x;a) = 1-\mathbf{e}^{-x^{2a}}, \tag{3}$$

Suppose a system has N subsystems functioning independently at a given time t where N has the well-known zero truncated Poisson (ZTP) distribution with parameter λ. The probability mass function (PMF) of N is given by

$$p_{\text{ZTP}}^{(\lambda)}\left(N=n\right) = \frac{\mathbf{e}^{-\lambda}\lambda^{n}}{n!\left(-\mathbf{e}^{-\lambda}+1\right)}\Big|_{(n=1,2,\dots)}, \tag{4}$$

For the ZTP random variable (r.v.), the expected value $E\ (N \mid \lambda)$ and the variance $Var(N \mid \lambda)$ are given by

$$\mathbf{E}(N \mid \lambda) = \lambda / \left(-\mathbf{e}^{-\lambda}+1\right)$$

and

$$Var(N \mid \lambda) = \frac{\lambda+\lambda^{2}}{-\mathbf{e}^{-\lambda}+1} - \left(\frac{\lambda}{-\mathbf{e}^{-\lambda}+1}\right)^{2},$$

respectively. Suppose that the failure time of each subsystem has the RLL. Let Y_i denote the failure time of the ith subsystem and let

$$X = \min\{Y_1, Y_2, \cdots, Y_N\},$$

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then the conditional CDF of X given N is

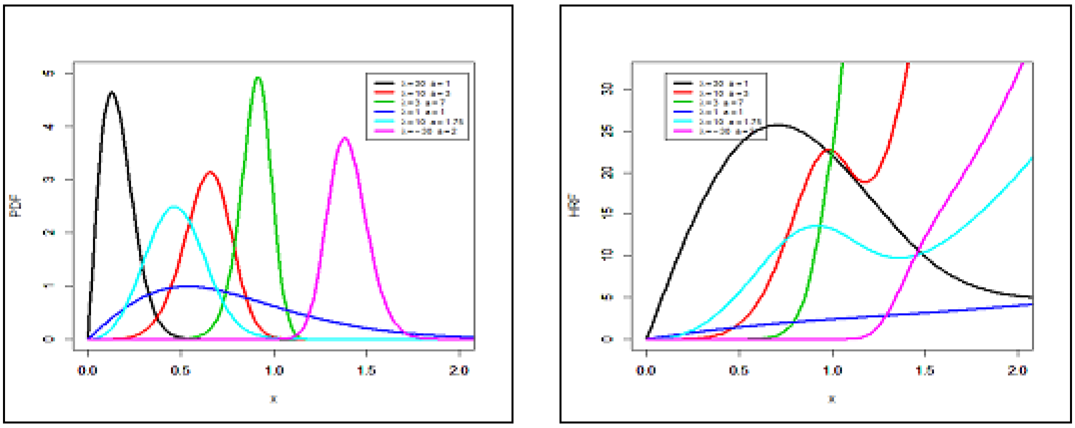
$$F\left(x\,|\,N\right)=1-\Pr\left(X>x\,|\,N\right)=1-\left[1-H_{\mathrm{RLL}}(x;a)\right]^{N}.$$

Therefore, the new model called the Poisson RLL (PRLL) with CDF can be expressed as

$$F_{\mathrm{PRLL}}^{(\lambda,a)}(x)=\frac{1-\mathbf{e}^{-\lambda\left(1-\mathbf{e}^{-x^{2a}}\right)}}{1-\mathbf{e}^{-\lambda}}, \tag{5}$$

with the corresponding PDF as

$$f_{\mathrm{PRLL}}^{(\lambda,a)}(x)=2\lambda a\frac{x^{a-1}\left(1+x^{a}\right)\mathbf{e}^{-\left[x^{2a}+\lambda\left(1-\mathbf{e}^{-x^{2a}}\right)\right]}}{\left(1-\mathbf{e}^{-\lambda}\right)\left[1-\left(1+x^{a}\right)^{-1}\right]}. \tag{6}$$



(a) **(b)**

Figure 1: PDFs and HRFs plots for the PRLL model

The PRLL density can be left-skewed, right-skewed, unimodal and symmetric (see Figure 1 (a)), whereas the PRLL HRF can be upside down then bathtub or upside down or increasing or upside down then increasing (see Figure 1 (b)).

Some useful extensions of the LL model can be found in Brito *et al.* (2017), Merovci *et al.* (2017), Hamedani *et al.* (2017), Cordeiro *et al.* (2018), Hamedani *et al.* (2018), Korkmaz *et al.* (2018), Ibrahim (2019), Hamedani *et al.* (2019), Aboray and Butt (2019), Korkmaz *et al.* (2019) and Yousof *et al.* (2019).

2. MATHEMATICAL PROPERTIES

Useful expansions

After some algebra the PDF in (6) can be expressed as

$$f_{\mathrm{PRLL}}^{(\lambda,a)}(x)=\sum_{r=0}^{\infty}\upsilon_{r}\,\mathbf{g}_{\mathrm{LL}}(x;a,1+r), \tag{7}$$

where

$$\upsilon_{r}=\frac{(-1)^{r}}{(1+r)!\Gamma\left(2j+k+2-r\right)}\times\sum_{j,k=0}^{\infty}\frac{2\lambda^{1+h}\left(-1\right)^{j}\Gamma\left(3+2j+k\right)\Gamma\left(2j+k+2\right)}{j!k!\left(-\mathbf{e}^{-\lambda}+1\right)\Gamma\left(2j+3\right)}\times\sum_{h,i=0}^{\infty}\frac{\left(-1\right)^{h+i}\Gamma\left(h+1\right)\left(i+1\right)^{j}}{i!\Gamma\left(h+1-i\right)},$$

and

$$\mathbf{g}_{\mathrm{LL}}(x;a,1+r)=a(1+r)x^{a-1}\left(1+x^{a}\right)^{-r-2}$$

is the LL density with parameters a and $(1+r)$. Similarly, the CDF of the PRLL can also be expressed as

$$F_{\text{PRLL}}^{(\lambda,a)}(x)=\sum_{r=0}^{\infty}\upsilon_r\ \mathbf{G}_{\text{LL}}(x;a,1+r),$$

where

$$\mathbf{G}_{\text{LL}}(x;a,1+r)=1-\left(1+x^a\right)^{-(1+r)}$$

is the LL CDF with parameters a and $(1+r)$.

Quantile and random number generation

The quantile function (QF) of X, where $X\sim$ PRLL (λ,a), is obtained by inverting (5) as

$$Q(u)=\left(-\ln\left\{1+\frac{1}{\lambda}\ln\left[-u\left(1-\mathbf{e}^{-\lambda}\right)+1\right]\right\}\right)^{\frac{1}{2a}}.$$

Simulating the PRLL r.v. is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the r.v. $X=Q(U)$ follows (5).

Moments

The $r^{(th)}$ ordinary moment of X, say $\mu_r^{'}$, follows from (7) as

$$\mu_n^{'}=\mathbf{E}\left(X^n\right)=\sum_{r=0}^{\infty}\upsilon_r\ (1+r)\ B\left((1+r)-na^{-1},1+na^{-1}\right)|_{[n<(1+r)a]}, \tag{8}$$

Taking $n=1$ in (8) gives the mean of X. The mean of X is calculated numerically for some selected values of parameters using equation (8) and results are presented in Table 1. The $n^{(th)}$ incomplete moment of X is defined by

$$m_n(t)=\int_{-\infty}^{t}x^r\ f(x)dx,$$

we can write from (7)

$$m_n(t)=\sum_{r=0}^{\infty}\upsilon_r\ (1+r)\ B\left(t^a;(1+r)-na^{-1},1+na^{-1}\right)|_{[n<(1+r)a]},$$

where

$$B(\xi_1,\xi_2)=\int_0^{\infty}x^{\xi_1-1}(1+x)^{-(\xi_1+\xi_2)}dx$$

and

$$B(q;\xi_1,\xi_2)=\int_0^{q}x^{\xi_1-1}(1+x)^{-(\xi_1+\xi_2)}dx$$

are the beta and the incomplete beta functions of the second type, respectively.

The moment generating function (MGF)

The MGF of , say $M_X(t)=\mathbf{E}\left[\exp(tX)\right]$, can be obtained via (7)

$$M_X(t;\lambda,a)=\sum_{r=0}^{\infty}\upsilon_r\ \ M_X\left(t;a,(1+r)\right),$$

where $M_X\left(t;a,(1+r)\right)$ is the MGF of the BrXII distribution with parameters a, $(1+r)$. Paranaíba *et al.* (2011) derived a simple way for getting MGF of the three parameter BrXII model. Similarly the MGF, say $M_X(t;\lambda,a)$, of the PRLL (λ,a) model. $\forall\ t<0$, we can write

$$M_X(t;a)=a\int_0^{\infty}y^{a-1}\left(1+y^a\right)^{-2}\exp(yt)\,dy.$$

Next, we require the Meijer **G** function defined by

$$\mathbf{G}_{(p,q)}^{(m_1,n)}\left(x \,|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right)=\frac{1}{2\pi i}\int_{(\mathbf{L})}\frac{\prod_{j=1}^{m_1}\Gamma\left(b_j+t\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-t\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j+t\right)\prod_{j=m_1+1}^{p}\Gamma\left(1-b_j-t\right)}x^{-t}dt,$$

where $i=\left(-1\right)^{\frac{1}{2}}$ is the complex unit and $\left(\mathbf{L}\right)$ denotes the integration path (see Gradshteyn and Ryzhik, 2000, Sec. 9.3). The Meijer G-function contains as particular cases many integrals with elementary and special functions (*et al*., 1986). We now assume that $a=m_1/b$, where m_1 and b are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. We have the following result, which holds for m_1 and b positive integers, $\mu>-1$ and $p>0$ (et al., 1992, p. 21),

$$\mathbf{I}\left(p,\mu,m_1,m_2\right)|_0^{\infty}=\int_0^{\infty}\exp\left(-px\right)x^{\mu}\left(1+x^{\frac{m_1}{\beta}}\right)^{m_2}dx$$

$$=\boldsymbol{\tau}\left(p,\mu,m_1,m_2\right)\left[\mathbf{G}_{(1+m_1,1)}^{(1,1+m_1)}\left(\frac{m_1^{m_1}}{p^{m_1}} \,|\, \begin{matrix}\Delta\left(m_1,-\mu\right),\Delta\left(1,m_2+1\right) \\ \Delta\left(1,0\right)\end{matrix}\right)\right],$$

where φ

$$\boldsymbol{\tau}\left(p,\mu,m_1,m_2\right)=\frac{m_1^{\mu+\frac{1}{2}}}{p^{\mu+1}\Gamma\left(-m_2\right)}\left(2\pi\right)^{-\frac{m_1-1}{2}}$$

and

$$\Delta\left(\varphi_1,\varphi_2\right)=\frac{\varphi_2}{\varphi_1},\frac{\varphi_2+1}{\varphi_1},\dots,\frac{\varphi_2+\varphi_1}{\varphi_1}.$$

then

$$M_X\left(t;a\right)|_{(t<0)}=m_1\left[\mathbf{I}\left(-t,m_1-1,m_1,-2\right)\right].$$

So, the MGF of X can be expressed as

$$M_X\left(t;\lambda,a\right)=m_1\sum_{r=0}^{\infty}\upsilon_r\left[\mathbf{I}\left(-t,\frac{m_1}{1+r}-1,\frac{m_1}{1+r},-\left(2+r\right)\right)\right].$$

Numerical analysis for mean, skewness and kurtosis of the PRLL distribution

The skewness (Skew) and kurtosis (Kurt) measures can be calculated from the ordinary moments using well-known relationships. The mean, variance, skew and kurt of the PRLL distribution are computed numerically for some selected values of parameter λ and a using the R software. From Table 1, we conclude that:

1. The skew of the PRLL distribution is always positive.
2. The kurtosis of the PRLL distribution can be less than 3 and more than 3.
3. The mean of X increases as λ increases.
4. The mean of X decreases as a decreases.

Table 1: Mean, variance, skewness and kurtosis of the PRLL distribution.

λ	a	mean	variance	skew	kurt
−10	0.1	0.001488	0.8898579	766.4538	638005.6
−8		0.0606214	36.91333	119.5657	15488.92
−6		2.326098	1433.597	19.1665	398.4723
−4		80.62927	44070.78	2.798808	9.888308
−3		447.2091	82274.13	0.3211897	1.756453
−10	0.20	213.7065	141326.7	1.243133	2.640785
	0.15	0.0755137	57.50894	106.9408	11880.12
	0.10	0.0014879	0.8898579	766.4538	638005.6
	0.05	0.000366	0.1750539	1579.388	2821510
	0.01	0.0001386	0.06788482	2541.372	7290883
−5	0.01	0.3031502	151.1797	54.14657	3301.037
	0.05	1.766393	913.3627	22.30285	557.2926
	0.10	13.92195	8480.119	7.668886	64.91905

3. PARAMETER ESTIMATION

Consider the estimation of the unknown parameters (λ, a) of the PRLL model from the complete samples by maximum likelihood (ML) method. Suppose that $x_1, \cdots, x_n$ be a random sample from the PRLL model with parameter vector $\Xi = (\lambda, a)^{\tau}$. The log-likelihood function ($\ell_n(\Xi)$) for Ξ is given by

$$\ell_n(\Xi) = n\log 2 + n\log\lambda + n\log a - n\log\left[-\mathbf{e}^{-\lambda} + 1\right]$$
$$+(a-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log s_i + \sum_{i=1}^{n}\log\left(1 - s_i^{-1}\right)$$
$$-\lambda\sum_{i=1}^{n}(1 - z_i) + \sum_{i=1}^{n}\log z_i,$$

where $s_i = 1 + x_i^a$ and $z_i = \exp\left[-\left(s_i - 1\right)^2\right]$. The above $\ell_n(\Xi)$ can be maximized numerically via `SAS` (`PROC NLMIXED`) or `R` (`optim`) or `Ox` program (via sub-routine `MaxBFGS`), among others. The components of the score vector $\mathbf{U}(\Theta) = \frac{\partial \ell}{\partial \Xi} = \left(\frac{\partial \ell_n(\Xi)}{\partial \lambda}, \frac{\partial \ell_n(\Xi)}{\partial a}\right)^{?}$ are easily to be derived.

4. APPLICATIONS

We provide four applications to illustrate the importance, potentiality and flexibility of the PRLL model. For these data, we compare the PRLL distribution, with BrXII, Marshall-Olkin BrXII (MOBrXII), Topp Leone BrXII (TLBrXII), Zografos-Balakrishnan BrXII (ZBBrXII), Five Parameters beta BrXII (FBBrXII), BBrXII, B exponentiated BrXII (BEBrXII), Five Parameters Kumaraswamy BrXII (FKwBrXII) and KwBrXII distributions, all those models are given in Yousof et al. (2018), Altun et al. (2018a and b), Aboray and Butt (2019) and Yousof *et al.* (2019).

Data set **I:**
{0.98, 5.560, 5.08, 0.390, 1.57, 3.19, 4.90, 2.930, 2.85, 2.77, 2.76, 1.730, 2.48, 3.680, 1.08, 3.220, 3.75, 3.22, 3.70, 2.740, 2.73, 2.50, 3.60, 3.110, 3.27, 2.870, 1.47, 3.11, 4.42, 2.40, 3.15, 2.67, 3.310, 2.81, 2.560, 2.17, 4.910, 1.59, 1.18, 2.480, 2.03, 1.69, 2.430, 3.39, 3.56, 2.830, 3.68, 2.00, 3.510, 0.85, 1.610, 3.28, 2.950, 2.81, 3.15, 1.920, 1.84, 1.220,

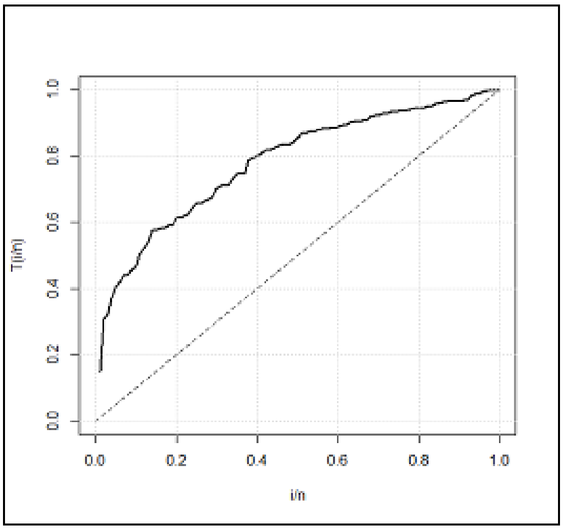
2.17, 1.61, 2.120, 3.09, 2.97, 4.2, 2.35, 1.410, 1.59, 1.120, 1.69, 2.79, 1.890, 1.87, 3.39, 3.33, 2.550, 3.68, 3.19, 1.71, 1.250, 4.7, 2.88, 2.960, 2.55, 2.59, 2.97, 1.57, 2.170, 4.38, 2.030, 2.820, 2.53, 3.310, 2.38, 1.360, 0.81, 1.170, 1.84, 1.80, 2.050, 3.65}called breaking stress data. This data set consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006).

Data set **II**:
{0.10, 0.33, 0.44, 0.560, 0.59, 0.720, 0.74, 0.77, 0.920, 0.93, 0.96, 1.00, 1.00, 1.02, 1.05, 1.070, 0.7, 1.080, 1.08, 1.080, 1.09, 1.12, 1.13, 1.150, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.510, 2.53, 2.540, 2.54, 2.780, 2.93, 3.270, 3.42, 3.47, 3.610, 4.020, 4.32, 4.58, 5.550} called survival times. In this application, we work with the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal (1960).

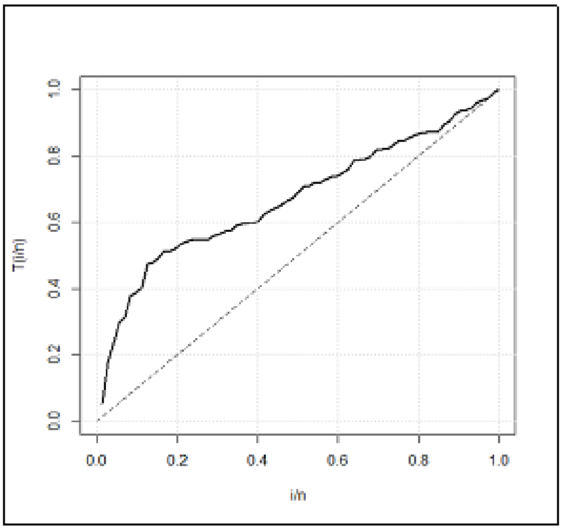
Data set **III:**
{5.90, 20.4, 14.90, 16.2, 17.20, 7.8, 6.10, 9.20, 10.20, 9.6, 13.30, 8.5, 21.60, 18.5, 5.10, 6.70, 17, 8.60, 9.7, 39.20, 35.7, 15.70, 9.7, 10, 4.10, 36, 8.50, 8, 9.20, 26.2, 21.9,16.70, 21.3, 35.4, 14.3, 8.50, 10.6, 19.10, 20.5, 7.1, 7.70, 18.1, 16.5, 11.9, 7, 8.6,12.5, 10.30, 11.2, 6.10, 8.4, 11, 11.60, 11.9, 5.2, 6.80, 8.9, 7.1, 10.80} called taxes revenue data. The actual taxes revenue data (in 1000 million Egyptian pounds).

Data set **IV:**
{65, 156, 121, 4, 39, 100, 134, 16, 108, 143, 56, 16, 22, 3, 4, 2, 3, 8, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 4, 3, 30, 4, 43} called leukaemia data. This real data set gives the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia.

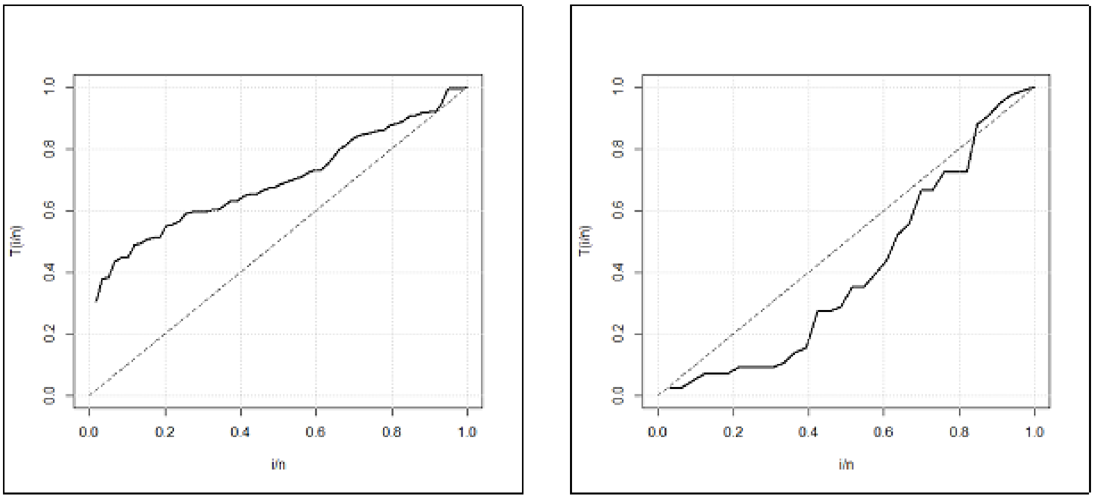
The total time test (TTT) plot for the 4 real data sets is presented in Figure 2. These plots indicate that the empirical HRFs of data sets **I**, **II** and **III** are increasing and bathtub for data set **IV**.



Data sets **I**



Data sets **II**



Data sets **III** Data sets **IV**

Figure 2: TTT plots for the 4 data sets.

We consider the following goodness-of-fit statistics: the Akaike information criterion (AICr), Bayesian information criterion (BICr), Hannan-Quinn information criterion (HQICr), consistent Akaike information criterion (CAICr), where

$$\mathrm{AICr} = -2\ell\left(\widehat{\Xi}\right) + 2k,$$

$$\mathrm{BICr} = -2\ell\left(\widehat{\Xi}\right) + k\log\left(n\right),$$

$$\mathrm{HQICr} = -2\ell\left(\widehat{\Xi}\right) + 2k\log\left[\log\left(n\right)\right]$$

and

$$\mathrm{CAICr} = -2\ell\left(\widehat{\Xi}\right) + 2kn/\left(n-k-1\right),$$

where k is the number of parameters, n is the sample size, $-2\ell\left(\widehat{\Xi}\right)$ is the maximized log-likelihood. Generally, the smaller these statistics are, the better the fit. Tables 2-5 gives the MLEs and standard errors, confidence interval (in parentheses) with AICr, BICr, CAICr and HQICr values for the four data sets respectively. Figures 3-6 gives the estimated PDF, P-P plot, estimated HRF and Kaplan-Meier survival plot for the four data sets respectively. Based on the values in Tables 2-5 and Figure 3-5 the PRLL model provides the best fits as compared to BrXII other models in the four applications with small values for BICr, AICr, CAICr and HQICr.

Table 2: MLEs and standard errors, confidence interval (in parentheses) with AICr, BICr, CAICr and HQICr values for the data set **I**.

Model	$\hat{\lambda}, \hat{\theta}, \hat{a}, \hat{b}, \hat{\gamma}$	AICr, BICr, CAICr, HQICr
BrXII	—,—, 5.941, 0.187,— —,—, (1.279) ,(0.044),— —,—, (3.43,8.45),(0.10,0.27),—	382.94, 388.15, 383.06, 385.05
MOBrXII	—,—, 1.192,4.834,838.73 —,—, (0.952),(4.896),(229.34) —,—, 0, 3.06),(0, 14.43),(389.22,1288.24)	305.78, 313.61, 306.03, 308.96
TLBrXII	—,—, 1.350,1.061,13.728 —,—, 0.378) ,(0.384) ,(8.400) —,—, (0.61, 2.09), (0.31,1.81) ,(0, 30.19)	323.52, 331.35, 323.77, 326.70
KwBrXII	48.103 ,79.516 ,0.351 ,2.730,— (19.348) ,(58.186) ,(0.098) ,(1.077) ,— (10.18,86.03) ,(0,193.56) ,(0.16,0.54), (0.62,4.84),—	303.76, 314.20, 304.18, 308.00
BBrXII	359.683 ,260.097 ,0.175 ,1.123 ,— (57.941) ,(132.213),(0.013),(0.243),— (246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6),—	305.64, 316.06, 306.06, 309.85
BEBrXII	0.381, 11.949, 0.937, 33.402, 1.705 (0.078), (4.635), (0.267), (6.287),(0.478) (0.23,0.53) ,(2.86,21), (0.41,1.5), (21,45), (0.8,2.6)	305.82, 318.84, 306.46, 311.09
FBBrXII	0.421, 0.834, 6.111, 1.674, 3.450 (0.011), (0.943), (2.314), (0.226), (1.957) (0.4,0.44), (0. 2.7), (1.57, 10.7), (1.23, 2.1), (0, 7)	304.26, 317.31, 304.89, 309.56
FKwBrXII	0.542,4.223, 5.313, 0.411, 4.152 (0.137), (1.882), (2.318), (0.497), (1.995) (0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)	305.50, 318.55, 306.14, 310.80
ZBBrXII	123.101,—,0.368, 139.247,— (243.011), —, (0.343), (318.546),— (0, 599.40), —,(0, 1.04), (0, 763.59),—	302.96, 310.78, 303.21, 306.13
PRLL	−9.568, —,0.537,—,— (1.079), —,(0.022),—,— (−11.8,−7.4), —,(0.49,0.58),—,—	**291.2, 296.4, 291.3, 293.3**

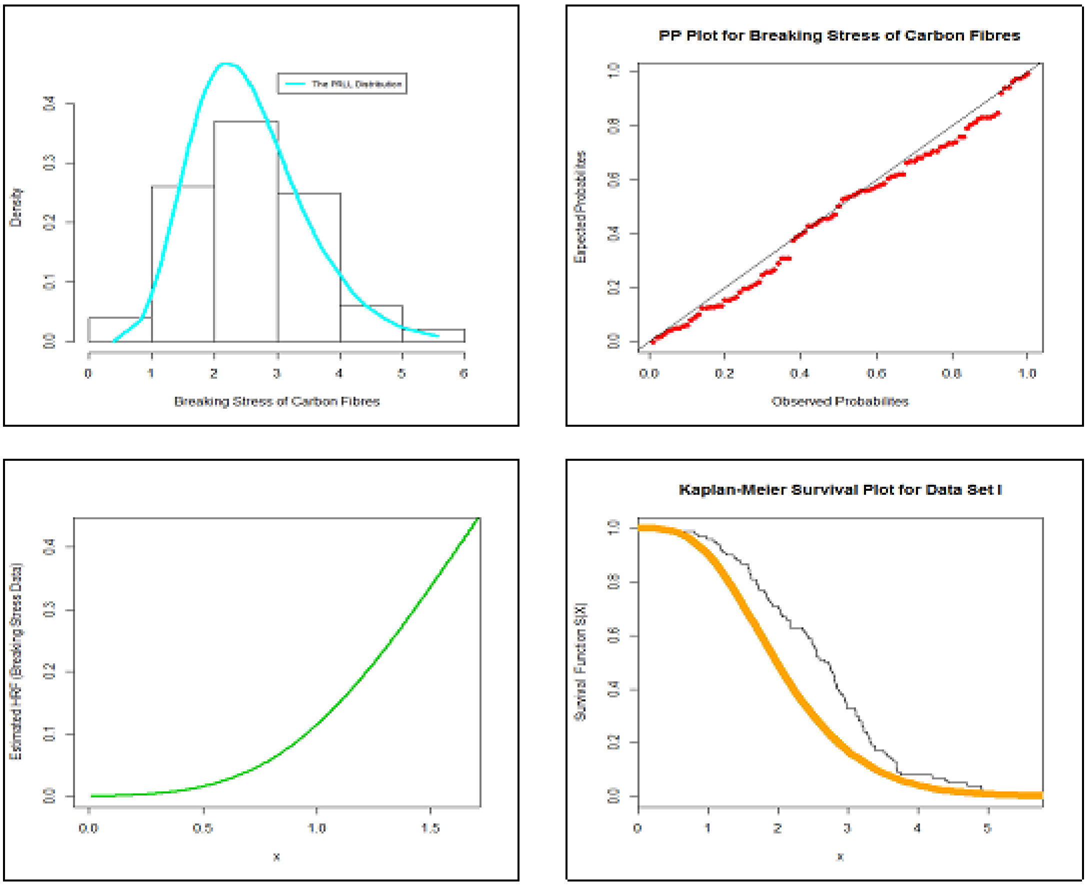


Figure 3: Estimated PDF, P-P plot, Estimated HRF and Kaplan-Meier Survival plot for data set I.

Table 3: MLEs and standard errors, confidence interval (in parentheses) with AICr, BICr, CAICr and HQICr values for the data set II.

Model	$\hat{\lambda}, \hat{\theta}, \hat{a}, \hat{b}, \hat{\gamma}$	AICr, BICr, CAICr, HQICr
BrXII	—,—, 3.102, 0.465, — —,—, (0.538), (0.077),— —,—, (2.05,4.16), (0.31,0.62),—	209.60, 214.15, 209.77, 211.40
MOBrXII	—,—, 2.259,1.533, 6.760 —,—, (0.864), (0.907), (4.587) —,—, (0.57,3.95), (0,3.31), (0, 15.75)	209.74, 216.56, 210.09, 212.44
TLBrXII	—,—, 2.393,0.458,1.796 —,—, (0.907), (0.244),(0.915) —,—, (0.62,4.17),(0, 0.94),(0.002,3.59)	211.80, 218.63, 212.15, 214.52
KwBrXII	14.105,7.424, 0.525, 2.274,— (10.805), (11.850), (0.279),(0.990),— (0, 35.28), (0.30.65), (0, 1.07),(0.33, 4.21),—	208.76, 217.86, 209.36, 212.38
BBrXII	2.555, 6.058,1.800,0.294, — (1.859), (10.391), (0.955),(0.466),— (0, 6.28), (0, 26.42), (0, 3.67),(0, 1.21), —	210.44, 219.54, 211.03, 214.06
BEBrXII	1.876,2.991, 1.780, 1.341, 0.572 (0.094), (1.731), (0.702), (0.816), (0.325) (1.7,2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)	212.10, 223.50, 213.00, 216.60
FBBrXII	0.621, 0.549,3.838, 1.381, 1.665 (0.541), (1.011), (2.785), (2.312), (0.436) (0, 1.7), (0, 2.5), (0, 9.3), (0, 5.9), (0.8, 4.5)	206.80, 218.20, 207.71, 211.30
FKwBrXII	0.558,0.308, 3.999, 2.131, 1.475 (0.442), (0.314), (2.082), (1.833), (0.361) (0, 1.4), (0, 0.9), (0, 3.1), (0, 5.7), (0.76, 2.2)	206.50, 217.90, 207.41, 211.00
PRLL	−3.943, —,0.542,—,— (0.571), —,(0.0334),—,— (−5.04,−2.76), —,(0.48,0.6),—,—	**204.8, 209.4, 205.0, 206.7**

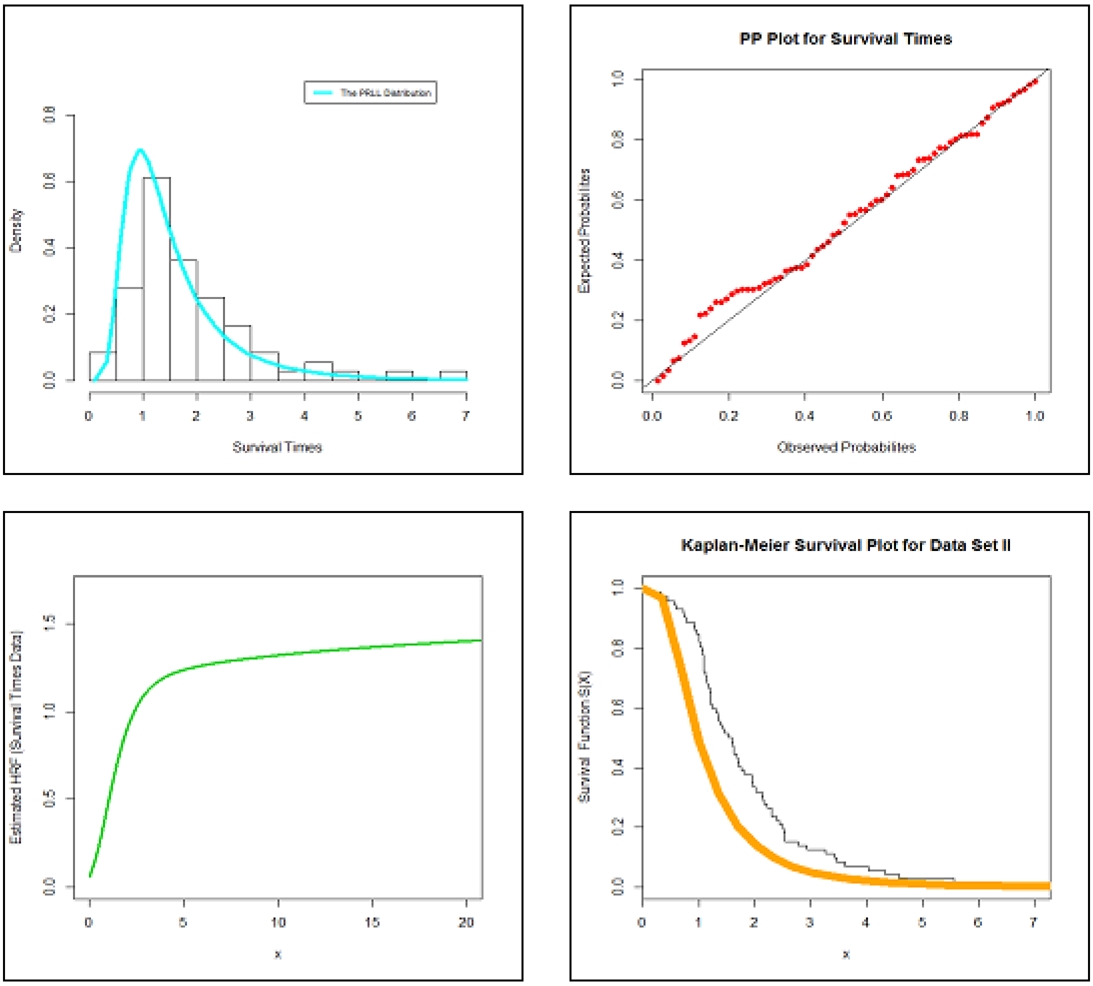


Figure 4: Estimated PDF, P-P plot, Estimated HRF and Kaplan-Meier Survival plot for data set III.

Table 4: MLEs and standard errors, confidence interval (in parentheses) with AICr, BICr, CAICr and HQICr values for the data set **III**.

Model	$\widehat{\lambda}, \widehat{\theta}, \widehat{a}, \widehat{b}, \widehat{\gamma}$	AICr, BICr, CAICr, HQICr
BrXII	—,—, 5.615, 0.072,— —,—, (15.048), (0.194),— —,—, (0, 35.11), (0, 0.45),—	518.46, 522.62, 518.67, 520.08
MOBrXII	—,—, 8.017, 0.419, 70.359 —,—, (22.083), (0.312), (63.831) —,—, (0, 51.29), (0, 1.03), (0, 195.47)	387.22, 389.38, 387.66, 389.68
TLBrXII	—,—, 91.320, 0.012, 141.073 —,—, (15.071), (0.002), (70.028) —,—, (61.78,120.86) (0.008, 0.02) (3.82,278.33)	385.94, 392.18, 386.38, 388.40
KwBrXII	18.130, 6.857, 10.694, 0.081,— (3.689), (1.035), (1.166), (0.012),— (10.89,25.36), (4.83,8.89), (8.41,12.98), (0.06,0.10),—	385.58, 393.90, 386.32, 388.86
BBrXII	26.725, 9.756, 27.364, 0.020,— (9.465), (2.781), (12.351), (0.007),— (8.17,45.27), (4.31,15.21), (3.16,51.57), (0.006,0.03),—	385.56, 394.10, 386.30, 389.10
BEBrXII	2.924, 2.911, 3.270, 12.486, 0.371 (0.564), (0.549), (1.251), (6.938), (0.788) (1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)	387.04, 397.42, 388.17, 391.09
FBBrXII	30.441, 0.584, 1.089, 5.166, 7.862 (91.745), (1.064), (1.021), (8.268), (15.036) (0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)	386.74, 397.14, 387.87, 390.84
FKwBrXII	12.878, 1.225, 1.665, 1.411, 3.732 (3.442), (0.131), (0.034), (0.088), (1.172) (6.13,19.62), (0.97,1.48), (1.56,1.73), (1.24,1.58), (1.43,6.03),—	386.96, 397.36, 388.09, 391.06
PRLL	−39.893, —,0.2864,—,— (8.784),—,(0.0126),—,— (−55.9,−23.9),—,(0.266,0.314),—,—	**382.1, 392.87, 385.3, 387.8**

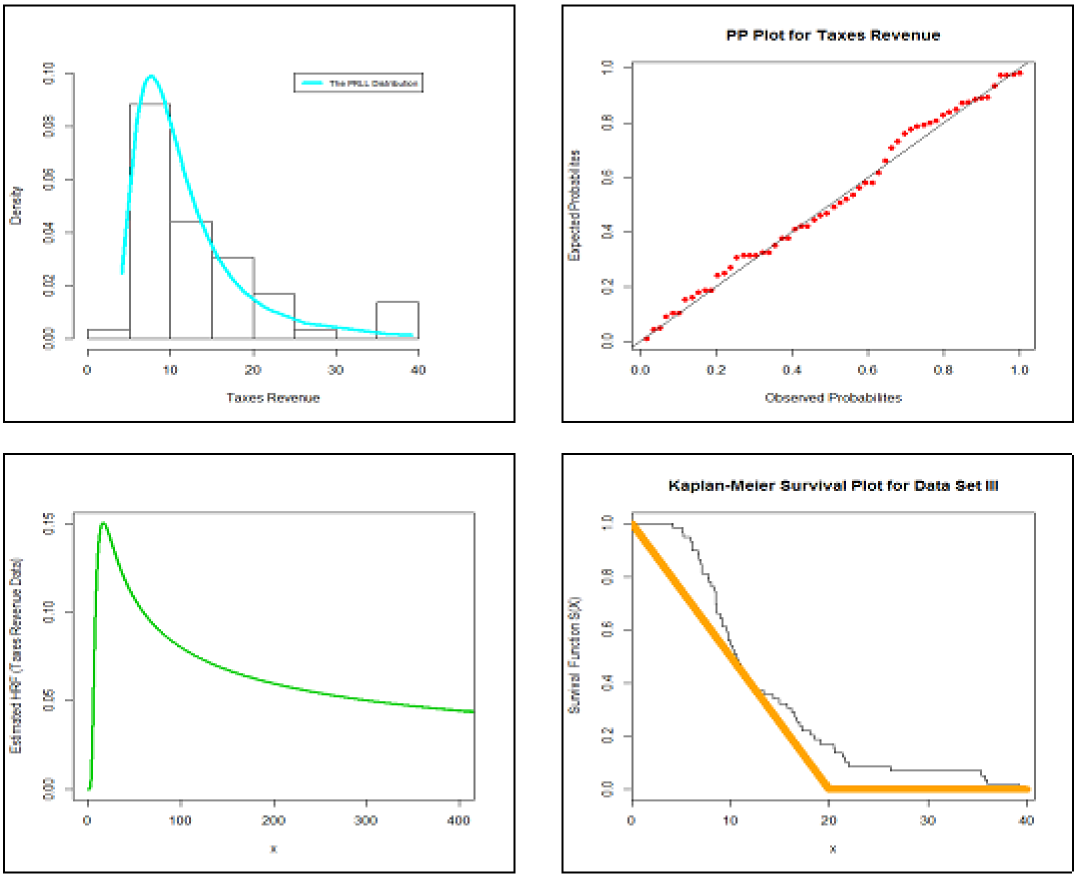


Figure 5: Estimated PDF, P-P plot, Estimated HRF and Kaplan-Meier Survival plot for data set III.

Table 5: MLEs and standard errors, confidence interval (in parentheses) with AICr, BICr, CAICr and HQICr values for the data set **IV**.

Model	$\hat{\lambda}, \hat{\theta}, \hat{a}, \hat{b}, \hat{\gamma}$	AICr, BICr, CAICr, HQICr
BrXII	—,—, 58.711,0.006,— —,—, (42.382), (0.004),— —,—, (0, 141.78), (0, 0.01),—	328.20, 331.19, 328.60, 329.19
MOBrXII	—,—, 11.838, 0.078, 12.251 —,—, (4.368), (0.013), (7.770) —,—, (0, 141.78), (0, 0.01), (0, 27.48)	315.54, 320.01, 316.37, 317.04
TLBrXII	—,—,0.281, 1.882 ,50.215 —,—, (0.288), (2.402), (176.50) —,—, (0, 0.85), (0, 6.59), (0, 396.16)	316.26, 320.73, 317.09, 317.76
KwBrXII	9.201, 36.428, 0.242,0.941,— (10.060), (35.650), (0.167), (1.045),— (0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99),—	317.36, 323.30, 318.79, 319.34
BBrXII	96.104, 52.121, 0.104, 1.227,— (41.201), (33.490), (0.023), (0.326),— (15.4,176.8),(0, 117.8), (0.6, 0.15), (0.59,1.9),—	316.46, 322.45, 317.89, 318.47
BEBrXII	0.087, 5.007, 1.561, 31.270, 0.318 (0.077), (3.851), (0.012), (12.940), (0.034) (0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)	317.58, 325.06, 319.80, 320.09
FBBrXII	15.194, 32.048, 0.233, 0.581, 21.855 (11.58), (9.867), (0.091), (0.067), (35.548) (0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)	317.86, 325.34, 320.08, 320.36
FKwBrXII	14.732, 15.285, 0.293, 0.839, 0.034 (12.390), (18.868), (0.215), (0.854), (0.075) (0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)	317.76, 325.21, 319.98, 320.26
ZBBrXII	41.973,—,0.157, 44.263,— (38.787),—,(0.082), (47.648),— (0, 117.99),—,(0, 0.32,) (0, 137.65),—	313.86, 318.35, 314.39, 315.36
PRLL	−8.409, —,0.1576,—,— (1.622),—, (0.0116),—,— (−12.6,−6.2),—, (0.14,0.18),—,—	**312.3, 315.3, 312.7, 313.3**

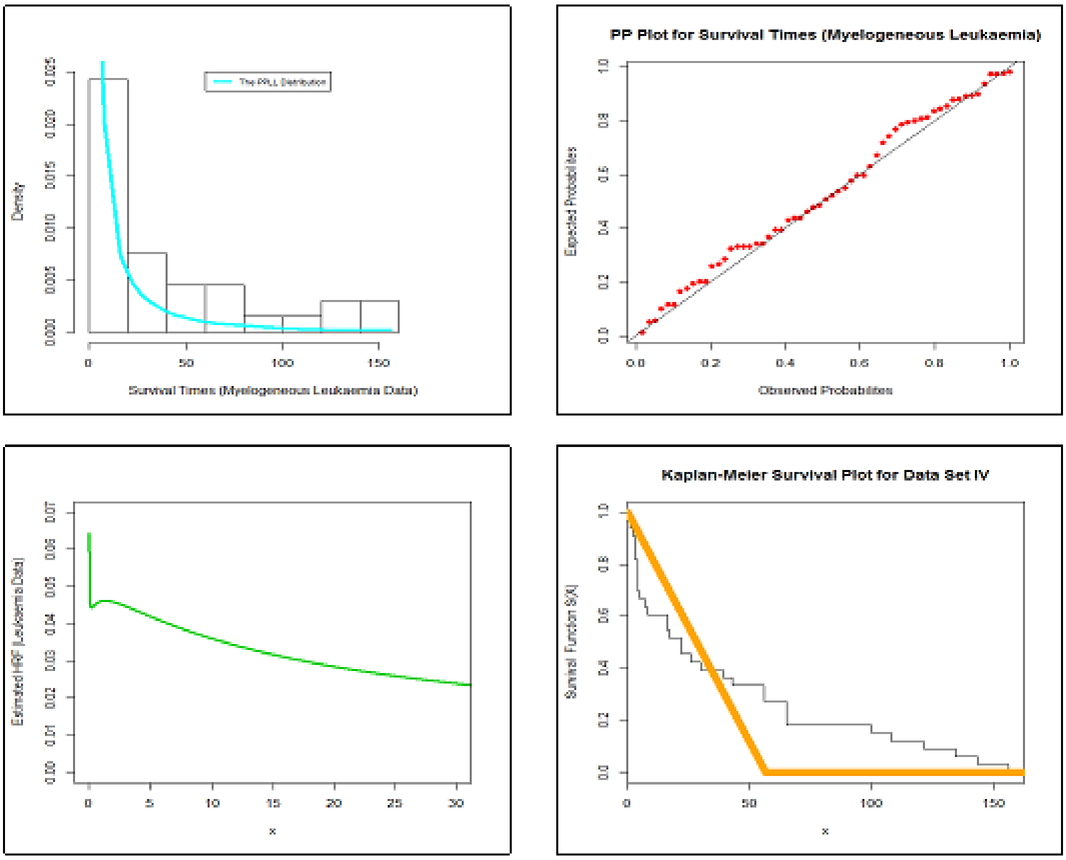


Figure 6: Estimated PDF, P-P plot, Estimated HRF and Kaplan-Meier Survival plot for data set IV.

5. CONCLUSIONS

In this work, we derive a new two parameter continuous compound log-logistic distribution with a strong motivation and wide application. The new model is constructed using the zero truncated Poisson distribution. Some of the fundamental properties of the new model are derived. Four applications to real data sets are provided for illustrating the wide applicability of the new model. The new two parameter model is the best among other ten competitive models which have two parameters or more. We estimated the unknown parameters via the method of maximum likelihood. The new log-logistic model provides the small values for AICr, BICr, CAICr and HQICr. The new model is much better than the Topp Leone Burr XII, Burr XII, Marshall--Olkin BrXII, Kumaraswamy Burr XII, beta Burr XII, five parameter beta Burr XII, beta exponentiated Burr XII, five parameters Kumaraswamy Burr XII and the Zografos-Balakrishnan Burr XII models in modeling the four data sets.

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