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On p-k Gamma Distribution

M. R. MAHMOUD AND R. M. MANDOUH*

Institute of Statistical Studies and Research, Cairo University, Egypt.

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ABSTRACT

Using p-k gamma function introduced by Gehlot (2017), a p-k gamma distribution with four parameters is constructed as a generalization of the p-k-gamma distribution with three parameters, the k-gamma distribution (Rahman et al., 2014) and the gamma distribution. Some properties of the new distribution are studied. Also, the estimation of the parameters is discussed using the method of maximum likelihood. The asymptotic variance-covariance matrix is obtained. Finally, a numerical study is provided.

***Keywords**: The p-k Gamma function; the k Gamma function; the p-k Gamma distribution; the k Gamma distribution; maximum likelihood estimates; asymptotic variance-covariance matrix.*

1. INTRODUCTION

In applied statistical work, popular distributions are modified and/or generalized using different directions. These directions are interested in deriving new distributions of univariate continuous distributions by introducing one or more additional shape parameter(s) to the baseline distribution. The Gamma (the Pearson type III) distribution is certainly among the most popular distributions. So we introduce a new general form of the gamma distribution through p-k gamma function defined by Gehlot (2017). Gehlot(2017) defined p-k gamma function as follows:

$${}_p\Gamma_k(m) = \int_0^\infty x^{m-1} e^{-x^k/p}\, dx, \quad p, k > 0, and\ Re(m) > 0. \tag{1}$$

This function contains the k-gamma function(with parameters m and k) as a particular case for p=k (Diaz and Pariguan, 2007) and the gamma function(with parameter m) when p=k=1. In the next section we review some definitions and notation used to obtain our results.

2. BASIC DEFINITIONS AND NOTATION

The Pochhammer symbol: The Pochhammer symbol or the rising factorial function $(m)_n$ takes the form
$$(m)_n = m(m+1)(m+2)\ldots(m+n-1); for\ n \geq 1, (m)_0 = 1, m \neq 0. \tag{2}$$

The Gamma Function: Let $m \in \mathbb{C}$; the Euler gamma function $\Gamma(m)$ is defined by
$$\Gamma(m) = \lim_{n\to\infty} \frac{n!\, n^{m-1}}{(m)_n}. \tag{3}$$
and the integral representation of gamma function is
$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx, \qquad Re(m) > 0. \tag{4}$$

An important and useful property of the gamma function is given by
$$\Gamma(m+1) = m\Gamma(m), \tag{5}$$

and it is related to the Pochhammer symbol through
$$(m)_n = \frac{\Gamma(m+n)}{\Gamma(m)}. \tag{6}$$

Corresponding Author: R. M. Mandouh*,
Institute of Statistical Studies and Research, Cairo University, Egypt.

The Beta Function: The beta function $B(m_1, m_2)$ is given by

$$B(m_1, m_2) = \int_0^1 x^{m_1-1}(1-x)^{m_2-1}\,dx, \qquad Re(m_1) > 0, Re(m_2) > 0. \tag{7}$$

The beta function can be expressed in terms of gamma function as follows

$$B(m_1, m_2) = \frac{\Gamma(m_1)\Gamma(m_2)}{\Gamma(m_1 + m_2)}. \tag{8}$$

For more details about these special functions and others such as hypergeometric function see Rainville (1960).

The Zeta Function: The Riemann's zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \;\; Re(s) > 1. \tag{9}$$

For more details about Riemann's zeta function and other special functions see Temme (1996).

The Hurwitz zeta function is given by

$$\zeta(m, s) = \sum_{n=0}^{\infty} \frac{1}{(n+m)^s}, \;\; m > 0. \tag{10}$$

when m=1 it gives the Riemann's zeta function defined in (9) (see Andrews *et al*. (1999).

Díaz and Pariguan (2007) introduced k-Pochhammer symbol $(m)_{n,k}$, k-Gamma function $\Gamma_k(m)$, k-Beta function $B_k(m)$. They proved several identities for $(m)_{n,k}$, $\Gamma_k(m)$ and $B_k(m)$. These identities were considered as a general form of identities satisfied by the classical Pochhammer symbol, gamma function and beta function. They also provided the integral representation for the $\Gamma_k(m)$ and $B_k(m)$.

The k-Pochhammer symbol: Let $m \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}^+$, the k-Pochhammer symbol is given by

$$(m)_{n,k} = m(m+k)(m+2k) \ldots\ldots\ldots (m+(n-1)k).$$

The k-Gamma Function: For k > 0, the k-gamma function Γ_k is

$$\Gamma_k(m) = \lim_{n\to\infty} \frac{n!\, k^n (nk)^{\frac{m}{k}-1}}{(m)_{n,k}}, \qquad m \in \mathbb{C}|k\mathbb{Z}^-. \tag{11}$$

with an integral form of k-Gamma function given by

$$\Gamma_k(m) = \int_0^{\infty} x^{m-1} e^{-x^k/k} dx, \qquad for\; m \in \mathbb{C}, \qquad Re(m) > 0. \tag{12}$$

Some Properties of k-gamma function:

$\Gamma_k(m+k) = m\Gamma_k(m)$.

$$(m)_{n,k} = \frac{\Gamma_k(m+nk)}{\Gamma_k(m)}.$$

$$\Gamma_k(k) = 1.$$

$$\Gamma_k(m) = a^{\frac{m}{k}} \int_0^{\infty} x^{m-1} e^{-\frac{x^k}{k}a} dx\,, for\; a \in \mathbb{R}.$$

For more properties of k-gamma function see Díaz and Pariguan (2007).

The k-Beta Function: The k-beta function $B_k(m_1, m_2)$ is given by

$$B_k(m_1, m_2) = \frac{\Gamma_k(m_1)\Gamma_k(m_2)}{\Gamma_k(m_1 + m_2)}, \qquad Re(m_1) > 0, \qquad Re(m_2) > 0. \tag{13}$$

The k-beta function satisfies the following identities:

$$B_k(m_1,m_2) = \int_0^{\infty} x^{m_1-1}(1+x^k)^{-\frac{m_1+m_2}{k}}\,dx.$$

$$B_k(m_1,m_2) = \frac{1}{k}\int_0^{1} x^{m_1/k-1}(1-x^k)^{\frac{m_2}{k}-1}\,dx.$$

$$B_k(m_1,m_2) = \tfrac{1}{k}B(m_1/k, m_2/k). \tag{14}$$

$$B_k(m_1,m_2) = \frac{(m_1+m_2)}{m_1 m_2}\prod_{n=0}^{\infty}\frac{nk(nk+m_1+m_2)}{(nk+m_1)(nk+m_2)}.$$

The k-Zeta Function: The k-zeta function is defined by

$$\zeta_k(m,s) = \sum_{n=0}^{\infty}\frac{1}{(m+ik)^s}, \qquad for\ k,m>0\ and\ s>0. \tag{15}$$

The k-zeta function satisfies the following identities:

$\zeta_k(m,2) = \partial_m^2(log\Gamma(m)).$

$\partial_m^2(\partial_s\zeta_k)|_{s=0} = -\partial_m^2(log\Gamma(m)).$

$$\partial_m^r(\partial_s\zeta_k(m,s)) = -m(s)_r\sum_{i=0}^{\infty}\frac{i^r}{(m+ik)^{r+s}}.$$

Also, Díaz and Pariguan (2007) discussed Hypergeometric function in view of the k-Pochhammer symbol. For more details see their article.

Gelton (2017) introduced the two parameter Pochhammer symbol, two parameter gamma function and two parameter beta function and named them, as p-k Pochhammer symbol, ${}_p(m)_{n,k}$, p-k gamma function, ${}_p\Gamma_k$ and p-k beta function, ${}_pB_k\,(m,n)$ respectively.

The p-k Pochhammer symbol: Let $m\in\mathbb{C}; k,p\in\mathbb{R}^+\ and\ Re(m)>0, n\in\mathbb{N}$, the The p-k Pochhammer symbol (i.e. Two Parameter Pochhammer Symobol), ${}_p(m)_{n,k}$ takes the form

$${}_p(m)_{n,k}, = (\frac{mp}{k})(\frac{mp}{k}+p)(\frac{mp}{k}+2p)\ldots\ldots\ldots(\frac{mp}{k}+(n-1)p).$$

and the relation between p-k Pochhammer symbol, k-Pochhammer symbol and classical Pochhammer symbol is

$${}_p(m)_{n,k}, = \left(\frac{p}{k}\right)^n (m)_{n,k} = (p)^n(\frac{m}{k})_n.$$

The p-k Gamma Function: Let $m\in\mathbb{C}/k\mathbb{Z}^-;\quad k,p\in\mathbb{R}^+-0$ and Re(m) > 0, n ∈ ℕ, the p-k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k\,(m)$ is given by

$${}_p\Gamma_k\,(m) = \frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{m}{k}-1}}{{}_p(m)_{n,k}\,(m)}, \tag{16}$$

with the following integral representation of p-k Gamma function

$${}_p\Gamma_k\,(m) = \int_0^{\infty} x^{m-1}e^{-x^k/p}\,dx. \tag{17}$$

The relation between p-k gamma function, k- gamma function and classical gamma function is given by

$${}_p\Gamma_k\,(m) = \left(\frac{p}{k}\right)^{\frac{m}{k}}\Gamma_k\,(m) = \frac{p^{\frac{m}{k}}}{k}\Gamma\left(\frac{m}{k}\right). \tag{18}$$

and the relation between p-k Pochhammer symbol and p-k gamma function take the form

$${}_p(m)_{n,k} = \frac{{}_p\Gamma_k\,(m+nk)}{{}_p\Gamma_k\,(m)}.$$

The p-k Beta Function: The p-k Beta function ${}_pB_k\,(m_1,m_2)$ is

$${}_pB_k\,(m_1,m_2) = \frac{{}_p\Gamma_k\,(m_1)\,{}_p\Gamma_k\,(m_2)}{{}_p\Gamma_k\,(m_1+m_2)}; Re(m_1)>0, Re(m_2)>0. \tag{19}$$

and the ${}_pB_k\,(m_1, m_2)$ function satisfies the following identities

$$ {}_pB_k\,(m_1, m_2) = \frac{1}{k}\int_0^1 x^{\frac{m_1}{k}-1}(1-x)^{\frac{m_2}{k}-1}\,dx. $$

$$ {}_pB_k\,(m_1, m_2) = \frac{1}{k}\int_0^1 \frac{x^{\frac{m_1}{k}-1}(1-x)^{\frac{m_2}{k}-1}}{(x+1)^{\frac{m_1+m_2}{k}}}\,dx. $$

$$ {}_pB_k\,(m_1, m_2) = \frac{1}{k}\int_0^\infty x^{\frac{m_1}{k}-1}(1+x^k)^{-\frac{m_1+m_2}{k}}\,dx. $$

$$ {}_pB_k\,(m_1, m_2) = \frac{1}{k}B(\frac{m_1}{k},\frac{m_2}{k}). \tag{20} $$

The p-k Psi Function: The logarithmic derivative of the p-k Gamma function is known as p-k Psi function, ${}_p\psi_k\,(m)$

$$ {}_p\psi_k\,(m) = \frac{d}{dm}\ln\left[{}_p\Gamma_k\,(m)\right] = \frac{1}{{}_p\Gamma_k\,(m)}\frac{d}{dm}\left[{}_p\Gamma_k\,(m)\right]. \tag{21} $$

and

$$ \ln\left[{}_p\Gamma_k\,(m)\right] = \int_1^m {}_p\psi_k\,(m)\,dx. \tag{22} $$

Some properties of ${}_p\psi_k\,(m)$ are given by

$$ {}_p\psi_k\,(m) = \frac{lnp}{k} + \psi\left(\frac{m}{k}\right). \tag{23} $$

$$ {}_p\psi_k\,(m) = \frac{lnp}{k} - \gamma - \frac{k}{m} + m\sum_{i=1}^{\infty}\frac{1}{i(m+ik)}. \tag{24} $$

$$ {}_p\psi_k\,(m) = \frac{lnp}{k} - \gamma + (m-k)\sum_{i=0}^{\infty}\frac{1}{(i+1)(m+ik)}. \tag{25} $$

where γ is Euler's constant and ψ(m) is classical Psi function. The r th derivative of p-k Psi function, ${}_p\psi_k\,(m)$ obtained in terms of k-Zeta function, $\zeta_k(m,r)$ as follows

$$ \frac{d^r}{dm^r}\left[\ln\left[{}_p\gamma_k\,(m)\right]\right] = \frac{d^{r-1}}{dm^{r-1}}\,{}_p\psi_k\,(m) = (-1)^r k(r-1)!\,\zeta_k(m,r),\ \text{for } r \geq 2 \tag{26} $$

where k-Zeta function is defined by

$$ \zeta_k(m,r) = \sum_{i=0}^{\infty}\frac{1}{(m+ik)^r}. \tag{27} $$

For more details about p-k Pochhammer symbol, p-k Gamma function, p-k Beta function, p-k Psi function and p-k Hypergeometric function see Gehlot (2017).

Using p-k gamma function studied by Gehton(2017), we introduce p-k gamma distribution with four parameters in the following section.

3. p-k Gamma Distribution

We say that the random variable X follows a p-k gamma distribution with four parameters m, λ, k and p if it has a density function of the form

$$ f(x) = \frac{\lambda^{-m}}{{}_p\Gamma_k\,(m)}x^{m-1}e^{-(x/\lambda)^k/p},\ m,p,k\ \ and\ \lambda > 0, \tag{28} $$

From (2) if λ = 1, we get p-k-gamma distribution with three parameters. If p=k and λ = 1, we have k-gamma distribution with two parameters (Rahman *et al*. (2017). If p=k=1, we have gamma distribution with two parameter. So the results obtained in this paper are general form for three distributions. Plots of the pdf of the p-k gamma distribution for different parameter values are given in figure (1).

The density (2) of the p-k gamma distribution is decreasing, for 0 < m < 1 and unimodal, for m > 1, with mode at the point

$$ x = (\frac{(m-1)p\lambda^k}{k})^{1/k} \tag{29} $$

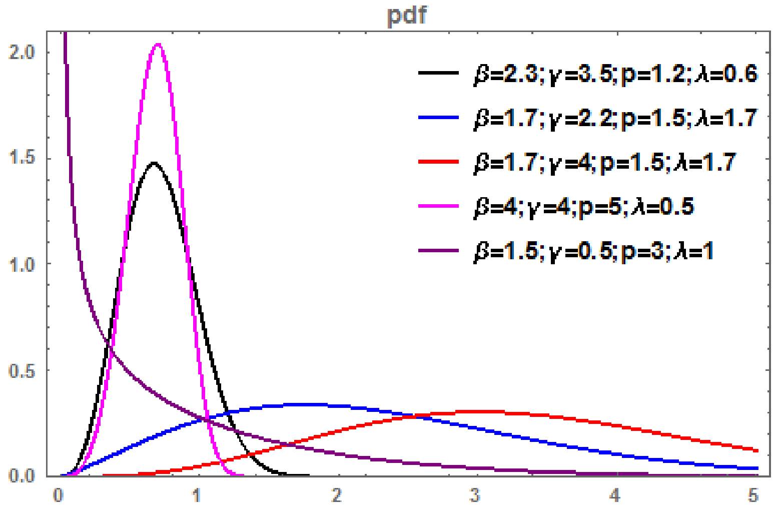

Figure-1: Probability density function of p-k Gamma distribution for different values of the parameters

The CDF of p-k gamma distribution takes the form

$$F(x) = \frac{\lambda^{-m}}{p^{\Gamma_k(m)}} \int_0^x x^{m-1} e^{-(x/\lambda)^k/p}\, dx$$
$$= \frac{1}{\Gamma(m/k)} \gamma(m/k, (x/k)^k/p). \tag{30}$$

where $\gamma(m/k, (x/k)^k/p)$ is incomplete gamma function. One can note that cdf does not have closed form but one can put it in expansion form as follow:

$$F(x) = \frac{1}{p^{\Gamma_k(m)}} \sum_{i=0}^{\infty} \frac{(-1)^i p^i x^{m+ik}}{i!\lambda^{m+ik}(m+ik)}. \tag{31}$$

The Hazard Rate Function of p-k gamma distribution: The hazard rate function of p-k gamma distribution is given by

$$h(x) = \frac{\frac{\lambda^{-m}}{p^{\Gamma_k(m)}} x^{m-1} e^{-(x/\lambda)^k/p}}{1 - \frac{\lambda^{-m}}{p^{\Gamma_k(m)}} \int_0^x x^{m-1} e^{-(x/\lambda)^k/p} dx}, \quad m, p, k \ \ and \ \ \lambda > 0. \tag{32}$$

Figure (2) illustrates possible shapes of (32) for selected parameter values. The shape appears increasing for (m > 1), constant for (p, k = 1), decreasing for (m < 1 and k < 1) and bathtub for (m < 1 and k < 1.)

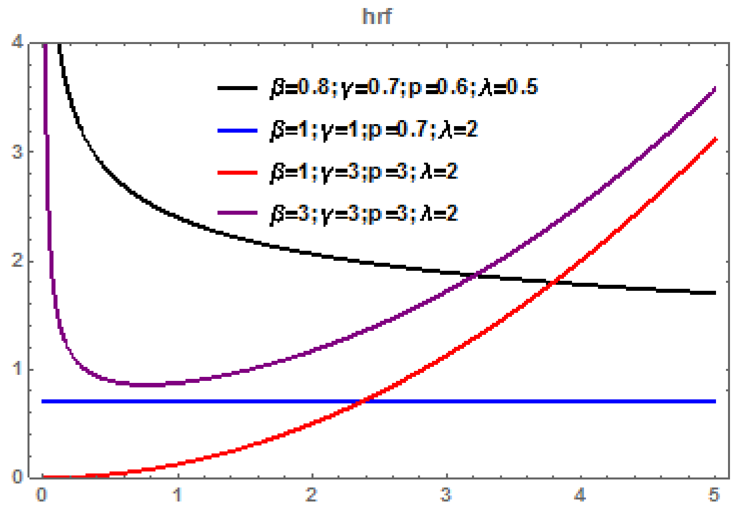

Figure-2: Hazard function of p-k Gamma distribution for different values of the parameters

The r th moment of p-k gamma distribution is given by:

$$E(X^r) = \frac{\lambda^{-m}}{{}_p\Gamma_k(m)}\int_0^\infty x^{m+r-1}e^{-(x/\lambda)^k/p}\,dx$$

$$= \frac{\lambda^{r}}{{}_p\Gamma_k(m)}\int_0^\infty y^{m+r-1}e^{-y^k/p}\,dx$$

$$= \frac{\lambda^r\,{}_p\Gamma_k(m+r)}{{}_p\Gamma_k(m)} \qquad (33)$$

Using (18), we have

$$E(X^r) = \frac{\lambda^r (p/k)\Gamma_k(m+r)}{\Gamma_k(m)} \qquad (34)$$

$$= \frac{\lambda^r p^{r/k}\Gamma\left(\frac{m+r}{k}\right)}{\Gamma(m/k)} \qquad (35)$$

So the mean is equal to $\frac{\lambda\, p^{1/k}\Gamma\left(\frac{m+r}{k}\right)}{\Gamma(m/k)}$ and the variance is equal to $\frac{\lambda^2 p^{2/k}}{\Gamma^2((m)/k)}\left[\Gamma((m+2)/k)\Gamma(m/k) - \Gamma^2((m+1)/k)\right]$.

Table 1 shows the mean and the variance of p-k gamma distribution for $\lambda = 1$ and different values of parameters m, p, and k.

Table-1: Mean and variance for different values of the three parameters m, k, and m of the p-k Gamma model

Parameters		P=0.5		P=1		P=3	
m	k	Mean	Variance	Mean	Variance	Mean	Variance
0.5	0.5	0.5	1.25	2	20	18000	1620.00
	0.7	0.31126	0.292370	0.83784	2.11844	4.02495	48.8900
	1.0	0.25000	0.125000	0.50000	0.50000	1.50000	4.50000
	3.0	0.25273	0.063888	0.31843	0.10140	0.45925	0.21091
	6.0	0.28089	0.067950	0.31529	0.08561	0.37864	0.12347
	10.0	0.29808	0.073260	0.31948	0.08415	0.35658	0.10483
0.7	0.5	0.84000	2.43600	3.36000	38.9760	30.2400	3157.06
	0.7	0.47025	0.47294	1.26582	3.42684	6.08100	79.0855
	1.0	0.35000	0.17500	0.70000	0.70000	2.10000	6.30000
	3.0	0.31971	0.07040	0.40281	0.11176	0.58095	0.23246
	6.0	0.34949	0.07075	0.39220	0.08913	0.47111	0.12855
	10.0	0.36927	0.07491	0.39577	0.08605	0.44173	0.10719
1.0	0.5	1.50000	5.25000	6.00000	84.0000	54.0000	6804.00
	0.7	0.73799	0.80857	1.98651	5.85870	9.54317	135.209
	1.0	0.50000	0.25000	1.00000	1.00000	3.00000	9.00000
	3.0	0.40119	0.07420	0.50547	0.11778	0.72901	0.24500
	6.0	0.42877	0.06889	0.48128	0.08680	0.57798	0.12518
	10.0	0.45025	0.07102	0.48256	0.08159	0.57798	0.10164
3.0	0.5	10.5000	78.7500	42.0000	1260.00	378.000	102060
	0.7	3.17534	4.88635	8.54739	35.4054	41.0615	817.097
	1.0	1.50000	0.75000	3.00000	3.00000	9.00000	27.0000
	3.0	0.70876	0.06636	0.89298	0.10533	1.28790	0.21910
	6.0	0.68063	0.04221	0.76398	0.05319	0.91749	0.07671
	10.0	0.69182	0.03718	0.74147	0.04272	0.82757	0.053199

One can notice that for fixed values of m and k, and $p \geq 1$ the mean and the variance are decreasing functions of k. For fixed m ($m < 1$) and p ($p < 1$), the mean and the variance first decrease ($k \leq 1$) then increases as k increases ($k < 1$). The mean and the variance are increasing functions of p when m and k are fixed. Also, the mean and the variance are increasing functions of m when k and p are fixed.

Also, using the formula (35) we will obtain the skewness and kurtosis of p-k gamma distribution for various combinations of the three parameters and these are given in table 2. So, one can note that for fixed m and p, the skewness decreases as k increases. For fixed m and k, the skewness is fixed when p increases. Also, the skewness is decreasing function of m when k and p are fixed. The same conclusion for kurtosis for $m < 1$ but there is no clear pattern for the kurtosis when $m \geq 1$.

Table-2: Skewness and Kurtosis for different values of the three parameters m, k, and m of the p-k Gamma model

Parameters		p=0.5		p=1		p=3	
m	k	Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
0.5	0.5	6.61876	84.7200	6.61876	84.7200	6.61876	84.7200
	0.7	4.18245	29.5209	4.18245	29.5209	4.18245	29.5209
	1.0	2.82843	12.0000	2.82843	12.0000	2.82843	12.0000
	3.0	1.16211	0.83789	1.16211	0.83789	1.16211	0.83789
	6.0	0.82333	-0.32380	0.82333	-0.32380	0.82333	-0.32380
	10.0	0.71749	-0.63487	0.71749	-0.63487	0.71749	-0.63487
0.7	0.5	5.36870	54.7027	5.36870	54.7027	5.36870	54.7027
	0.7	3.49837	20.5420	3.49837	20.5420	3.49837	20.5420
	1.0	2.39046	8.57143	2.39046	8.57143	2.39046	8.57143
	3.0	0.88141	0.20896	0.88141	0.20896	0.88141	0.20896
	6.0	0.52782	-0.71982	0.52782	-0.71982	0.52782	-0.71982
	10.0	0.40994	-0.97092	0.40994	-0.97092	0.40994	-0.97092
1.0	0.5	4.30201	34.4082	4.30201	34.4082	4.30201	34.4082
	0.7	2.89489	13.9844	2.89489	13.9844	2.89489	13.9844
	1.0	2.00000	6.00000	2.00000	6.00000	2.00000	6.00000
	3.0	0.63278	-0.15848	0.63278	-0.15848	0.63278	-0.15848
	6.0	0.25310	-0.86963	0.25310	-0.86963	0.25310	-0.86963
	10.0	0.114609	-1.05757	0.114609	-1.05757	0.114609	-1.05757
3.0	0.5	2.23121	8.76571	2.23121	8.76571	2.23121	8.76571
	0.7	1.62537	4.34224	1.62537	4.34224	1.62537	4.34224
	1.0	1.15470	2.00000	1.15470	2.00000	1.15470	2.00000
	3.0	0.16810	-0.27054	0.16810	-0.27054	0.16810	-0.27054
	6.0	0.29480	-0.33438	0.29480	-0.33438	0.29480	-0.33438
	10.0	0.54854	-0.18327	0.54854	-0.18327	0.54854	-0.18327

The Moment Generating Function of p-k gamma distribution

Here, we will obtain the moment generating function of the new distribution as follows:

$$E(e^{tx}) = \frac{\lambda^{-m}}{{}_p\Gamma_k(m)} \int_0^\infty x^{m-1} e^{-\left(\frac{((x/\lambda)^k}{p} - tx\right)} dx$$

$$= \sum_{i=0}^{\infty} \frac{\lambda^i t^i \, {}_p\Gamma_k(m+i)}{i! \, {}_p\Gamma_k(m)} \tag{36}$$

$$= \sum_{i=0}^{\infty} \frac{\lambda^i t^i (p/k)^i \Gamma_k(m+i)}{i! \, \Gamma_k(m)} \tag{37}$$

$$= \sum_{i=0}^{\infty} \frac{\lambda^i t^i (p)^{i/k} \Gamma((m+i)/k)}{i! \, \Gamma(m/k)} \tag{38}$$

Differentiating (36-38) r th times with respect to t and putting t equal to zero, we will obtain the r th moment of p-k gamma distribution.

Shannon's Entropy

To measure variation of uncertainty, Shannon in 1948 defined the entropy of a random variable X to be $\eta_X = E[-lnf(x)]$. Entropy has various applications in many fields such as science, engineering, and economics. Using (2), we get the following form of Shannon's entropy of p-k gamma:

$$\begin{aligned}
\eta_X &= E[-\ln\left(\frac{\lambda^{-m}}{{}_p\Gamma_k(m)} x^{m-1} e^{-(x/\lambda)^k/p}\right)] \\
&= E[mln(\lambda) + \ln\left({}_p\Gamma_k(m)\right) - (m-1)\ln(x) + (x/\lambda)^k/p] \\
&= mln(\lambda) + \ln({}_p\Gamma_k(m)) - (m-1)E[\ln(x)] + E[(x/\lambda)^k/p] \\
&= mln(\lambda) + \ln({}_p\Gamma_k(m)) - (m-1)E[\ln(x)] + \frac{1}{p}\frac{{}_p\Gamma_k(m+k)}{{}_p\Gamma_k(m)}
\end{aligned} \tag{39}$$

To compute $E[\ln(x)]$, we first compute $E(x^w)$ and differentiate it with respect to w then put $w = 0$. i.e. we have

$$\begin{aligned}\eta_X &= mln(\lambda) + {}_p\Gamma_k(m) - (m-1)\,{}_p\psi_k(m) + E[(x/\lambda)^k/p] \\ &= mln(\lambda) + {}_p\Gamma_k(m) - (m-1)\,{}_p\psi_k(m) + \frac{m}{kp}\left({}_p\Gamma_k(m)\right).\end{aligned} \quad (40)$$

For λ = 1 and p=k, we have Shannon's entropy of k-gamma distribution with parameters m and k and for p=k=1, we have Shannon's entropy of gamma distribution with parameters m and λ.

Functions of p-k Gamma Random Variables

If X and Y are independent, have p-k gamma distributions with pdfs given by (28) then $(x/\lambda)^k + (y/\lambda)^k/p$ has a gamma distribution $(\frac{n+m}{k}, p)$.

If X and Y are independent, have p-k gamma distributions with pdfs given by (28) then $\frac{(x/\lambda)^k}{(x/\lambda)^k+(y/\lambda)^k}$ has a p-k beta distribution with pdf $f(z) = \frac{1}{k\ {}_p|B_k(m,n)} z^{\frac{m}{k}-1}(1-z)^{\frac{n}{k}-1}$ or has $\frac{1}{k}$ Beta distribution(m/ k,n/ k).

3.1 Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n follows p-k gamma distribution. To estimate the four unknown parameters (p, k, m, λ) of the new distribution we obtain the likelihood function with the form:

$$L = \prod_{i=1}^{n} \frac{\lambda^{-m}}{{}_p\Gamma_k(m)} \prod_{i=1}^{n}(x_i^{\,m-1}) \prod_{i=1}^{n} e^{-(x_i/\lambda)^k/p}, \quad (41)$$

Taking the logarithm- of likelihood function, we get

$$lnL = -mln[\lambda] - nln\left[{}_p\Gamma_k(m)\right] + (m-1)\sum_{i=1}^{n} \ln[x_i] - \sum_{i=1}^{n} \frac{(x_i/\lambda)^k}{p}. \quad (42)$$

The first derivatives of the log-likelihood function are given as follows

$$\begin{aligned}\frac{\partial lnL}{\partial m} &= -n\,{}_p\psi_k(m) + \sum_{i=1}^{n} \ln[x_i], \\ \frac{\partial lnL}{\partial p} &= -\frac{nm}{kp} + \sum_{i=1}^{n} \frac{(x_i/\lambda)^k}{p^2}, \\ \frac{\partial lnL}{\partial k} &= n\left(\frac{1}{k} + \frac{m}{k^2}\left(\ln[p] + \psi\left(\frac{m}{k}\right)\right)\right) - \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\lambda}\right)^k}{p} \ln[x_i], \\ \frac{\partial lnL}{\partial \lambda} &= -\frac{n}{\lambda} + \sum_{i=1}^{n} \frac{k}{\lambda} \frac{\left(\frac{x_i}{\lambda}\right)^k}{p}.\end{aligned} \quad (43)$$

where ψ(.) the logarithmic derivative of gamma function is known as Psi function (digamma function).

Equating (43) to zero and solving them numerically, one can obtain the estimates of the unknown parameters. Now, the second order derivative of log-likelihood function are obtained as follows

$$\begin{aligned}\frac{\partial^2 lnL}{\partial n^2} &= -n\,{}_p\psi'_k(m) = -\frac{n}{k^2}\,\psi'\left(\frac{m}{k}\right) = -nk\zeta_k(m,2), \\ \frac{\partial^2 lnL}{\partial nk} &= nk^2\left(mln[p] + m\psi\left(\frac{m}{k}\right)\right), \\ \frac{\partial^2 lnL}{\partial mp} &= -\frac{nm}{pk}, \\ \frac{\partial^2 lnL}{\partial p^2} &= \frac{nm}{k^2 p} - 2\sum_{i=1}^{n} \left(\frac{x_i}{\lambda}\right)^k /p^3, \\ \frac{\partial^2 lnL}{\partial pk} &= \frac{nm}{k^2 p} + \sum_{i=1}^{n} x_i^{\,k}/p^2 \ln[x_i], \\ \frac{\partial^2 lnL}{\partial pm} &= -\frac{n}{kp},\end{aligned}$$

$$\frac{\partial^2 lnL}{\partial k^2} = -n(-2\frac{m}{k^3}\ln[p] + \frac{1}{k^2} + 2\frac{m}{k^3}\psi\left(\frac{m}{k}\right) - \frac{m^2}{k^4}\psi'\left(\frac{m}{k}\right) - \frac{1}{p}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k \ln[x_i]), \tag{44}$$

$$\frac{\partial^2 lnL}{\partial km} = n(\frac{\ln[p]}{k^2} + \frac{1}{k^2}\psi\left(\frac{x}{k}\right)),$$

$$\frac{\partial^2 lnL}{\partial kp} = \frac{nm}{pk^2} - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k / p^2 \ln[x_i]),$$

$$\frac{\partial^2 lnL}{\partial p\lambda} = -\frac{n}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k / p^2,$$

$$\frac{\partial^2 lnL}{\partial k\lambda} = \frac{k}{\lambda} + \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k / pln[x_i] + \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k / p,$$

$$\frac{\partial^2 lnL}{\partial \lambda^2} = \frac{n}{\lambda^2} + \frac{k(k+1)}{\lambda^2}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^k}{p}.$$

where $\psi'(.)$ the first derivative of Psi function and $\psi^{(n)}(.)$ the nth derivative of Psi function which is known as PolyGamma and $\zeta_k(.,.)$ k-Zeta function defined by $\zeta_k(m,r) = \sum_{i=1}^{\infty}\frac{1}{(x+nk)^r}$. and the information matrix is given by

$$J(\theta) = -\begin{bmatrix} I_{m,m} & I_{m,p} & I_{m,k} & I_{m,\lambda} \\ I_{p,m} & I_{p,p} & I_{p,k} & I_{p,k} \\ I_{k,m} & I_{k,p} & I_{k,k} & I_{k,\lambda} \\ I_{\lambda,m} & I_{\lambda,p} & I_{\lambda,k} & I_{\lambda,\lambda} \end{bmatrix} \tag{45}$$

where I is the expected value of the second derivative of the log-likelihood with respect to the parameters. Inverting the information matrix and replacing the unknown parameters by their mles to obtain the asymptoticvariance-covariance matrix of $(\widehat{m}, \hat{p}, \hat{k}, \widehat{\lambda})$. 100(1−γ) % approximate confidence intervals for the parameters m, p, k and λ are respectively,

$\left(\widehat{m} \pm z_{\frac{\gamma}{2}}\sqrt{var(\widehat{m})}\right)$,

$\left(\hat{p} \pm z_{\frac{\gamma}{2}}\sqrt{var(\hat{p})}\right)$,

$\left(\hat{k} \pm z_{\frac{\gamma}{2}}\sqrt{var(\hat{k})}\right)$, *and*

$(\hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{var(\hat{\lambda})})$.

3.2 Numerical Study

Now, we generate 1000 random samples of p-k gamma distribution with size 50 to study the behavior of the mle via absolute value of relative bias (ARbias) and scaled root mean square error (SRmse). With different actual values of the four parameters one can note that the ARbias and the SRmse **for λ** decreases when λ < 1 than for λ > 1. **For k** (with m, p, and λ < 1), the ARbias decreases when k > 1 than k < 1 while the SRmse increases for k > 1 than k < 1 and vise verse when m, p, and λ > 1. **For m** (if k, p, and λ < 1) the ARbias and SRmse increases for m > 1 than m < 1 but if k, p, and λ > 1 the ARbias increases but the SRmse decreases for m > 1. **For p** m, k, and λ < 1 the ARbias and the SRmse increase for p > 1 than p < 1 but if k, p, and λ < 1 the ARbias and the SRmse decrease for p > 1.

Table-3: MLEs, RAbias and SRmse of the four unknown parameters of the p-k gamma distribution for various values of the parameters at sample size = 50

Actual value				m	k	p	λ
m,	k,	p,	λ	Mean			
0.5	0.5	0.5	1.5	1.082280	0.285767	0.338020	0.330207
0.5	0.5	0.5	0.5	0.964707	0.292705	0.246071	0.364204
3.5	1.5	1.5	1.5	2.543320	1.405990	0.544112	0.717068
1.5	0.5	0.5	0.5	1.06616	0.255315	0.178401	0.491731
0.5	1.5	0.5	0.5	1.62158	0.576532	0.590480	0.561303
0.5	0.5	1.5	0.5	1.19404	0.328340	0.403676	0.501633
3.0	3.0	0.5	0.5	2.29170	0.416443	0.428648	0.148639
3.0	3.0	3.0	0.5	3.96368	0.580175	0.22023	0.509545
3.0	3.0	3.0	3.0	0.849434	0.788447	1.39960	1.13084

				RAbias			
0.5	0.5	0.5	1.5	1.164560	0.437730	0.428467	0.774653
0.5	0.5	0.5	0.5	0.929414	0.414589	0.507859	0.271593
3.5	1.5	1.5	1.5	0.695548	0.598289	0.637259	0.521955
1.5	0.5	0.5	0.5	1.172330	0.829790	0.643199	0.016539
0.5	1.5	0.5	0.5	0.081051	0.153065	0.180960	0.122607
0.5	0.5	1.5	0.5	1.388080	0.343319	0.730883	0.003266
3.0	3.0	0.5	0.5	0.236092	0.861186	0.142704	0.702721
3.0	3.0	3.0	0.5	0.321226	0.806608	0.926590	0.019895
3.0	3.0	3.0	3.0	0.716855	0.737184	0.533468	0.623053
				SRmse			
0.5	0.5	0.5	1.5	1.636170	0.533277	0.570789	0.778332
0.5	0.5	0.5	0.5	1.363130	0.501656	0.615331	0.377077
3.5	1.5	1.5	1.5	0.935574	1.019900	0.664488	0.653378
1.5	0.5	0.5	0.5	1.566090	0.832005	0.679550	0.400326
0.5	1.5	0.5	0.5	0.578083	2.522110	1.146430	1.234466
0.5	0.5	1.5	0.5	1.625490	0.445592	0.764855	0.938695
3.0	3.0	0.5	0.5	0.788617	0.863460	0.393248	0.773012
3.0	3.0	3.0	0.5	0.788199	0.810226	0.987246	0.746850
3.0	3.0	3.0	3.0	0.718401	0.740945	0.578034	0.636148

4. SUMMARY

In this paper we introduced a new distribution via two parameter gamma function called p-k gamma function. Some properties of the new distribution are derived and estimation of unknown parameters is discussed.

REFERENCES

1. G. E. Andrews, R. Askey and R. Roy (1999). Special Functions, Cambridge University Press.
2. R. Diaz and E. Pariguan (2017). On Hypergeometric Functions and Pohchammer k-Symbol, Divulgaciones Mathematicas, 15(2): 179-192.
3. K. S. Gehlot, (2017). Two Parameter Gamma Function and It's Properties, https: //arxiv.org/pdf/1701.01052, 11-Nov-2018.
4. S. Mubeen, N. Sadiq and F. Shaheen, Properties of k-gamma, k-beta and k-psi functions. Bothalia Journal, 4(2014): 371-379. 13
5. G. Rahman, S. Mubeen, A. Rehman and M. Naz, (2014). On k-Gamma and k-Beta Distributions and Moment Generating Functions, Journal of Probability and Statistics, Volume 2014, Article ID 982013, 6 pages.
6. E. D. Rainvile (1960). Special Functions. the Macmillan, New York, USA.
7. N. M. Temme (1996). Special Functions: An Introduction to the Classical Functions of Mathematical Physics, John Wiley and Sons, Inc.

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