International Journal of Mathematical Archive-10(4), 2019, 1-8
Available online through www.ijma.info ISSN 2229 – 5046

GENERALIZED COUPLED FIXED POINT THEOREMS ON BIPOLAR METRIC SPACES

ANIMESH GUPTA* AND GANESH KUMAR SONI

Department of Mathematics,
Swami Vivekanand Govt. P. G. College, Narsinghpur, India.

(Received On: 24-01-19; Revised & Accepted On: 02-04-19)

ABSTRACT

In this article, certain coupled fixed point theorems, which can be considered as generalizations of Banach fixed point theorem, are extended to bipolar metric spaces. Also, some results which are related to these theorems are obtained. Finally, it is given an example which presents the applicability of obtained results.

2000 Mathematics Subject Classification: *46A80, 47H10, 54H25.*

Key words and phrases: *Bipolar metric space, coupled fixed point, completeness.*

INTRODUCTION

Since the year 1922, Banachs contraction principle, when many authors saw that these fixed point theorems can be utilized to investigate existence and uniqueness of solutions of periodic boundary value problems, difierential equations and nonlinear integral equations, these theorems attracted their attention. And, they extended these theorems to various generalizations of metric spaces as cone, partial and modular, e.g. [1, 2, 4, 5, 7, 9, 11, 12].

In literature, the notion of coupled fixed point has been introduced by Guo and Lakshmikantham [6] in 1987. Afterward, Bhaskar and Lakshmikantham [3] introduced certain coupled fixed point theorems in partially ordered metric spaces. We express a series of definitions of some fundamental notions related to bipolar metric spaces.

Definition 1: A bipolar metric space is a triple $(X,\ Y,\ d)$ such that $X, Y \neq \phi$, and $d : X \times Y \rightarrow R^+$ is a function satisfying the properties
(B0) if $d\ (x,\ y) = 0$, then $x = y$,
(B1) if $x = y$, then $d\ (x,\ y) = 0$,
(B2) if, $y \in X \cap Y$, then $d\ (x,\ y) = d\ (y,\ x)$,
(B3) $d(x_1, y_2) \leq d(x_1,\ y_1) + d(x_2,\ y_1) + d(x_2,\ y_2)$,
for all $(x,\ y),\ (x_1,\ y_1),\ (x_2,\ y_2)\ \in X \times Y$, where R^+ symbolises the set of all non-negative real numbers. Then d is called a bipolar metric on the pair $(X,\ Y\ )$.

Definition 2: Let $(X_1,\ Y_1)$ and $(X_2,\ Y_2)$ be pairs of sets and given a function $: X_1 \cup Y_1 \rightarrow X_2 \cup Y_2$. If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq$ Y_2, we call f a covariant map from $(X_1,\ Y_1)$ to $(X_2,\ Y_2)$ and denote this with $f : (X_1,\ Y_1) \rightarrow (X_2,\ Y_2)$. If $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$, then we call f a contravariant map from $(X_1,\ Y_1)$ to $(X_2,\ Y_2)$ and write $f : (X_1,\ Y_1) \rightarrow \rightarrow (X_2,\ Y_2)$. In particular, if d_1 and d_2 are bipolar metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we sometimes use the notations $f : (X_1,\ Y_1,\ d_1) \rightarrow (X_2,\ Y_2,\ d_2)$ and $f : (X_1,\ Y_1,\ d_1) \rightarrow (X_2,\ Y_2,\ d_2)$.

Definition 3: Let $(X,\ Y,\ d)$ be a bipolar metric space. A point $u \in X \cup Y$ is called a left point if $u \in X$, a right point if $u \in Y$ and a central point if it is both left and right point. Similarly a sequence (x_n) on the set X is called a left sequence and a sequence (y_n) on Y is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence. A sequence (u_n) is said to be convergent to a point u, iff (u_n) is a left sequence, u is a right point and $lim_{n\rightarrow\infty} d(u_n,\ u) = 0$, or (u_n) is a right sequence, u is a left point and $lim_{n\rightarrow\infty} d(u,\ u_n) = 0$. A bisequence $(x_n,\ y_n)$ on $(X,\ Y,\ d)$ is a sequence on the set $X \times Y$. If the sequences (x_n) and (y_n) are convergent, then the bisequence $(x_n,\ y_n)$ is said to be convergent, and if (x_n) and (y_n) converge to a common point, then $(x\_n,\ y\_n)$ is called biconvergent. $(x_n,\ y_n)$ is a Cauchy bisequence, if $lim_{n,m\rightarrow\infty} d(x_n,\ y_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Corresponding Author: **Animesh Gupta***

Definition 4: Let $(X_1,\ Y_1,\ d_1)$ and $(X_2,\ Y_2,\ d_2)$ be bipolar metric spaces.
(1) A map $f:\ (X_1,\ Y_1,\ d_1) \to (X_2,\ Y_2,\ d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_1\ (x_0,\ y) < \delta$ implies $d_2\big(f\ (x_0),\ f\ (y)\big) < \epsilon$ all $y \in Y_1$.
(2) A map $f:\ (X_1,\ Y_1,\ d_1) \to (X_2,\ Y_2,\ d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_1(x,\ y_0) < \delta$ implies $d_2\big(f\ (x),\ f\ (y_0)\big) < \epsilon$ for all $x\ \in X_1$.
(3) A map f is called continuous, if it is left-continuous at each point $x\ \in X_1$ and right-continuous at each point $y\ \in\ Y_1$.
(4) A contravariant map $f:\ (X_1, Y_1, d_1) \to\to\ (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $f:\ (X_1, Y_1, d_1) \to \big(Y_2, X_2,\ \bar{d}_2\big)$.

It can be seen from the Definition \ref(def4) that a covariant or a contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous if and only if $(u_n) \to\ v$ on (X_1, Y_1, d_1) implies $\big(f\ (u_n)\big) \to\ f\ (v)$ on (X_2, Y_2, d_2).

In this paper, we extend certain coupled fixed point theorems, which can be considered as generalization of Banach fixed point theorem, to bipolar metric spaces. Also, we obtain some results which are related to these theorems. Finally, we give an example which presents the applicability of our obtained results.

COUPLED FIXED POINT THEOREMS IN BIPOLAR METRIC SPACES

Let Δ denote the class of all functions $\beta:\ [0,\infty\ ) \to\ [0,1)$ which satisfy the following conditions,

$$\beta\ (t_n) \to\ 1 \quad \text{implies} \quad t_n\ =\ 0\,.$$

The following are examples of some functions belonging to \Delta.
(1) $\beta(t) = \mathrm{k}$ for $s, t\ \in [0,\infty\ )$, where $\mathrm{k} \in [0,\ 1)$.
(2) $\beta(t) = \frac{log\ (1+kt)}{kt},\ t\ >\ 0.$

Now, we will prove our main result.

Definition 5: Let (X, Y, d) be a bipolar metric space, $F:(X^2, Y^2) \to\ \ (X, Y)$ be a covariant mapping. $(a, b) \in X^2 \cup\ Y^2$ is said to be a coupled fixed point of F if $F(a, b) = a$ and $F(b, a) = b$.

Let Δ denote the class of those functions $\beta:\ [0,\infty\ ) \to [0,1)$ which satisfy the following condition

$$\beta\ (t_n) \to\ 1 \quad \text{implies} \quad t_n\ =\ 0\,.$$

Theorem 6: Let (X, Y, d) be a complete bipolar metric space, $F:(X^2, Y^2) \to\ (X, Y)$ be a covariant mapping and $\beta_1, \beta_2 \in \Delta$. If F satisfies the condition

$$d\big(F(a,b), F(p,q)\big) \le \beta_1\ (d(a,p))d(a,p) + \beta_2(d(b,q))d(b,q), \tag{1}$$

for all $a, b\ \ \in\ X, p, q\ \in\ Y$, then $F:\ X^2 \cup Y^2 \to\ X \cup\ Y$ has a unique coupled fixed point.

Proof: Let $a_0, b_0\ \in\ X$ and $p_0, q_0\ \in\ Y$. We take $a_1, b_1\ \in\ X$ and $p_1, q_1\ \in\ Y$ with $a_1 =\ \ F(a_0, b_0), b_1\ =\ F(b_0, a_0), p_1\ =\ F(p_0, q_0), q_1\ =\ F(q_0, p_0)$ .And similarly, we take $a_2, b_2 \in X$ and $p_2, q_2\ \in\ Y$ with $a_2\ =\ F(a_1, b_1), b_2\ =\ F(b_1, a_1), p_2\ =\ F(p_1, q_1), q_2\ =\ F(q_1, p_1)$.

In this way, we obtain bisequences (a_n, b_n) and (p_n, q_n) with

$$a_{n+1}\ =\ F(a_n, b_n),\ \ b_{n+1}\ =\ F(b_n, a_n),$$
$$p_{n+1}\ =\ F(p_n, q_n) \text{ and } q_{n+1}\ =\ F(q_n, p_n)$$

for all $n\ \in\ N^+$. From (1), we get

$$\begin{aligned} d(a_n, p_{n+1}) &= d\big(F(a_{n-1}, b_{n-1}), F(p_n, q_n)\big),\\ &\le \beta_1\big(d(a_{n-1}, p_n)\big)d(a_{n-1}, p_n) + \beta_2\big(d(b_{n-1}, q_n)\big)d(b_{n-1}, q_n) \end{aligned} \tag{2}$$

and

$$\begin{aligned} d(b_n, q_{n+1}) &= d\big(F(b_{n-1}, a_{n-1}), F(q_n, p_n)\big),\\ &\le \beta_1\big(d(b_{n-1}, q_n)\big)d(b_{n-1}, q_n) + \beta_2\big(d(a_{n-1}, p_n)\big)d(a_{n-1}, p_n) \end{aligned} \tag{3}$$

for all $n \in\ N^+$. Let

$$e_n\ =\ d(a_n, p_{n+1}) + d(b_n, q_{n+1})$$

for all $n\ \ \in\ \ N^+$. Combining (2) and (3), we observe that

$$\begin{aligned} e_n &= d(a_n, p_{n+1})\ + d(b_n, q_{n+1})\\ &\le\ \beta_1\big(d(a_{n-1}, p_n)\big)d(a_{n-1}, p_n) + \beta_2\big(d(b_{n-1}, q_n)\big)d(b_{n-1}, q_n)\\ &\quad + \beta_1\big(d(b_{n-1}, q_n)\big)d(b_{n-1}, q_n) +\ \beta_2\big(d(a_{n-1}, p_n)\big)d(a_{n-1}, p_n)\\ &= (\beta_1 + \beta_2)\Big(\big(d(a_{n-1}, p_n) + d(b_{n-1}, q_n)\big)\Big)\ \big(d(a_{n-1}, p_n) + d(b_{n-1}, q_n)\big)\\ &\le e_{n-1}. \end{aligned}$$

Then we get

$$0 \leq e_n \leq e_{n-1} \leq e_{n-2} \leq \cdots e_0. \tag{4}$$

On the other hand,

$$\begin{aligned} d(a_{n+1}, p_n) &= d\big(F(a_n, b_n), F(p_{n-1}, q_{n-1})\big), \\ &\leq \beta\_1\big(d(a_n, p_{n-1})\big)d(a_n, p_{n-1}) + \beta_2\big(d(b_n, q_{n-1})\big)d(b_n, q_{n-1}) \end{aligned} \tag{5}$$

and

$$\begin{aligned} d(b_{n+1}, q_n) &= d\big(F(b_n, a_n), F(q_{n-1}, p_{n-1})\big), \\ &\leq \beta_1\big(d(b_n, q_{n-1})\big)d(b_n, q_{n-1}) + \beta_2\big(d(a_n, p_{n-1})\big)d(a_n, p_{n-1}) \end{aligned} \tag{6}$$

for all $n \in N^+$ and $\lambda < 1$. Let

$$s_n = d(a_{n+1}, p_n) + d(b_{n+1}, q_n)$$

for all n ∈ N^+. Combining (5) and (6), we observe that

$$\begin{aligned} s_n &= d(a_{n+1}, p_n) + d(b_{n+1}, q_n) \\ &\leq \beta_1\big(d(a_n, p_{n-1})\big)d(a_n, p_{n-1}) + \beta_2\big(d(b_n, q_{n-1})\big)d(b_n, q_{n-1}) \\ &\quad + \beta_1\big(d(b_n, q_{n-1})\big)d(b_n, q_{n-1}) + \beta_2\big(d(a_n, p_{n-1})\big)d(a_n, p_{n-1}) \\ &= (\beta_1 + \beta_2)(d(a_n, p_{n-1}) + d(b\_n, q\_(n-1)))(d(a_n, p_{n-1}) + d(b_n, q_{n-1}) \\ &= s_{n-1}. \end{aligned}$$

Then similar to Equation (4), we obtain that

$$0 \leq s_n \leq s_{n-1} \leq s_{n-2} \leq \cdots \ s_0. \tag{7}$$

Moreover,

$$\begin{aligned} d(a_n, p_n) &= d\big(F(a_{n-1}, b_{n-1}), F(p_{n-1}, q_{n-1})\big), \\ &\leq \beta_1\big(d(a_{n-1}, p_{n-1})\big)d(a_{n-1}, p_{n-1}) + \beta_2\big(d(b_{n-1}, q_{n-1})\big)d(b_{n-1}, q_{n-1}) \end{aligned} \tag{8}$$

and

$$\begin{aligned} d(b_n, q_n) &= d\big(F(b_{n-1}, a_{n-1}), F(q_{n-1}, p_{n-1})\big), \\ &\leq \beta_1\big(d(b_{n-1}, q_{n-1})\big)d(b_{n-1}, q_{n-1}) + \beta\_2\big(d(a_{n-1}, p_{n-1})\big)d(a_{n-1}, p_{n-1}) \end{aligned} \tag{9}$$

for all $n \in N^+$. Therefore, let

$$t_n = d(a_n, p_n) + d(b_n, q_n)$$

for all $n \in N^+$. Combining (8) and (9), we observe that

$$\begin{aligned} t_n &= d(a_n, p_n) + d(b_n, q_n) \\ &\leq \beta_1\big(d(a_{n-1}, p_{n-1})\big)d(a_{n-1}, p_{n-1}) + \beta_2\big(d(b_{n-1}, q_{n-1})\big)d(b_{n-1}, q_{n-1}) \\ &\quad +\beta_1\big(d(b_{n-1}, q_{n-1})\big)d(b_{n-1}, q_{n-1}) + \beta_2\big(d(a_{n-1}, p_{n-1})\big)d(a_{n-1}, p_{n-1}) \\ &= (\beta_1 + \beta_2)\big(d(a_{n-1}, p_{n-1}) + d(b_{n-1}, q_{n-1})\big)\big(d(a_{n-1}, p_{n-1}) + d(b_{n-1}, q_{n-1})\big) \\ &= t_{n-1}. \end{aligned}$$

Thus, we obtain that

$$0 \leq t_n \leq t_{n-1} \leq t_{n-2} \leq \cdots \ t_0. \tag{10}$$

Using the property (B3), we get

$$\begin{aligned} d(a_n, p_m) &\leq d(a_n, p_{n+1}) + d(a_{n+1}, p_{n+1}) + \cdots + d(a_{m-1}, p_m), \\ d(b_n, q_m) &\leq d(b_n, q_{n+1}) + d(b_{n+1}, q_{n+1}) + \cdots + d(b_{m-1}, q_m) \end{aligned} \tag{11}$$

and

$$\begin{aligned} d(a_m, p_n) &\leq d(a_m, p_{m-1}) + d(a_{m-1}, p_{m-1}) + \cdots + d(a_{n+1}, p_n), \\ d(b_m, q_n) &\leq d(b_m, q_{m-1}) + d(b_{m-1}, q_{m-1}) + \cdots + d(b_{n+1}, q_n) \end{aligned} \tag{12}$$

for each $m, n \in N$, $n < m$. Then, from (4), (7) , (10), (11) and (12), we have

$$\begin{aligned} d(a_n, p_m) + d(b_n, q_m) &\leq \big(d(a_n, p_{n+1}) + d(b_n, q_{n+1})\big) \\ &\quad +\big(d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})\big) + \cdots \\ &\quad +\big(d(a_{m-1}, p_{m-1}) + d(b_{m-1}, q_{m-1})\big) \\ &\quad +\big(d(a_{m-1}, p_m) + d(b_{m-1}, q_m)\big), \\ &= e_n + t_{n+1} + e_{n+1} + \cdots + t_{m-1} + e_{m-1}, \\ &< e_0 + t_0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} d(a_m, p_n) + d(b_m, q_n) &\leq \big(d(a_m, p_{m-1}) + d(b_m, q_{m-1})\big) \\ &\quad +\big(d(a_{m-1}, p_{m-1}) + d(b_{m-1}, q_{m-1})\big) + \cdots \\ &\quad +\big(d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})\big) \\ &\quad +\big(d(a_{n+1}, p_n) + d(b_{n+1}, q_n)\big), \\ &= s_n + t_{m-1} + t_{m-1} + \cdots + t_{n+1} + s_{n+1}, \\ &< s_0 + t_0 \end{aligned} \tag{14}$$

for $n < m$. Since, for an arbitrary $\epsilon > 0$, there exists n_0 such that $(e_0 + t_0) < \frac{\epsilon}{3}$ and $(s_0 + t_0) < \frac{\epsilon}{3}$ from (13) and (14) we have

$$d(a_n, p_m) + d(b_n, q_m) < \frac{\epsilon}{3}$$

for each $n, m \geq n_0$. Then (a_n, p_n) and (b_n, q_n) are Cauchy bisequences. Because of completeness of (X, Y, d), there exist a, b ∈ X and $p, q \in Y$ with

$$lim_{n\to\infty} a_n = p\ ,\ lim_{n\to\infty} b_n = q\ , lim_{n\to\infty}\, p_n = a,\ \ and\ lim_{n\to\infty}\, q_n = b\,. \tag{15}$$

Then there exists $n_1 \in N$ with

$$d(a_n, p) < \frac{\epsilon}{3},\ \ d(b_n, q) < \frac{\epsilon}{3},\ \ d(a, p_n) < \frac{\epsilon}{3}, and\ \ d(b, q_n) < \frac{\epsilon}{3}$$

for all $n \geq n_1$ and every $\epsilon > 0$. Since (a_n, p_n) and (b_n, q_n) are Cauchy bisequences, we get $d(a_n, p_n) < \frac{\epsilon}{3}$ *and* $d(b_n, q_n) < \frac{\epsilon}{3}$ So, from (1), we have

$$\begin{aligned} d(F(a,b),p) &\leq d(F(a,b),p_{n+1}) + d(a_{n+1},p_{n+1}) + d(a_{n+1},p) \\ &= d\big(F(a,b),F(p_n,q_n)\big) + d(a_{n+1},p_{n+1}) + d(a_{n+1},p) \\ &\leq \beta_1\big(d(a,p_n)\big)d(a,p_n) + \beta_2\, b\big(d(b,q_n)\big)d(b,q_n) + d(a_{n+1},p_{n+1}) + d(a_{n+1},p) \\ &< \beta_1\left(\frac{\epsilon}{3}\right)\frac{\epsilon}{3} + \beta_1\left(\frac{\epsilon}{3}\right)\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \frac{\epsilon}{3} + 2\frac{\epsilon}{3} \\ &< \epsilon \end{aligned}$$

for each $n \in N$ and $\lambda < 1$. Then $d(F(a,b),p) = 0$. Hence, $F(a,b) = p$. Similarly, we get

$F(b,a) = q,\ \ F(p,q) = a\ \ and\ \ F(q,p) = b$. On the other hand, from (15) we get

$$d(a,p) = d(\ lim_{n\to\infty}\, p_n, lim_{n\to\infty}\ a_n) = lim_{n\to\infty} d(a_n,p_n) = 0$$

and

$$d(b,q) = d(lim_{n\to\infty} q_n, \qquad lim_{n\to\infty} b_n) = lim_{n\to\infty} d(b_n,q_n) = 0.$$

So, $a = p$ and $b = q$. Therefore, $(a,b) \in X^2 \cap Y^2$ is a coupled fixed point of F.

Now, to show the uniqueness, we begin by taking another coupled fixed point $(a^*, b^*) \in X^2 \cup Y^2$. If $(a^*, b^*) \in X^2$, then we get

$$d(a^*,a) = d\big(F(a^*,b^*),F(a,b)\big) \leq kd(a^*,a) + ld(b^*,b)$$

and

$$d(b^*,b) = d\big(F(b^*,a^*),F(b,a)\big) \leq kd(b^*,b) + ld(a^*,a).$$

Therefore, we have

$$d(a^*,a) + d(b^*,b) \leq \lambda\left(d(a^*,a) + d(b^*,b)\right). \tag{16}$$

Since $\lambda < 1$, by (16) this means that $d(a^*,a) + d(b^*,b) = 0$. So, we obtain that $a^* = a$ and $b^* = b$. Similarly, if $(a^*, b^*) \in Y^2$, we have $a^* = a$ and $b^* = b$. Then (a,b) is a unique coupled fixed point of F.

The following corollary is obtained, if we take $\beta_1(t) = k$ and $\beta_2\,(t) = l$ in Theorem 6.

Corollary 7: Let (X, Y, d) be a complete bipolar metric space, $F : (X^2, Y^2) \to (X, Y)$ be a covariant mapping and k, l be non-negative constants. If F satisfies the condition

$$d\big(F(a,b),F(p,q)\big) \leq kd(a,p) + ld(b,q), \quad k + l < 1 \tag{17}$$

for all $a, b \in X, p, q \in Y$, then $F : X^2 \cup Y^2 \to X \cup Y$ has a unique coupled fixed point.

The following corollary is obtained, if we take equal the constants k, l in Theorem 6.

Corollary 8: Let $(X,\ Y,\ d)$ be a complete bipolar metric space, $F : (X^2,\ Y^2) \to (X,\ Y)$ be a covariant mapping and k, l be non-negative constants. If the condition

$$d\big(F(a,\ b),\ F(p,\ q)\big) \leq \frac{k}{2}\left[\big(d(a,\ p) + d(b,\ q)\big),\right]$$

$k < 1$ holds for all $a,\ b \in X, p,\ q \in Y$, then $F :\ X^2 \cup Y^2 \to X \cup Y$ has a unique coupled fixed point.

Now, we express another generalization of coupled fixed point theorem in bipolar metric spaces.

Definition 9: Let $(X,\ Y,\ d)$ be a bipolar metric space, $a \in X, p \in Y$ and $F :\ (X \times Y,\ Y \times X) \to (X,\ Y)$ be a covariant mapping. $(a,\ p)$ is said to be a coupled fixed point of F if $F(a,\ p) = a$ and $F(p,\ a) = p$.

Theorem 10: Let $(X,\ Y,\ d)$ be a complete bipolar metric space, $F : (X \times Y,\ Y \times X) \to (X,\ Y)$ be a covariant mapping and $\beta_1\ ,\beta_2 \in \Delta$. If the condition

$$d\big(F(a,\ p),\ F(q,\ b)\big) \leq \beta_1 a\big(d(a,\ q)\big)d(a,\ q) + \beta_2 b\big(d(b,\ p)\big)d(b,\ p), \tag{18}$$

holds for all $a, b \in X, p, q \in Y$, then $F: (X \times Y) \cup (Y \times X) \to X \cup Y$ has a unique coupled fixed point.

Proof: Similar to the proof of Theorem 6, we define bisequences (a_n, p_n) and (b_n, q_n) as follows,

$$a_{n+1} = F(a_n, b_n), b_{n+1} = F(b_n, a_n), \ p_{n+1} = F(p_n, q_n) \text{ and } q_{n+1} = F(q_n, p_n)$$

for all $n \in N^+$. Let $k + l = \lambda$. From \ref (18) , we get

$$\begin{aligned} d(a_n, q_{n+1}) &= d\big(F(a_{n-1}, p_{n-1}), F(q_n, b_n)\big), \\ &\le \beta_1\big(d(a_{n-1}, q_n)\big)d(a_{n-1}, q_n) + \beta_2\big(d(b_n, p_{n-1})\big)d(b_n, p_{n-1}) \end{aligned} \tag{19}$$

$$\begin{aligned} d(a_{n+1}, q_n) &= d\big(F(a_n, p_n), F(q_{n-1}, b_{n-1})\big), \\ &\le \beta_1\big(d(a_n, q_{n-1})\big)d(a_n, q_{n-1}) + \beta_2\big(d(b_{n-1}, p_n)\big)d(b_{n-1}, p_n) \end{aligned} \tag{20}$$

$$\begin{aligned} d(b_n, p_{n+1}) &= d\big(F(b_{n-1}, q_{n-1}), F(p_n, a_n)\big), \\ &\le \beta_1\big(d(b_{n-1}, p_n)\big)d(b_{n-1}, p_n) + \beta_2\big(d(a_n, q_{n-1})\big)d(a_n, q_{n-1}) \end{aligned} \tag{21}$$

$$\begin{aligned} d(b_{n+1}, p_n) &= d\big(F(b_n, q_n), F(p_{n-1}, a_{n-1})\big), \\ &\le \beta_1\big(d(b_n, p_{n-1})\big)d(b_n, p_{n-1}) + \beta_2\big(d(a_{n-1}, q_n)\big)d(a_{n-1}, q_n) \end{aligned} \tag{22}$$

for all $n \in N^+$. Let

$$e_n = d(a_n, q_{n+1}) + d(b_n, p_{n+1})$$

for all $n \in N^+$. Using (19), (20), (21) and (22), we get

$$\begin{aligned} e_n &= d(a_n, q_{n+1}) + d(b_n, p_{n+1}) \\ &\le \beta_1\big(d(a_{n-1}, q_n)\big)d(a_{n-1}, q_n) + \beta_2\big(d(b_{n-1}, p_n)\big)d(b_{n-1}, p_n) \\ &\quad + \beta_1\big(d(b_{n-1}, p_n)\big)d(b_{n-1}, p_n) + \beta_2\big(d(a_{n-1}, q_n)\big)d(a_{n-1}, q_n) \\ &= (\beta_1 + \beta_2)\Big(\big(d(a_{n-1}, q_n) + d(b_{n-1}, p_n)\big)\Big)\big(d(a_{n-1}, q_n) + d(b_{n-1}, p_n)\big) \\ &< e_{n-1}. \end{aligned}$$

and

$$\begin{aligned} s_n &= d(a\_(n+1), q\_(n)) + d(b\_n, p\_(n+1)) \\ &\le \beta_1\big(d(a_n, q_{n-1})\big)d(a_n, q_{n-1}) + \beta_2\big(d(b_{n-1}, p_n)\big)d(b_{n-1}, p_n) \\ &\quad + \beta_1\big(d(b_{n-1}, p_n)\big)d(b_{n-1}, p_n) + \beta_2\big(d(a_n, q_{n-1})\big)d(a_n, q_{n-1}) \\ &= (\beta_1 + \beta_2)\big(d(a_n, q_{n-1}) + d(b_{n-1}, p_n)\big)\big(d(a_n, q_{n-1}) + d(b_{n-1}, p_n)\big) \\ &< s_{n-1}. \end{aligned}$$

Then we get

$$0 \le e_n \, e_{n-1} \le e_{n-2} \le \cdots \ e_0. \tag{23}$$

and

$$0 \le e_n \le e_{n-1} \le e_{n-2} \le \cdots e_0. \tag{24}$$

$$\begin{aligned} d(a_n, p_n) &= d\big(F(a_{n-1}, p_{n-1}), F(q_{n-1}, b_{n-1})\big), \\ &\le \beta_1\big(d(a_{n-1}, q_{n-1})\big)d(a_{n-1}, q_{n-1}) + \beta_2\big(d(b_{n-1}, p_{n-1})\big)d(b_{n-1}, p_{n-1}) \end{aligned} \tag{25}$$

and

$$\begin{aligned} d(b_n, q_n) &= d\big(F(b_{n-1}, p_{n-1}), F(p_{n-1}, a_{n-1})\big), \\ &\le \beta_1\big(d(b_{n-1}, p_{n-1})\big)d(b_{n-1}, p_{n-1}) + \beta_2\big(d(a_{n-1}, q_{n-1})\big)d(a_{n-1}, q_{n-1}) \end{aligned} \tag{26}$$

for all $n \in N^+$. Let

$$t_n = d(a_n, q_n) + d(b_n, p_n)$$

for all $n \in N^+$. Combining (25) and (26), we observe that

$$\begin{aligned} t_n &= d(a_n, q_n) + d(b_n, p_n) \\ &\le \beta_1\big(d(a_{n-1}, q_{n-1})\big)d(a_{n-1}, q_{n-1}) + \beta_2\big(d(b_{n-1}, p_{n-1})\big)d(b_{n-1}, p_{n-1}) \\ &\quad + \beta_1\big(d(b_{n-1}, p_{n-1})\big)d(b_{n-1}, p_{n-1}) + \beta_2\big(d(a_{n-1}, q_{n-1})\big)d(a_{n-1}, q_{n-1}) \\ &= (\beta_1 + \beta_2)\big(d(a_{n-1}, q_{n-1}) + d(b_{n-1}, p_{n-1})\big)\big(d(a_{n-1}, q_{n-1}) + d(b_{n-1}, p_{n-1})\big) \\ &< t_{n-1}. \end{aligned}$$

So we get

$$0 \le t_n \le t_{n-1} \le t_{n-2} \le \cdots \le t_0. \tag{27}$$

We obtain that

$$\begin{aligned} d(a_n, q_m) &\le d(a_n, q_{n+1}) + d(a_{n+1}, q_{n+1}) + \cdots + d(a_{m-1}, q_m), \\ d(b_n, p_m) &\le d(b_n, p_{n+1}) + d(b_{n+1}, p_{n+1}) + \cdots + d(b_{m-1}, p_m), \\ d(a_m, q_n) &\le d(a_m, q_{m-1}) + d(a_{m-1}, q_{m-1}) + \cdots + d(a_{n+1}, q_n), \\ d(b_m, p_n) &\le d(b_m, p_{m-1}) + d(b_{m-1}, p_{m-1}) + \cdots + d(b_{n+1}, p_n) \end{aligned} \tag{28}$$

for each $n, m \in N,\ n < m$. Thus, from (23), (24) (27) and (28) , we have

$$\begin{aligned} d(a_n, q_m) + d(b_m, p_n) &\leq \big(d(a_n, q_{n+1}) + d(b_{n+1}, p_n)\big) \\ &\quad +\big(d(a_{n+1}, q_{n+1}) + d(b_{n+1}, p_{n+1})\big) + \cdots \\ &\quad +\big(d(a_{m-1}, q_{m-1}) + d(b_{m-1}, p_{m-1})\big) \\ &\quad +\big(d(a_{m-1}, q_m) + d(b_m, p_{m-1})\big), \\ &= e_n + t_{n+1} + e_{n+1} + \cdots + t_{m-1} + e_{m-1}, \\ &< e_0 + t_0 \end{aligned} \tag{29}$$

and

$$\begin{aligned} d(a_m, q_n) + d(b_n, p_m) &\leq \big(d(a_m, q_{m-1}) + d(b_{m-1}, p_m)\big) \\ &\quad +\big(d(a_{m-1}, q_{m-1}) + d(b_{m-1}, p_{m-1})\big) + \cdots \\ &\quad +\big(d(a_{n+1}, q_{n+1}) + d(b_{n+1}, p_{n+1})\big) \\ &\quad +\big(d(a_{n+1}, q_n) + d(b_n, p_{n+1})\big), \\ &= s_{m-1} + t_{m-1} + \cdots + s_{n+1} + t_{n+1} + s_n, \\ &\leq e_0 + t_0 \end{aligned} \tag{30}$$

for $n < m$. Since, for an arbitrary $\epsilon > 0$, there exists n_0 such that $e_0 + t_0 < \frac{\epsilon}{3}$ and $s_0 + t_0 < \frac{\epsilon}{3}$ from (13) and (14) we have

$$d(a_n, q_m) + d(b_m, p_n) < \frac{\epsilon}{3}$$

for each $n, m \geq n_0$. Then (a_n, p_n) and (b_n, q_n) are Cauchy bisequences. Because of completeness of (X, Y, d), there exist $a, b \in X$ and $p, q \in Y$ with

$$lim_{n\to\infty}\, a_n = q\ , lim_{n\to\infty} b_n = p\ , lim_{n\to\infty} p_n = b,\ and\ lim_{n\to\infty} q_n = a. \tag{31}$$

Then there exists $n_1 \in N$ with

$$d(a_n, q) < \frac{\epsilon}{3},\ d(b_n, p) < \frac{\epsilon}{3},\ \ d(b, p_n) < \frac{\epsilon}{3}\ and\ d(a, q\_n) < \frac{\epsilon}{3}$$

for all $n \geq n_1$ and every $\epsilon > 0$. Since (a_n, q_n) and (b_n, p_n) are Cauchy bisequences, we get

$$d(a_n, q_n) < \frac{\epsilon}{3} \text{ and } d(b_n, p_n) < \frac{\epsilon}{3}.$$

So, from (18), we have

$$\begin{aligned} d(F(a,p), q) &\leq d(F(a,p), q_{n+1}) + d(a_{n+1}, q_{n+1}) + d(a_{n+1}, q) \\ &= d\big(F(a,q), F(p_n, b_n)\big) + d(a_{n+1}, q_{n+1}) + d(a_{n+1}, q) \\ &\leq \beta_1\big(d(a, q_n)\big)\, d(a, q_n) + \beta_2\big(d(b, p_n)\big) d(b, p_n) + d(a_{n+1}, q_{n+1}) + d(a_{n+1}, q) \\ &< \beta_1\left(\frac{\epsilon}{3}\right)\frac{\epsilon}{3} + \beta_2\ \left(\frac{\epsilon}{3}\right)\frac{\epsilon}{3} + \ \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \frac{\epsilon}{3} + 2\frac{\epsilon}{3} \\ &< \epsilon \end{aligned}$$

for each $n \in N$ and $\lambda < 1$. Then $d(F(a,p), q) = 0$. Hence, $F(a,p) = q$. Similarly, we get

$$d(a, q) = d(\ lim_{n\to\infty}\ q_n, lim_{n\to\infty} a_n) = lim_{n\to\infty} d(a_n, q_n) = 0$$

and

$$d(b, p) = d(\ lim_{n\to\infty}\ p_n,\ \ lim_{n\to\infty} b_n) = lim_{n\to\infty} d(b_n, q_n) = 0.$$

So, a = q and b = p. Therefore, $(a, p) \in X^2 \cap Y^2$ is a coupled fixed point of F. As in the proof of the Theorem 6, uniqueness of the coupled fixed point of F can be shown easily.

In the setting of $\beta_1(t)$ and $\beta_2(t)$ if we take $\beta_1(t) = k$ and $\beta_2\ (t) = l$ in Theorem 6.

Corollary 11: Let $(X,\ Y,\ d)$ be a complete bipolar metric space, $F : (X \times\ Y,\ Y\ \times X) \to (X,\ Y)$ be a covariant mapping and k, l be non-negative constants. If the condition

$$d\big(F(a,\ p),\ F(q,\ b)\big) \leq kd(a,\ q)\ +\ ld(b,\ p),\quad k + l\ < 1$$

holds for all $a,\ b \in X, p,\ q \in Y$, then $F :\ (X \times Y\ ) \cup\ (Y \times\ X)\backslash to\ X \cup\ Y$ has a unique coupled fixed point.

Corollary 12: Let (X, Y, d) be a complete bipolar metric space. $F :\ (X\ \backslash times\ Y, Y\ \backslash times\ X) \to (X, Y)$ be a covariant mapping and k, l be non-negative constants. If the condition

$$d\big(F(a,p), F(q,b)\big) \leq \frac{k}{2}\ \ (d(a,q)\ +\ d(b,p)),\qquad k < 1$$

holds for all $a, b \in X,\ \ p, q \in\ Y$, then $F : (X \times\ Y\ ) \cup (Y \times\ X) \to\ X \cup\ Y$ has a unique coupled fixed point.

Example 13: Let $U_n(R)$ and $L_n(R)$ be the sets of all $n\times n$ upper and lower triangular matrices over R, respectively. A function $d: U_n(R)\times L_n(R)\to R^+$ be defined as

$$d(A,B)=\Sigma_{i,j=1}^{n}|a_{ij}-b_{ij}|$$

for all $A=(a_{ij})_{n\times n}\in U_n(R)$ and $B=(b_{ij})_{n\times n}\in L_n(R)$. Then it is apparent that $(U_n(R),L_n(R),d)$ is a complete bipolar metric space. We take a covariant mapping $F:(U_n(R)^2,L_n(R)^2)\to(U_n(R),L_n(R))$ such as

$$F(A,B)=\left(\frac{r}{2r+1}\ a_{ij}+\frac{r}{3r+1}\ b_{ij}\right)_{n\times n}$$

where $\left(A=(a_{ij})_{n\times n},\ B=(b_{ij})_{n\times n}\right)\in U_n(R)^2\cup L_n(R)^2$.
Then we get

$$\begin{aligned} d\big(F(A,B),\ F(C,D)\big) &= d\left(\left(\tfrac{a_{ij}+b_{ij}}{3}\right)_{n\times n},\ \left(\tfrac{c_{ij}+d_{ij}}{3}\right)_{n\times n}\right)\\ &= \Sigma_{i,j=1}^{n}\left|\left(\tfrac{r}{(2r+1)}a_{ij}+\tfrac{r}{3r+1}\ b_{ij}-\tfrac{r}{2r+1}c_{ij}-\tfrac{r}{3r+1}\ d_{ij}\right)\right|\\ &\le \Sigma_{i,j=1}^{n}\left(\left(\tfrac{r}{2r+1}\right)\left|\tfrac{(a_{ij}-c_{ij})}{3}\right|+\left(\tfrac{r}{3r+1}\right)\left|\tfrac{(b_{ij}-d_{ij})}{3}\right|\right)\\ &= \left(\frac{r}{2r+1}\right)d(A,C)+\left(\frac{r}{3r+1}\right)d(B,D) \end{aligned}$$

for all $A=(a_{ij})_{n\times n}, B=(b_{ij})_{n\times n}\in U_n(R)$ and $C=(c_{ij})_{n\times n}, D=(d_{ij})_{n\times n}\in L_n(R)$.
Therefore, the equation (17) is satisfied for $\beta_1(t)=\left(\frac{r}{2r+1}\right)$ *and* $\beta_2(t)=\left(\frac{r}{3r+1}\right)$. Then from Corollary 7, F has a unique coupled fixed point. It is obvious that the coupled fixed point is $(0_{n\times n},\ 0_{n\times n})\in U_n(R)\cap L_n(R)$ where $0_{n\times n}$ is the null matrix.

On the other hand, if

$$F:(U_n(R)^2,L_n(R)^2)\to\big(U_n(R),L_n(R)\big)$$

is defined by $F(A,B)=\left(\frac{a_{ij}+b_{ij}}{3}\right)_{n\times n}$
where $\left(A=(a_{ij})_{n\times n},\ B=(b_{ij})_{n\times n}\right)\in U_n(R)^2\cup L_n(R)^2$. Then it can be observed that

$$\left(\frac{(r)}{(2r+1)}\right)d(A,C)+\left(\frac{r}{3r+1}\right)d(B,D).$$

Then F satisfies the equation (17) for k = 1. Therefore, coupled fixed points of F are both $(0_{n\times n},\ 0_{n\times n})\in U_n(R)\cap L_n(R)$ and $(I_n,\ I_n)\in U_n(R)\cap L_n(R)$ where $0_{n\times n}$ is the null matrix and I_n is the identity matrix. As it can be seen from this expression, F has not a unique coupled fixed point. Thus, the conditions $\beta_1(t)+\beta_2(t)<1$ where ($\beta_1(t)=k$ *and* $\beta_2(t)=l$) in Corollary 7 and $\beta_1(t)+\beta_2(t)<1$ in Theorem 6 are the most appropriate conditions for satisfying the uniqueness of coupled fixed point.

Example 14: Let $U_n(R)$ and $L_n(R)$ be the sets of all $n\times n$ upper and lower triangular matrices over R, respectively. A function $d: U_n(R)\times L_n(R)\to R^+$ be defined as

$$d(A,B)=\Sigma_{i,j=1}^{n}|a_{ij}-b_{ij}|$$

for all $A=(a_{ij})_{n\times n}\in U_n(R)$ and $B=(b_{ij})_{n\times n}\in L_n(R)$. Then it is apparent that $(U_n(R),L_n(R),d)$ is a complete bipolar metric space. We take a covariant mapping $F:(U_n(R)^2,L_n(R)^2)\to\big(U_n(R),L_n(R)\big)$ such as

$$F(A,B)=\left(\frac{a_{ij}+b_{ij}}{3}\right)_{n\times n}$$

where $\left(A=(a_{ij})_{n\times n},\ B=(b_{ij})_{n\times n}\right)\in U_n(R)^2\cup L_n(R)^2$.
Then we get

$$\begin{aligned} d\big(F(A,B),\quad F(C,D)\big) &= d\left(\left(\frac{a_{ij}+b_{ij}}{3}\right)_{n\times n},\quad \left(\frac{c_{ij}+d_{ij}}{3}\right)_{n\times n}\right)\\ &= \Sigma_{i,j=1}^{n}\left|\frac{a_{ij}+b_{ij}-c_{ij}-d_{ij}}{3}\right|\\ &\le \Sigma_{i,j=1}^{n}\left(\left|\frac{a_{ij}-c_{ij}}{3}\right|+\left|\frac{b_{ij}-d_{ij}}{3}\right|\right)\\ &= \frac{1}{3}\big(d(A,C)+d(B,D)\big) \end{aligned}$$

for all $A=(a_{ij})_{n\times n}, B=(b_{ij})_{n\times n}\in U_n(R)$ and $C=(c_{ij})_{n\times n}, D=(d_{ij})_{n\times n}\in L_n(R)$.
Therefore, the equation (17) is satisfied for $k=\frac{2}{3}$. Then from Corollary 8, F has a unique coupled fixed point. It is obvious that the coupled fixed point is $(0_{n\times n},\ 0_{n\times n})\in U_n(R)\cap L_n(R)$ where $0_{n\times n}$ is the null matrix.

On the other hand, if $F:(U_n(R)^2, L_n(R)^2) \to (U_n(R), L_n(R))$ is defined by $F(A,B) = \left(\frac{a_{ij}+b_{ij}}{3}\right)_{n\times n}$ where $\left(A = (a_{ij})_{n\times n},\ B = (b_{ij})_{n\times n}\right) \in U_n(R)^2 \cup L_n(R)^2$. Then it can be observed that

$$d\big(F(A,B),\ F(C,D)\big) \leq \frac{1}{2}\left(d(A,C) + d(B,D)\right).$$

Then F satisfies the equation (17) for k = 1. Therefore, coupled fixed points of F are both $(0_{n\times n},\ 0_{n\times n}) \in U_n(R) \cap L_n(R)$ *and* $(I_n,\ I_n) \in U_n(R) \cap L_n(R)$ where $0_{n\times n}$ is the null matrix and I_n is the identity matrix. As it can be seen from this expression, F has not a unique coupled fixed point. Thus, the conditions k < 1 in Corollary 8 and k + l < 1 in Theorem 6 are the most appropriate conditions for satisfying the uniqueness of coupled fixed point.

ACKNOWLEDGMENT

The authors express deep gratitude to the chief editor and referees for valuable comments and suggestions.

REFERENCES

1. M. Abbas, M. Ali Khan, S. Radenovic. Common coupled fixed point theorems in cone metric spaces for w- compatible mappings, Appl. Math. Comput. 217, 195-202, 2010.
2. I. Altun, H. Simsek. Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. 2010, 17 pages, Article ID 621492, 2010.
3. T.G. Bhaskar, V. Lakshmikantham. Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65, 1379-1393, 2006.
4. Y.J. Cho, B.E. Rhoades, R. Saadati, B. Samet, W. Shatanawi. Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type, Fixed Point Theory Appl. 2012(8), 1-14, 2012.
5. L.j. Ciric, V. Lakshmikantham. Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, Stoch. Anal. Appl. 27, 1246-1259, 2009.
6. D. Guo, V. Lakshmikantham. Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. 11, 623-632, 1987.
7. E. Karapinar. Coupled fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl. 59, 3656-3668, 2010.
8. V. Lakshmikantham, L.j. Ciric. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70, 4341-4349, 2009.
9. A. Mutlu, N. Yolcu, B. Mutlu. Fixed point theorems in partially ordered rectangular metric spaces, British Journal of Mathematics & Computer Science 15(2), 1-9, 2016.
10. A. Mutlu, U. Gurdal. Bipolar metric spaces and some fixed point theorems, J. Non- linear Sci. Appl. 9(9), 5362-5373, 2016.
11. A. Mutlu, K. Ozkan, U. Gurdal. Coupled fixed point theorem in partially ordered modular metric spaces and its an application, J. Comput. Anal. Appl. 25(2), 207-216, 2018.
12. N. Tahat, H. Aydi, E. Karapinar, W. Shatanawi. Common fixed points for singlevalued and multi-valued maps satisfying a generalized contraction in G-metric spaces, Fixed Point Theory Appl. 2012(48), 2012, doi: 10.1186/1687-1812-2012-48.

Source of support: Nil, Conflict of interest: None Declared.