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THE MODIFIED WEIBULL-G FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this paper, we introduce a new family called the modified weibull-G (MW-G) family of distributions generated from modified weibull distribution. This new family of distributions provides a lot of new and well known distributions as special cases. We discuss four special distributions for the new family. Some mathematical properties are derived including quantile function, moments, and order statistics. The estimation of the distribution parameters is performed by the maximum likelihood method. Applications show that the new family of distributions can provide a better fit than several existing distributions.

Key words: *modified weibull-G family, modified weibull distribution, quantile, moments, maximum likelihood method.*

1. INTRODUCTION

Applications in different areas of engineering, business economics, life time analysis shows that the data sets arising from various areas may require flexible distributions. So there is an increasing trend in developing generalized familiesof distributions by adding one or more additional parameters to the baseline distributions.

Some of well-known families are: generalized modified weibull (Carrasco et al. (2008)), beta modified weibull (Silva *et al.* (2010)), gamma- exponentiated weibull distributions (Pinho *et al.* (2012)), generalized beta-G family (Alexander *et al.* (2012)), exponentiated generalized-G family (Cordeiro *et al.* (2013)), transformed-transformer (T-X) family (Alzaatreh *et.al.* (2013)), weibull-G family (Bourguignon *et al.* (2014)), Mcdonald exponentiated modified weibull (Merovci and Elbatal (2015)), gamma exponentiated modified weibull (Pu *et al.* (2016)), Kumaraswamy weibull-G family (Hassan and Elgarhy (2016a)), exponentiated weibull-G family (Hassan and Elgarhy (2016b), generalized additive weibull-G (Hassan *et al.* (2017)), inverse weibull-G (Hassan and Nassr (2018)) and more.

The aim of this article, we use the T-X generator approach suggested by Alzaatreh *et al.* (2013) with X follows modified weibull (Sarhan and Zaindin (2009)) random variable to define a new family of distributions called the modified weibull- generator (MW-G) family of distributions with the hope it yields a better fit in certain practical situations. Additionally, we provide a comprehensive account of the mathematical properties of the proposed family of distributions. According to this approach, the cumulative distribution function (cdf) of a random variable X satisfy the relation:

$$F(x) = \int_0^{Q[G(x)]} v(t)\ dt\,, \tag{1}$$

where $T \in [a,b]$ is a random variable such that $-\infty \prec a \prec b \prec \infty$ with the probability density function (pdf), $v(t)$, and $Q[G(x)]$ be a function of the cdf of any random variable X.

The remainder of this article is organized as follows, In Secion 2, introduces the proposed modified weibull-G family of distributions model. Four special distributions of MW-G family are defined in Section 3. Some useful expansions for the pdf and cdf of MW-G family are obtained in Section 4. In the same section, explicit expressions for the quantile function, moments, and order statistics are derived. In Section 5, estimation of the distribution parameters is performed by maximum likelihood method. Section 6, provides two applications to two real data sets are discussed to illustrate the flexibility of the new family. Finally, some concluding remarks are presented in Section 7.

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2. THE MODIFIED WEIBULL-G FAMILY

The modified weibull distribution has pdf and cdf given by $g(t)=\left(\beta+\lambda\gamma t^{\gamma-1}\right)e^{-\left(\beta t+\lambda t^{\gamma}\right)}$, $G(t)=1-e^{-\left(\beta t+\lambda t^{\gamma}\right)}$ respectively.

Based on T-X generator, the cdf of modified weibull-G (MW-G) family is derived by replacing the generator $v(t)$ in (1) by modified weibull pdf, with $Q[G(x)]=G(x;\xi)/\overline{G}(x;\xi)$ as the following:

$$F(x)=\int_0^{\frac{G(x;\xi)}{\overline{G}(x;\xi)}}\left(\beta+\lambda\gamma t^{\gamma-1}\right)e^{-\left(\beta t+\lambda t^{\gamma}\right)}\,dt$$

$$=1-e^{-\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]} \tag{2}$$

where $G(x;\xi)$ is the baseline cdf, which depends on a parameter vector ξ, and $\overline{G}(.)=1-G(.)$. The corresponding pdf of the MW-G family can be written as follows:

$$f(x)=\left[\beta+\lambda\gamma\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma-1}\right]\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{2}}e^{-\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]},\ x,\beta,\lambda,\gamma\succ 0 \tag{3}$$

We shall write X ~MW-G$(\beta,\lambda,\gamma,\xi)$ to denote that the random variable X has pdf (3). For each G distribution, we define the MW-G distribution with three extra parameters β,λ and γ defined by the pdf (3). One can note the following special cases: for $\beta=0$, we have the Weibull-G (Bourguignon et al. (2014)). If $\beta=0$ and $\gamma=1$, we have the Exponential-G (Bourguignon et al. (2014)).

Furthermore, the reliability function; $\overline{\mathrm{F}}(\mathrm{x})$, and hazard rate function; $\mathrm{h}(\mathrm{x})$ are respectively, given by

$$\overline{F}(x)=1-F(x)=e^{-\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]},$$

and

$$h(x)=\frac{f(x)}{\overline{F}(x)}=\left[\beta+\lambda\gamma\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma-1}\right]\left[\overline{G}(x;\xi)\right]^{-2}g(x;\xi).$$

3. SPECIAL DISTRIBUTIONS

A number of new distributions can be deduced as special models from the MW-G family of distributions. Here, four special distributions, namely; modified weibull-unifom (MW-U), modified weibull- weibull (MW-W), modified weibull-lomax (MW-L), and modified weibull-fre'chet (MW-F) are introduced.

3.1 Modified Weibull-Uniform Distribution

As a first distribution, suppose the parent distribution is uniform in the interval $(0,a)$, $a>0$. Then $g(x;a)=1/a, 0<x<a<\infty, G(x;a)=x/a$. The cdf of a random variable X has MW-U distribution, say $X\sim MW-U(\beta,\lambda,\gamma,a)$ is given by:

$$F_{MWU}(x)=1-e^{-\left[\beta\left(\frac{x}{a-x}\right)+\lambda\left(\frac{x}{a-x}\right)^{\gamma}\right]},0<x<a<\infty,\ \beta,\lambda,\gamma>0,$$

The corresponding pdf is:

$$f_{MWU}(x)=\left[\beta+\lambda\gamma\left(\frac{x}{a-x}\right)^{\gamma-1}\right]\frac{a}{[a-x]^{2}}e^{-\left[\beta\left(\frac{x}{a-x}\right)+\lambda\left(\frac{x}{a-x}\right)^{\gamma}\right]},\ 0<x<a<\infty.$$

The reliability function and hazard rate function are respectively, given by

$$\overline{F}_{MWU}(x) = e^{-\left[\beta\left(\frac{x}{a-x}\right)+\lambda\left(\frac{x}{a-x}\right)^{\gamma}\right]}, \text{ and } h_{MWU}(x) = a(a-x)^{-2}\left[\beta+\lambda\gamma\left(\frac{x}{a-x}\right)^{\gamma-1}\right].$$

3.2 Modified Weibull-Weibull Distribution

The second distribution, consider the weibull pdf and cdf with parameter $\theta > 0, \alpha > 0$ are $g(x) = \alpha\,\theta\,x^{\alpha-1}e^{-\theta\,x^{\alpha}}$, and $G(x) = 1 - e^{-\theta\,x^{\alpha}}$, repectively. The cdf of a random variable X has MW-W distribution, say $X \sim MW - W(\beta,\lambda,\gamma,\theta,\alpha)$ is given by:

$$F_{MWW}(x) = 1 - e^{-\left[\beta\left(e^{\theta x^{\alpha}}-1\right)+\lambda\left(e^{\theta x^{\alpha}}-1\right)^{\gamma}\right]}, \quad x,\beta,\lambda,\gamma,\theta,\alpha > 0,$$

The corresponding pdf is:

$$f_{MWW}(x) = \alpha\,\theta\,x^{\alpha-1}\,e^{\theta x^{\alpha}}\left[\beta+\lambda\gamma\left(e^{\theta x^{\alpha}}-1\right)^{\gamma-1}\right]e^{-\left[\beta\left(e^{\theta x^{\alpha}}-1\right)+\lambda\left(e^{\theta x^{\alpha}}-1\right)^{\gamma}\right]}. \tag{4}$$

The density function (4) reduces to modified weibull- exponential distribution for $\alpha = 1$. For $\alpha = 2$, the density function (4) reduces to modified weibull-rayleigh distribution. The density function (4) includes the weibull-weibull distribution (Bourguignon *et al.* (2014)) when $\beta = 0$.

So the reliability function and hazard rate function are respectively, given by

$$\overline{F}_{MWW}(x) = e^{-\left[\beta\left(e^{\theta x^{\alpha}}-1\right)+\lambda\left(e^{\theta x^{\alpha}}-1\right)^{\gamma}\right]}, \text{ and } h_{MWW}(x) = \alpha\,\theta\,x^{\alpha-1}\,e^{\theta x^{\alpha}}\left[\beta+\lambda\gamma\left(e^{\theta x^{\alpha}}-1\right)^{\gamma-1}\right].$$

3.3 Modified Weibull-Lomax Distribution

The third distribution is modified weibull-lomax whose cdf is derived by substituting the following cdf $G(x) = 1 - \left(1+\frac{x}{\theta}\right)^{-\alpha}$, $\theta, \alpha > 0$ in (2) as follows:

$$F_{MWL}(x) = 1 - e^{-\left[\beta\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)+\lambda\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)^{\gamma}\right]}, \quad x,\beta,\lambda,\gamma,\theta,\alpha > 0.$$

The pdf of a random variable X having MW-W distribution, say $X \sim MW - L(\beta,\lambda,\gamma,\theta,\alpha)$ is given by:

$$f_{MWL}(x) = \left(\frac{\alpha}{\theta}\right)\left(1+\frac{x}{\theta}\right)^{\alpha-1}\left[\beta+\lambda\gamma\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)^{\gamma-1}\right]e^{-\left[\beta\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)+\lambda\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)^{\gamma}\right]}.$$

Moreover, the reliability function and hazard rate function are respectively, given by

$$\overline{F}_{MWL}(x) = e^{-\left[\beta\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)+\lambda\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)^{\gamma}\right]}, \text{ and}$$

$$h_{MWL}(x) = \left(\frac{\alpha}{\theta}\right)\left(1+\frac{x}{\theta}\right)^{\alpha-1}\left[\beta+\lambda\gamma\left(\left(1+\frac{x}{\theta}\right)^{\alpha}-1\right)^{\gamma-1}\right].$$

3.4 Modified Weibull-Fre'chet Distribution

Considering the baseline distribution is Fre'chet with pdf and cdf with parameter $a > 0, b > 0$ are $g(x) = a\,b^{a}\,x^{-(a+1)}e^{-(b/x)^{a}}$, and $G(x) = e^{-(b/x)^{a}}$, respectively. The cdf of a random variable X has MW-F distribution, say $X \sim MW - F(\beta,\lambda,\gamma,a,b)$ is

$$F_{MWF}(x) = 1 - e^{-\left[\beta\left(e^{(b/x)^{a}}-1\right)^{-1}+\lambda\left(e^{(b/x)^{a}}-1\right)^{-\gamma}\right]}, \quad x,\beta,\lambda,\gamma,a,b > 0,$$

The corresponding pdf is:

$$f_{MWF}\,(x)=\frac{a\,b^{a}\,x^{-(a+1)}\,e^{-(b/x)^{a}}}{\left(1-e^{-(b/x)^{a}}\right)^{2}}\left[\beta+\lambda\gamma\left(e^{(b/x)^{a}}-1\right)^{1-\gamma}\right]e^{-\left[\beta\left(e^{(b/x)^{a}}-1\right)^{-1}+\lambda\left(e^{(b/x)^{a}}-1\right)^{-\gamma}\right]}.$$

The reliability function and hazard rate function are respectively, given by

$$\overline{F}_{MWF}\,(x)=e^{-\left[\beta\left(e^{(b/x)^{a}}-1\right)^{-1}+\lambda\left(e^{(b/x)^{a}}-1\right)^{-\gamma}\right]},$$

and

$$h_{MWF}\,(x)=\frac{a\,b^{a}\,x^{-(a+1)}\,e^{-(b/x)^{a}}}{\left(1-e^{-(b/x)^{a}}\right)^{2}}\left[\beta+\lambda\gamma\left(e^{(b/x)^{a}}-1\right)^{1-\gamma}\right].$$

Figures 1 and 2 are illustrates possible shapes of the density functions for some modified weibull-g distributions.

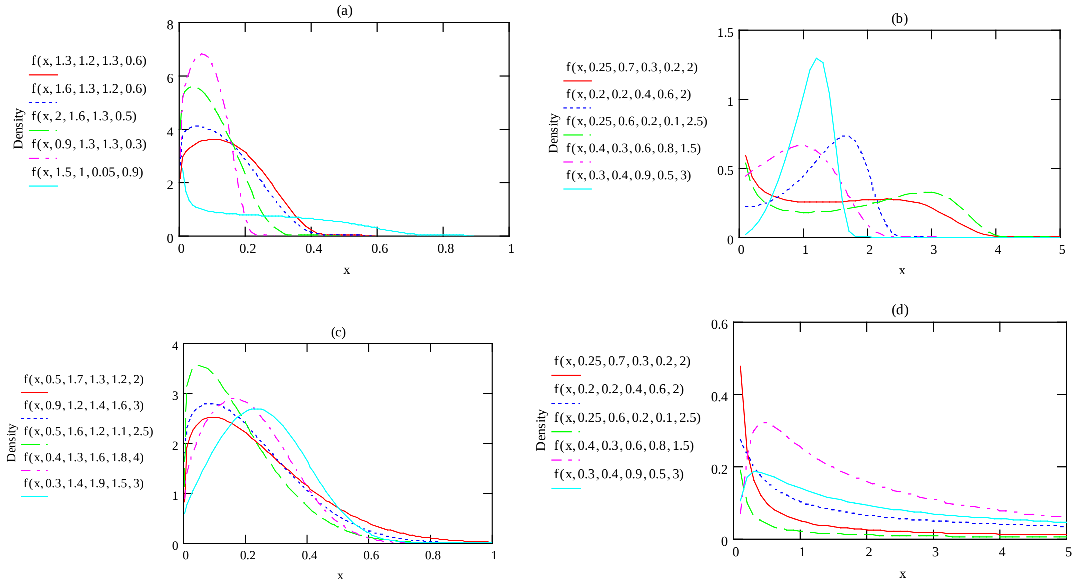

Figure-1: (a) MW-U(β,λ,γ,a), (b) MW-W$(\beta,\lambda,\gamma,\theta,\alpha)$, (c) MW-L$(\beta,\lambda,\gamma,\theta,\alpha)$, and (d) MW-F $(\beta,\lambda,\gamma,a,b)$ density plots for various values of parameters.

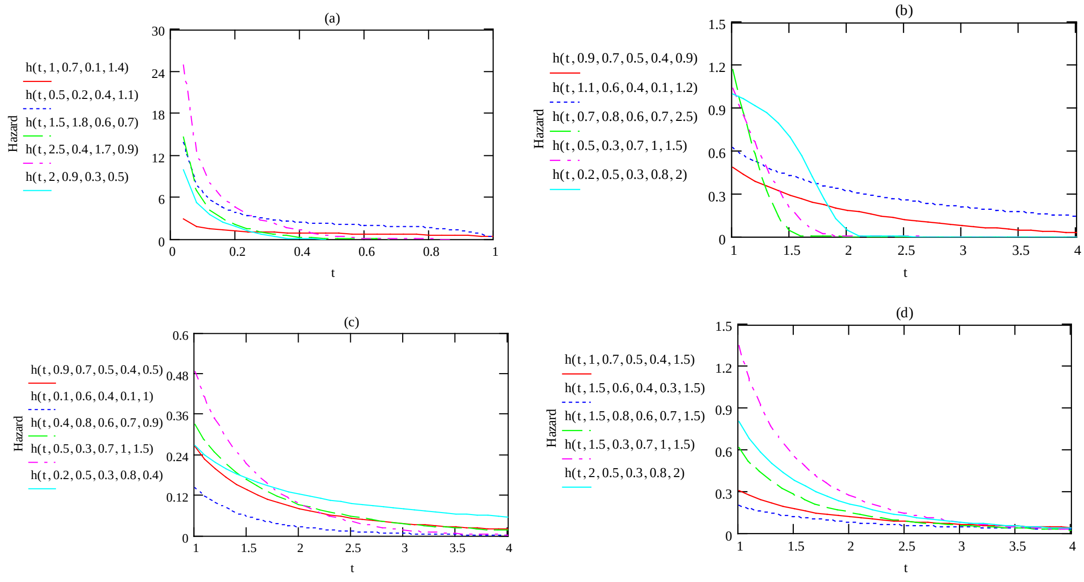

Figure-2: (a) MW-U(β,λ,γ,a), (b) MW-W$(\beta,\lambda,\gamma,\theta,\alpha)$, (c) MW-L $(\beta,\lambda,\gamma,\theta,\alpha)$, and (d) MW-F $(\beta,\lambda,\gamma,a,b)$ hazard plots for various values of parameters.

4. MATHEMATICAL PROPERTIES

In this section, we provide some of the basic mathematical properties of the MW-G family of distributions.

1.1 Useful Expansions

By using the power series representation for the exponential function twice, we have

$$e^{-\left[\beta\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]+\lambda\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\gamma}\right]}=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\beta^{i}\lambda^{j}}{i!\,j!}\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{i+\gamma j},$$

and

$$\left[\frac{g(x;\xi)}{\overline{G}(x;\xi)^{2}}\right]e^{-\left[\beta\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]+\lambda\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\gamma}\right]}=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\beta^{i}\lambda^{j}}{i!\,j!}g(x;\xi)\left[\frac{G(x;\xi)^{i+\gamma j}}{\overline{G}(x;\xi)^{i+\gamma j+2}}\right].$$

Let s be arbitrary positive integer, using the binomial theorem prove that

$$\left[\beta+\lambda\gamma\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\gamma-1}\right]^{s}=\sum_{t=0}^{s}\binom{s}{t}\beta^{s-t}(\lambda\gamma)^{t}\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{t(\gamma-1)},$$

hence

$$\left[\beta+\lambda\gamma\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma-1}\right]^{s}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{2}}e^{-\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]}=$$

$$\sum_{t=0}^{s}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{s}{t}\frac{\beta^{i+s-t}(\lambda)^{j+t}\gamma^{t}}{i!\,j!}g(x;\xi)\left[\frac{G(x;\xi)^{i+(j+t)\gamma-t}}{\overline{G}(x;\xi)^{i+(j+t)\gamma-t+2}}\right], \tag{5}$$

Using the negative binomial theorem, prove the following:

$$\left(\overline{G}(x;\xi)\right)^{-(i+(j+t)\gamma-t)-2}=\sum_{k=0}^{\infty}(-1)^{k}\binom{i+(j+t)\gamma-t+1+k}{k}G(x;\xi)^{k},$$

Therefore

$$\left[\beta+\lambda\gamma\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma-1}\right]^{s}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{2}}e^{-\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]}=\sum_{t=0}^{s}\sum_{i,j,k=0}^{\infty}w_{s,i,j,k}\,g(x;\xi)$$

$$G(x;\xi)^{i+(j+t)\gamma-t+k},$$

where $w_{s,i,j,k}=\frac{(-1)^{i+j+k}\beta^{i+s-t}\lambda^{j+t}\gamma^{t}}{i!\,j!}\binom{s}{t}\binom{i+(j+1)\gamma-t+k+1}{k}$.

In particular, For s=1 in the above expression, we get the MW-G density function as follows

$$f(x)=\sum_{i,j,k=0}^{\infty}w_{0,i,j,k}\,g(x;\xi)G(x;\xi)^{i+j\gamma+k}+\sum_{i,j,k=0}^{\infty}w_{1,i,j,k}\,g(x;\xi)G(x;\xi)^{i+(j+1)\gamma+k-1}. \tag{6}$$

Additionally, using binomial theorem for $[F(x)]^{s}$, s is a positive integer, becomes

$[F(x)]^{s}=\sum_{t=0}^{s}(-1)^{t}\binom{s}{t}e^{-t\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]}$. Applying exponential expansion for $e^{-t\left[\beta\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)+\lambda\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\gamma}\right]}$,

we obtain

$$[F(x)]^{s}=\sum_{t=0}^{s}\sum_{m,n=0}^{\infty}\frac{(-1)^{t+m+n}(t\beta)^{m}(t\lambda)^{n}}{m!\,n!}\binom{s}{t}\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{m+n\gamma}. \tag{7}$$

Using negative binomial theorem in (7), leads to

$$[F(x)]^s = \sum_{t=0}^{s} \sum_{m,n,u=0}^{\infty} \Psi_{t,m,n,u}\, G(x;\xi)^{m+n\gamma+u}, \tag{8}$$

where $\Psi_{t,m,n,u} = \dfrac{(-1)^{t+m+n+u} (t\beta)^m (t\lambda)^n}{m!\,n!} \binom{s}{t} \binom{m+n\gamma+u-1}{u}$.

4.2 Quantile and Median

The quantile function $x_q = Q(x_q) = F^{-1}(q)$ of MW-G$(\beta,\lambda,\gamma,\xi)$ family, if q has a Uniform distribution $U(0,1)$, is the real solution of the nonlinear equation:

$$\lambda\left[\frac{G(x_q;\xi)}{\overline{G}(x_q;\xi)}\right]^{\gamma} + \beta\left[\frac{G(x_q;\xi)}{\overline{G}(x_q;\xi)}\right] + \ln(1-q) = 0. \tag{9}$$

The above equation has no closed form solution in x_q, so we have to use a numerical technique to get the quantile. Put $q = 0.5$, in (9) one gets the median of MW-G$(\beta,\lambda,\gamma,\xi)$ family. For $\beta = 0$, (9) reduced to the quantile function of the W-G(λ,γ,ξ) as follows:

$$x_q = Q(x_q) = G^{-1}\left[\frac{\{(-1/\lambda)\ln(1-q)\}^{1/\gamma}}{1+\{(-1/\lambda)\ln(1-q)\}^{1/\gamma}}\right].$$

Example 1: consider the MW-W$(\beta,\lambda,\gamma,\theta,\alpha)$ distribution discussed in subsection (3.2) when $\beta = 0$. The quantile function of the W-W$(\lambda,\gamma,\theta,\alpha)$ distribution is obtained as follows:

$$x_q = Q(x_q) = \left[\left(\frac{-1}{\theta}\right)\ln\left[1 - \frac{\{(-1/\lambda)\ln(1-q)\}^{1/\gamma}}{\left(1+\{(-1/\lambda)\ln(1-q)\}^{1/\gamma}\right)}\right]\right]^{1/\alpha}.$$

Kenny and Keeping (1962) proposed skewness based on quartiles called the Bowley skewness which is defined as follows:

$$\mathrm{BS} = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Further, the Moors kurtosis (Moors (1988)) based on octiles is defined as follows:

$$\mathrm{MK} = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

where $Q(.)$ is the quantile function. Figure 4 illustrates plots of the skewness and kurtosis of the W-W distribution for some choices of the parameter γ as a function of α and fixed values of λ and θ. These plots indicate that the skewness and kurtosis decrease when $\gamma = 0.5, 0.8, 1, 2$ (increases) for fixed α, λ and θ.

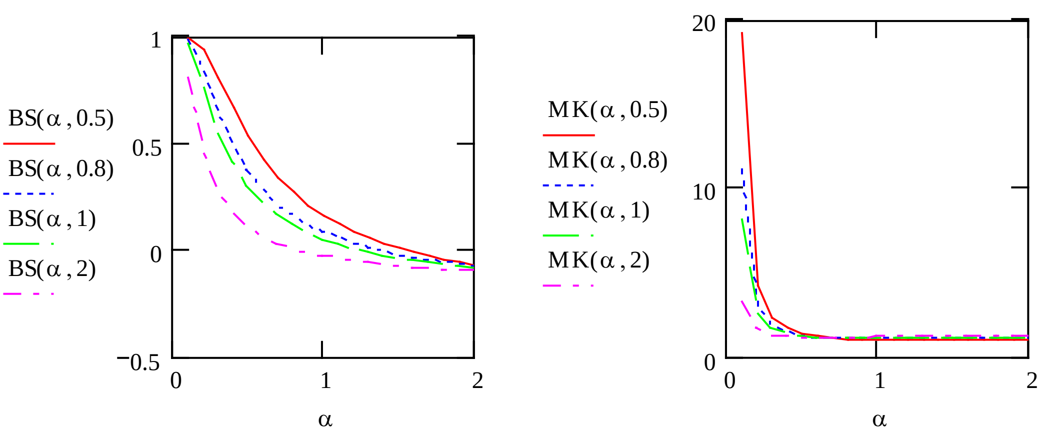

Figure-4: plots of skewness and kurtosis for W-W distribution based on quantile.

4.3 Moments

The moments of the MW-G family of distributions are derived. More specially, the moments of the MW-U distribution is obtained.

The r^th non-central moments of X has the MW-G, denoted by μ_r', is derived from (6) as follows:

$$\mu_r' = \sum_{i,j,k=0}^{\infty} w_{0,i,j,k} \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+j\gamma+k} dx + \sum_{i,j,k=0}^{\infty} w_{1,i,j,k} \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+(j+1)\gamma+k-1} dx, \quad r = 1,2,3,.. \tag{10}$$

Furthermore, the moment generating function of X, denoted by $M_X(t)$, is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$
$$= \sum_{r=0}^{\infty} \sum_{i,j,k=0}^{\infty} \frac{t^r}{r!} w_{0,i,j,k} \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+j\gamma+k} dx$$
$$+ \sum_{r=0}^{\infty} \sum_{i,j,k=0}^{\infty} \frac{t^r}{r!} w_{1,i,j,k} \int_{-\infty}^{\infty} x^r\, g(x;\xi) G(x;\xi)^{i+(j+1)\gamma+k-1} dx.$$

Example 2: Consider the MW-U distribution discussed in subsection (3.1). The r^th non-central moments of X has MW-U can be obtained from (10) with pdf and cdf as defined in subsection (3.1) as follows:

$$\mu_r' = \sum_{i,j,k=0}^{\infty} \left[\frac{w_{0,i,j,k}}{(i + j\gamma + k + r + 1)} \right] a^r + \sum_{i,j,k=0}^{\infty} \left[\frac{w_{1,i,j,k}}{(i + (j+1)\gamma + k + r)} \right] a^r.$$

In particular, the mean and variance of the MW-U distribution are obtained, respectively, as follows:

$$E(X) = \sum_{i,j,k=0}^{\infty} \left[\frac{w_{0,i,j,k}\; a}{(i + j\gamma + k + r + 1)} \right] + \sum_{i,j,k=0}^{\infty} \left[\frac{w_{1,i,j,k}\; a}{(i + (j+1)\gamma + k + r)} \right],$$

and

$$Var(X) = \left[\sum_{i,j,k=0}^{\infty} \left[\frac{w_{0,i,j,k}\; a^2}{(i + j\gamma + k + r + 1)} \right] + \sum_{i,j,k=0}^{\infty} \left[\frac{w_{1,i,j,k}\; a^2}{(i + (j+1)\gamma + k + r)} \right] \right] - [E(X)]^2.$$

4.4 Mean Deviation

The mean deviation about the mean and the mean deviation about the median measures the amount of scattering in a population. For a random variable X have pdf $f(x)$ and cdf $F(x)$ with mean μ and median M. The mean deviation about the mean and mean deviation about the median are, respectively, defined by:

$$\delta_1(X) = 2\mu F(\mu) - 2\int_{-\infty}^{\mu} x\, f(x)\; dx\;, \tag{11}$$

and

$$\delta_2(X) = \mu - 2\int_{-\infty}^{M} x\, f(x)\; dx\;. \tag{12}$$

Example 3: Consider the MW-U distribution discussed in subsection (3.1). The mean deviation about the mean and the mean deviation about the median of X has MW-U, respectively, can be obtained from (11) and (12) using (6) and cdf defined in subsection (3.1) assuming as follows:

$$\delta_1(X) = 2\mu \left[\begin{array}{l} \left(1 - e^{-\left[\beta\left(\frac{\mu}{a-\mu}\right) + \lambda\left(\frac{\mu}{a-\mu}\right)^{\gamma} \right]} \right) - \\ \left\{ \sum_{i,j,k=0}^{\infty} \left[\left[\frac{w_{0,i,j,k}\, (\mu/a)^{i+j\gamma+k+1}}{(i + j\gamma + k + 2)} \right] + \sum_{i,j,k=0}^{\infty} \left[\frac{w_{1,i,j,k}\, (\mu/a)^{i+(j+1)\gamma+k}}{(i + (j+1)\gamma + k + 1)} \right] \right] \right\} \end{array} \right],$$

and

$$\delta_2(X) = \mu - 2M\left[\sum_{i,j,k=0}^{\infty}\left[\frac{w_{0,i,j,k}\,(M/a)^{i+j\gamma+k+1}}{(i+j\gamma+k+2)}\right] + \sum_{i,j,k=0}^{\infty}\left[\frac{w_{1,i,j,k}\,(M/a)^{i+(j+1)\gamma+k}}{(i+(j+1)\gamma+k+1)}\right]\right].$$

where μ and M are the mean and median of the MW-U distribution.

4.5 Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(z)}$ be an ordered random sample for a random sample $X_1, X_2, \ldots, X_z$ of size z drawn from the MW-G family of distributions with pdf and cdf given by (6) and (8), respectively. The pdf of $X_{(r)}$, $r = 1,2,\ldots,z$ is given by

$$f_{X_{(r)}}(x) = \frac{1}{B(r, z-r+1)}\, f(x)\, F(x)^{r-1}\,[1-F(x)]^{z-r},$$

Using binomial series expansion of $[1-F(x)]^{z-r}$, we have

$$f_{X_{(r)}}(x) = \sum_{l=0}^{z-r}\frac{(-1)^l}{B(r, z-r+1)}\binom{z-r}{l}[F(x)]^{r+l-1}\, f(x), \tag{13}$$

Substituting from (6) and (8) by replacing $s = (r+l-1)$ into (13), we can get the pdf of $X_{(r)}$ as follows:

$$f_{X_{(r)}}(x) = \sum_{l=0}^{z-r}\sum_{t=0}^{r+l-1}\sum_{i,j,k,m,n,u=0}^{\infty}\eta_{l,t,m,n,u}\left[w_{0,i,j,k} + w_{1,i,j,k}G(x;\xi)^{\gamma-1}\right]g(x;\xi)G(x;\xi)^h, \tag{14}$$

where

$\eta_{l,t,m,n,u} = \dfrac{1}{B(r, z-r+1)}(-1)^l\dbinom{z-r}{l}\Psi_{t,m,n,u}$, $h = i + (j+n)\gamma + k + m + u$, and B(., .) is the beta function.

In particular, we define the smallest order statistics $X_{(1)}$ by substituting $r = 1$ in (14) and the largest order statistics $X_{(z)}$ by substituting $r = z$ in (14), respectively, as follows:

$$f_{X_{(1)}}(x) = \sum_{l=0}^{z-1}\sum_{t=0}^{l}\sum_{i,j,k,t,m,n,u=0}^{\infty}M_{l,t,m,n,u}\left[w_{0,i,j,k} + w_{1,i,j,k}G(x;\xi)^{\gamma-1}\right]g(x;\xi)G(x;\xi)^h,$$

and

$$f_{X_{(z)}}(x) = \sum_{t=0}^{z+l-1}\sum_{i,j,k,t,m,n,u=0}^{\infty}N_{t,m,n,u}\left[w_{0,i,j,k} + w_{1,i,j,k}G(x;\xi)^{\gamma-1}\right]g(x;\xi)G(x;\xi)^h,$$

where $M_{l,t,m,n,u} = z(-1)^l\dbinom{z-1}{l}\Psi_{t,m,n,u}$, and $N_{t,m,n,u} = z\ \Psi_{t,m,n,u}$.

5. PARAMETERS ESTIMATION

Here, we derive the maximum likelihood estimates (MLEs) of the unknown parameters for the new family of distributions based on complete samples. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n follows the MW-G(Θ), $\Theta = (\beta, \lambda, \gamma, \xi)^T$ be a parameter vector, with pdf (3). The log-likelihood function from density (3) is given by:

$$\ln L(\Theta) = \sum_{i=1}^{n}\ln\left[\beta + \lambda\gamma\, T(x_i;\xi)^{\gamma-1}\right] + \sum_{i=1}^{n}\ln g(x_i;\xi) - 2\sum_{i=1}^{n}\ln\overline{G}(x_i;\xi) - \beta\sum_{i=1}^{n}T(x_i;\xi)$$
$$-\lambda\sum_{i=1}^{n}T(x_i;\xi)^{\gamma}.$$

where $T(x_i;\xi) = \left[\dfrac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}\right]$.

The components of the score function $U(\Theta)=\left(U_{\beta},U_{\lambda},U_{\gamma},U_{\xi}\right)^{T}$ are

$$U_{\beta}=\sum_{i=1}^{n}V(x_i;\xi)-\sum_{i=1}^{n}T(x_i;\xi),$$

$$U_{\lambda}=\gamma\sum_{i=1}^{n}\left[V(x_i;\xi)T(x_i;\xi)^{\gamma-1}\right]-\sum_{i=1}^{n}T(x_i;\xi)^{\gamma},$$

$$U_{\gamma}=\lambda\sum_{i=1}^{n}\left[\frac{T(x_i;\xi)^{\gamma-1}\left[1+\gamma\ln T(x_i;\xi)\right]}{V(x_i;\xi)}\right]-\lambda\sum_{i=1}^{n}T(x_i;\xi)^{\gamma}\ln T(x_i;\xi),$$

and

$$U_{\xi}=\sum_{i=1}^{n}\left[\frac{\lambda\gamma(\gamma-1)T(x_i;\xi)^{\gamma-2}\left[\partial T(x_i;\xi)/\partial\xi\right]}{V(x_i;\xi)}\right]+\sum_{i=1}^{n}\left[\frac{\left[\partial g(x_i;\xi)/\partial\xi\right]}{g(x_i;\xi)}\right]$$
$$-2\sum_{i=1}^{n}\left[\frac{\left[\partial\overline{G}(x_i;\xi)/\partial\xi\right]}{\overline{G}(x_i;\xi)}\right]-\beta\sum_{i=1}^{n}\left[\partial T(x_i;\xi)/\partial\xi\right]-\lambda\gamma\sum_{i=1}^{n}\left[T(x_i;\xi)^{\gamma-1}\left[\partial T(x_i;\xi)/\partial\xi\right]\right],$$

where $V(x_i;\xi)=\left[\beta+\lambda\gamma\,T(x_i;\xi)^{\gamma-1}\right]^{-1}$. Setting $U_{\beta},U_{\lambda},U_{\gamma}$, and U_{ξ} equal to zero and solving the resulting system of non-linear equations simultaneously to obtain the MLEs, say $\hat{\Theta}=\left(\hat{\beta},\hat{\lambda},\hat{\gamma},\hat{\xi}\right)^{T}$ of $\Theta=\left(\beta,\lambda,\gamma,\xi\right)^{T}$ of the MW-G family of distributions. The resulting equations cannot be solved analytically and statistical software can be used to solve them numerically.

The elements of the information matrix are defined as follows:

$$U_{\beta\beta}=-\frac{\partial^2\ln L(\Theta)}{\partial\beta^2}=\left[V(x_i;\xi)\right]^2,$$

$$U_{\beta\lambda}=-\frac{\partial^2\ln L(\Theta)}{\partial\beta\partial\lambda}=\gamma\sum_{i=1}^{n}T(x_i;\xi)\left[V(x_i;\xi)\right]^2,$$

$$U_{\beta\gamma}=-\frac{\partial^2\ln L(\Theta)}{\partial\beta\partial\gamma}=\lambda T(x_i;\xi)^{\gamma-1}\left[1+\gamma\ln T(x_i;\xi)\right],$$

$$U_{\beta\xi_k}=-\frac{\partial^2\ln L(\Theta)}{\partial\beta\partial\xi_k}=\lambda\gamma(\gamma-1)\sum_{i=1}^{n}\left(T(x_i;\xi)^{\gamma-2}\left[V(x_i;\xi)\right]^2T_k'(x_i;\xi)\right)+\sum_{i=1}^{n}T_k'(x_i;\xi),$$

$$U_{\lambda\lambda}=-\frac{\partial^2\ln L(\Theta)}{\partial\lambda^2}=\gamma^2\sum_{i=1}^{n}T(x_i;\xi)^{2(\gamma-1)}\left[V(x_i;\xi)\right]^2,$$

$$U_{\lambda\gamma}=-\frac{\partial^2\ln L(\Theta)}{\partial\lambda\partial\gamma}=-\beta\sum_{i=1}^{n}T(x_i;\xi)^{\gamma-1}\left[V(x_i;\xi)\right]^2\left[1+\gamma\ln T(x_i;\xi)\right]+\sum_{i=1}^{n}T(x_i;\xi)^{\gamma}\ln T(x_i;\xi),$$

$$U_{\lambda\xi_k}=-\frac{\partial^2\ln L(\Theta)}{\partial\lambda\partial\xi_k}=-\beta\gamma(\gamma-1)\sum_{i=1}^{n}T(x_i;\xi)^{\gamma-2}\left[V(x_i;\xi)\right]^2T_k'(x_i;\xi)+\gamma\sum_{i=1}^{n}T(x_i;\xi)^{\gamma-1}T_k'(x_i;\xi),$$

$$U_{\gamma\gamma}=-\frac{\partial^2\ln L(\Theta)}{\partial\gamma^2}=-\sum_{i=1}^{n}V(x_i;\xi)\left(\lambda T(x_i;\xi)^{\gamma-1}\ln T(x_i;\xi)\left[2+\gamma\ln T(x_i;\xi)\right]\right)$$
$$+\lambda\sum_{i=1}^{n}T(x_i;\xi)^{\gamma}\left(\ln T(x_i;\xi)\right)^2,$$

$$U_{\gamma\xi_k}=-\frac{\partial^2\ln L(\Theta)}{\partial\gamma\partial\xi_k}=-\beta\lambda\sum_{i=1}^{n}T(x_i;\xi)^{\gamma-2}\left[V(x_i;\xi)\right]^2T_k'(x_i;\xi)\left[(2\gamma-1)+\left(\gamma^2-\gamma\right)\ln T(x_i;\xi)\right]$$
$$-\lambda^2\gamma^2\sum_{i=1}^{n}T(x_i;\xi)^{2\gamma-3}\left[V(x_i;\xi)\right]^2T_k'(x_i;\xi)+\lambda\sum_{i=1}^{n}T(x_i;\xi)^{\gamma-1}T_k'(x_i;\xi)\left[1+\gamma\ln T(x_i;\xi)\right],$$

and

$$U_{\xi_k\xi_l} = -\frac{\partial^2 \ln L(\Theta)}{\partial\xi_k \partial\xi_l} = -\sum_{i=1}^{n} \frac{g''_{kl}(x_i;\xi)}{g(x_i;\xi)} + \sum_{i=1}^{n} \frac{g'_k(x_i;\xi)\, g'_l(x_i;\xi)}{\left(g(x_i;\xi)\right)^2} + 2\sum_{i=1}^{n} \frac{\bar{G}''_{kl}(x_i;\xi)}{\bar{G}(x_i;\xi)}$$

$$-2\sum_{i=1}^{n} \frac{\bar{G}'_k(x_i;\xi)\bar{G}'_l(x_i;\xi)}{\left(\bar{G}(x_i;\xi)\right)^2} + \beta\sum_{i=1}^{n} T''_{kl}(x_i;\xi) + \lambda\gamma\sum_{i=1}^{n} T(x_i;\xi)^{\gamma-2} \begin{bmatrix} (\gamma-1)T'_k(x_i;\xi)T'_l(x_i;\xi) \\ +T(x_i;\xi)T''_{kl}(x_i;\xi) \end{bmatrix}$$

$$-\lambda\gamma(\gamma-1)\sum_{i=1}^{n} T(x_i;\xi)^{\gamma-2} V(x_i;\xi) \begin{bmatrix} \dfrac{(\gamma-2)T'_k(x_i;\xi)T'_l(x_i;\xi)}{T(x_i;\xi)} + T''_{kl}(x_i;\xi) \\ -\lambda\gamma(\gamma-1)T(x_i;\xi)^{\gamma-2} V(x_i;\xi) T''_{kl}(x_i;\xi) \end{bmatrix},.$$

where $T'_k(.;\xi) = \partial T(.;\xi)/\partial\xi_k$ and $T''_{kl}(.;\xi) = \partial^2 T(.;\xi)/\partial\xi_k\, \partial\xi_l$.

The $100(1-\eta)\%$ approximate two sided confidence intervals for the parameters β, λ, γ, and ξ of the MW-G distribution are respectively given by:

$$\hat{\beta} \pm z_{\eta/2}\sqrt{\mathrm{var}(\hat{\beta})},\ \hat{\lambda} \pm z_{\eta/2}\sqrt{\mathrm{var}(\hat{\lambda})},\ \hat{\gamma} \pm z_{\eta/2}\sqrt{\mathrm{var}(\hat{\gamma})}, \text{ and } \hat{\xi} \pm z_{\eta/2}\sqrt{\mathrm{var}(\hat{\xi})},$$

where $z_{(\eta/2)}$ is the upper $(\eta/2)-th$ percentile of the standard normal distribution and The $Var(.)'s$ denote the diagonal elements of inverting the information matrix and replace the unknown parameters by their MLEs $\hat{\Theta}$ of the model parameters.

6. APPLICATION

We present two real data sets to illustrate the flexibility of MW-W as special distribution of MW-G family of distributions and two distributions called: Additive Weibull (AW, Almalki and Yuan (2013)), New Modified Weibull (NMW, Doostmoradi *et al.* (2014)) for data modeling. The model selection is carried out using -2 log-likelihood function (-2lnL), Akaike InformationCcriterion (AIC), Corrected Akaike Information Criteria (CAIC), Bayesian Information Criterion (BIC), and Kolomogrov Smirnov (KS) statistic. However, the better distribution corresponds to the smaller values of AIC, CAIC, BIC, KS criteria.

Data Set 1:
The first data set is provided in Murthy et al. (2004) about time between failures for repairable item. The data are listed as the following:
1.43, 0.11, 0.63, 0.71, 0.77, 1.23, 2.63, 1.49, 1.24, 3.46, 2.46, 1.97, 0.59, 0.74, 1.86, 1.23, 0.94, 1.17, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37.

The MLEs of MW-W parameters are: $\hat{\beta} = 0.280,\ \hat{\lambda} = 0.540,\ \hat{\gamma} = 0.350,\ \hat{\alpha} = 1.473,$ and $\hat{\theta} = 0.123.$ In Table (1), we list the values of -2lnL, AIC, CAIC, BIC, and KS statistic. We observed that the MW-W distribution has the smallest -2lnL, AIC, CAIC, BIC and KS values compared with those values of the other distributions. So, the MW-W is the better fit than the other distributions to this data.

Table-1: Model selection criteria for the first data set

Model	-2lnL	AIC	CAIC	BIC	KS
NMW	242.501	250.051	251.651	255.656	0.942
AW	159.642	167.642	169.242	173.246	0.283
MW-W	145.130	155.130	157.630	162.136	0.223

Data Set 2:
The second data set represents 34 observations of vinyl chloride data obtained from clean up gradient ground-water monitoring wells in mg/L. the data are obtained from Bhaumik et al. (2009) and recorded as follows:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

The MLEs of MW-W parameters are: $\hat{\beta} = 0.211,\ \hat{\lambda} = 0.523,\ \hat{\gamma} = 0.291,\ \hat{\alpha} = 1.159,$ and $\hat{\theta} = 0.14.$ Results in Table (2) , indicate that the MW-W distribution is the better fit than the other competitive distributions for this data set based on the selected criteria.

Table-2: Model selection criteria for the second data set

Model	-2lnL	AIC	CAIC	BIC	KS
NMW	314.807	322.807	324.186	328.912	1
AW	221.798	229.798	231.178	235.904	0.982
MW-W	190.306	200.306	202.306	207.938	0.598

7. CONCLUSION

In this paper, we introduce a new family of univariate distribution based on the modified weibull distribution as a new generator. Many new sub-models are obtained and four special distributions are provided. Mathematical properties of the new MW-G family are derived. We give explicit closed form expressions for the moments, and distribution of order statistics. The maximum likelihood method of estimation is employed to obtain the distribution parameters and the observed fisher information matrix is derived. We fit MW-W as special sub-model to two real data sets in order to explain the flexibility of this new family.

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