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RAINBOW TOTAL CONNECTION NUMBER OF SOME WHEEL RELATED GRAPHS

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ABSTRACT

A path P connecting two vertices u and v in a totally colored graph G is called a rainbow total-path between u and v if all elements in $V(P) \cup E(P)$*, except for u and v, are assigned distinct colors. A total-colored graph is rainbow total-connected if it has a rainbow total-path between every two vertices. The rainbow total-connection number of a graph G is the minimum colors such that G is rainbow total-connected. In this paper, we gave the rainbow total-connection number of sunflower graph and lotus inside circle graph.*

Keywords: *total-colored graph; rainbow total-connection number; sunflower graph; lotus inside circle.*

I. INTRODUCTION

Chartrand *et al.* [2008] introduced the concept rainbow coloring. They determined rainbow connection number of the cycle, path, tree and wheel graphs. Since then many are studying the concept. Please see Li *et al.* [2013] and Sun et al. [2012]. Li *et al.* [2013] studied the rainbow connection numbers of line graphs in the light of particular properties of line graphs and gave two sharp upper bounds for rainbow connection number of a line graph. While, Sun *et al.* [2012] investigated the rainbow connection number of the line graph, middle graph and total graph of a connected triangle-free graph and obtained three (near) sharp upper bounds in terms of the number of vertex-disjoint cycles of the original graph. Continuing the study of rainbow coloring, Uchizawa *et al.* [2011] introduced and studied the rainbow total-connection number of graphs. Later, Sun [2013, 2015] also studied rainbow total-connection number. They characterized the rainbow total-connection number of trees, and gave the rainbow total-connection number of cycles, path and wheels. In this paper, we gave the rainbow total-connection number of some wheel related graphs. In particular, we gave the rainbow total-connection number of sunflower graphs, lotus inside circle and helms.

A graph *G* is an ordered pair (V, E) where *V* is a non-empty finite set and *E* is a family of two element subsets of *V*. The elements of *V* are called *vertices* and the elements of *E* are called *edges*. If $\{u,v\}$ is an edge, then we say that vertices *u* and *v* are *adjacent*, and that *u* and *v* are *incident* to $\{u,v\}$. We write edge $\{u,v\}$ concisely as uv. The *path* $P_n = (v_1, v_2, \ldots, v_n)$ is the graph with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. The *cycle* $C_n = [v_1, v_2, \ldots, v_n]$ is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. A *complete* graph K_n is the graph with *n* vertices and any two vertices is connected by an edge. The *complement* of a graph *G*, denoted by $\overline{G}$, is the graph with the same vertices as *G* and two vertices in $\overline{G}$ are adjacent if they are not adjacent in *G*.

A total coloring of a graph $G = (V, E)$ is a function *f* from $V \cup E$ to a set *C* whose elements are called *colors*. In this case, we say that $G(f)$ is totally colored. A path *P* connecting two vertices *u* and *v* in a totally colored graph *G* is called a rainbow total-path between *u* and *v* if all the elements in $[V(P) \cup E(P)] \setminus \{u,v\}$ are assigned distinct colors. The total-colored graph is *rainbow total-connected* if it has a rainbow total-path in between every two vertices. The *rainbow total-connection number* of a graph *G* is the minimum colors such that *G* is rainbow total-connected.

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The following classes of graphs are found Ponraj *et al.* [2015]. The *graph lotus inside circle*, denoted by LC_n, is the graph of order $2n+1$ obtained by joining each vertex u_i of the star $K_{1,n}=(\{u\},\varnothing)+(\{u_1,u_2,\ldots,u_n\},\varnothing)$ to vertices w_i and $w_{i+1(\bmod n)}$ of the cycle $C_n=[w_1,w_2,\ldots,w_n]$. The *helm* H_n of order $2n+1$ is the graph obtained from $W_n=(\{u\},\varnothing)+[w_1,w_2,\ldots,w_n]$ by attaching pendant edges v_iw_i for every $i=1,2,\ldots,n$. The *sunflower* graph SF_n of order $2n+1$ obtained by adding vertices w_i joined by edges to vertices v_i and $v_{i+1(\bmod n)}$ of the $W_n=(\{v\},\varnothing)+[v_1,v_2,\ldots,v_n]$ for every $i=1,2,\ldots,n$.

Hereafter, please refer to Yellen *et al.* [2000] for concepts that are used but were not discussed in this paper.

In this study, we determined the rainbow total-connection number of lotus inside circle and sunflower graphs.

II. RESULTS

This section presents the results of this study.

A. Total Rainbow Connection Number of Lotus Inside a Circle

This subsection gives the total rainbow connection number of lotus inside a circle graph. Remark 1 states that the rainbow connection number of a graph *G* is greater than or equal twice its diameter.

Remark 1: *Let $G=(V,E)$ be a graph with diameter d. Then $rtc(G)\geq 2d-1$.*

To see this, let $u,v\in V$ such that the distance in between *u* and *v*, $d(u,v)$, is equal to *d*. Note that any rainbow path connecting *u* and *v* requires $2d-1$ colors. Hence, $rtc(G)=2d-1$.

Theorem 2.2: *Let $C_n=[w_1,w_2,\ldots,w_n]$ be a cycle of order n, and $K_{1,n}=(\{u_0\},\varnothing)+(\{u_1,u_2,\ldots,u_n\},\varnothing)$ be a star of order $n+1$. Let LC_n be the graph lotus inside circle obtained by joining each vertex u_i to vertices w_i and $w_{i+1(\bmod n)}$. Then*

$$rtc(LC_n)=\begin{cases}4 & , \text{ if } n=3,4\\ 6 & , \text{ if } n=5,6\\ 7 & , \text{ if } n\geq 7\ .\end{cases}$$

Proof:
For $n=3$, define $f_3:V(LC_3)\cup E(LC_3)\to\{1,2,3,4\}$ as follows:

Table-1: *Images of the elements of $V(LC_3)\cup E(LC_3)$*

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	1	u_0u_1	4	u_2w_3	4
u_1	3	u_0u_2	4	u_3w_3	4
u_2	3	w_3w_1	2	u_3w_1	2
u_3	3	u_0u_3	2	w_1w_2	2
w_1	1	u_1w_1	2	w_2w_3	2
w_2	1	u_1w_2	4		
w_3	1	u_2w_2	2		

Then f_3 is a total rainbow 4-coloring of LC_3. Hence, $rtc(LC_3)\leq 4$. Since the diameter of of LC_3 is equal to 2, by Remark 1, we must have, $rtc(LC_3)=4$. For $n=4$, define $f_4:V(LC_4)\cup E(LC_4)\to\{1,2,3,4\}$ as follows:

Table-2: *Images of the elements of* $V(LC_4) \cup E(LC_4)$

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	3	u_0u_1	2	u_3w_4	4
u_1	3	u_0u_2	2	u_4w_4	3
u_2	4	u_0u_3	4	u_4w_1	3
u_3	3	u_0u_4	1	w_1w_2	4
u_4	4	u_1w_1	2	w_2w_3	3
w_1	1	u_1w_2	4	w_3w_4	4
w_2	1	u_2w_2	2	w_4w_1	3
w_3	1	u_2w_3	3		
w_4	1	u_3w_3	2		

Then f_4 is a total rainbow 4-coloring of LC_4. Hence, $rtc(LC_3) \le 4$. Since the diameter of of LC_4 is equal to 2, by Remark 1, we must have, $rtc(LC_4) = 4$. For $n = 5$, define $f_5 : V(LC_5) \cup E(LC_5) \to \{1,2,3,4,5,6\}$ as follows:

Table-3: *Images of the elements of* $V(LC_5) \cup E(LC_5)$

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	6	u_0u_1	1	u_4w_4	1
u_1	5	u_0u_2	1	u_4w_5	3
u_2	5	u_0u_3	2	u_5w_5	1
u_3	6	u_0u_4	2	u_5w_1	2
u_4	5	u_0u_5	3	w_1w_2	2
u_5	5	u_1w_1	3	w_2w_3	1
w_1	6	u_1w_2	2	w_3w_4	3
w_2	6	u_2w_2	3	w_4w_5	2
w_3	6	u_2w_3	2	w_5w_1	1
w_4	4	u_3w_3	3		
w_5	4	u_3w_4	3		

Then f_5 is a total rainbow 6-coloring of LC_5. Hence, $rtc(LC_5) \le 6$. Since the diameter of of LC_5 is equal to 3, by Remark 1, we must have, $rtc(LC_5) = 6$. For $n = 6$, define $f_6 : V(LC_6) \cup E(LC_6) \to \{1,2,3,4,5,6\}$ as follows:

Table-4: *Images of the elements of* $V(LC_6) \cup E(LC_6)$

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	4	u_0u_1	2	u_4w_5	1
u_1	4	u_0u_2	2	u_5w_5	3
u_2	6	u_0u_3	3	u_5w_6	3
u_3	5	u_0u_4	3	u_6w_6	2
u_4	4	u_0u_5	1	u_6w_1	2
u_5	6	u_0u_6	1	w_1w_2	1
u_6	5	u_1w_1	1	w_2w_3	3
w_1	5	u_1w_2	1	w_3w_4	2
w_2	4	u_2w_2	3	w_4w_5	1
w_3	6	u_2w_3	3	w_5w_6	3
w_4	5	u_3w_3	2	w_6w_1	2
w_5	4	u_3w_4	2		
w_6	6	u_4w_4	1		

Then f_6 is a total rainbow 6-coloring of LC_6. Hence, $rtc(LC_6) \le 6$. Since the diameter of of LC_6 is equal to 3, by Remark 1, we must have, $rtc(LC_6) = 6$. For $n = 7$, define $f_7 : V(LC_7) \cup E(LC_7) \to \{1, 2, \ldots, 8\}$ as follows:

Table-5: *Images of the elements of* $V(LC_7) \cup E(LC_7)$

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	5	u_0u_1	1	u_5w_5	4
u_1	8	u_0u_2	2	u_5w_6	4
u_2	7	u_0u_3	1	u_6w_6	3
u_3	8	u_0u_4	2	u_6w_7	3
u_4	7	u_0u_5	1	u_7w_7	4
u_5	8	u_0u_6	2	u_7w_1	4
u_6	7	u_0u_7	3	w_1w_2	1
u_7	8	u_1w_1	4	w_2w_3	1
w_1	8	u_1w_2	4	w_3w_4	1
w_2	8	u_2w_2	3	w_4w_5	1
w_3	7	u_2w_3	3	w_5w_6	1
w_4	8	u_3w_3	4	w_6w_7	1
w_5	7	u_3w_4	4	w_7w_1	1
w_6	8	u_4w_4	3		
w_7	7	u_4w_5	3		

Then f_7 is a total rainbow 7-coloring of LC_7. It can be shown that a total rainbow coloring of LC_7 can not have a fewer than 7 colors. Hence, $rtc(LC_7) = 7$.

For $n \ge 8$ and n is even, Then we define $f_n : V(LC_n) \cup E(LC_n) \to \{1, 2, \ldots, 7\}$ as follows:

$$f_n(x) = \begin{cases} 1 & , \text{ if } x = w_i w_{i+1 (\bmod n)} \text{ or } x = u_0 u_i \text{ with } i \text{ odd} \\ 2 & , \text{ if } x = u_0 u_i \text{ with } i \text{ even} \\ 3 & , \text{ if } x = u_i w_i \text{ or } x = u_i w_{i+1 (\bmod n)}, \text{ with } i \text{ odd} \\ 4 & , \text{ if } x = u_i w_i \text{ or } x = u_i w_{i+1 (\bmod n)}, \text{ with } i \text{ even} \\ 5 & , \text{ if } x = w_i \text{ or } x = u_{i+1 (\bmod n)}, \text{ with } i \text{ odd} \\ 6 & , \text{ if } x = u_0 \\ 7 & , \text{ if } x = w_i \text{ or } x = u_{i+1}, \text{ with } i \text{ even} \end{cases}$$

Let $w, v \in V(LC_n)$ and consider the following cases:

Case-1: $\deg(w) = 4$

If $\deg(w) = 4$, then $w = w_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase-1: $\deg(v) = 4$

If $\deg(v) = 4$, then $v = w_j$ for some $j = 1, 2, \ldots, n$ with $j \ne i$. Note that if *i* and *j* have the same parity, then $(w_i, u_i, u_0, u_{j-1}, w_j)$ is a rainbow path connecting *w* and *v*, and if *i* and *j* have the different parity, then $(w_i, u_i, u_0, u_j, w_j)$ is a rainbow path connecting *w* and *v*.

Subcase-2: $\deg(v) = 3$

If $\deg(v) = 3$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. If $i = j - 1$, then (u_i, w_j) is a rainbow path connecting *w* and *v*, and if $i = j$, then (u_i, w_j) is a rainbow path connecting *w* and *v*. If *i* and *j* have the same parity with $j \ne j-1, j$, then (w_i, u_{i-1}, u_0, u_j) is a rainbow path connecting *w* and *v*, and if *i* and *j* have the different parity with $j \ne j-1, j$, then (w_i, u_i, u_0, u_j) is a rainbow path connecting *w* and *v*.

Subcase-3: $\deg(v) = n$

If $\deg(v) = n$, then $v = u_0$. Note that (w_i, u_i, u_0) is a rainbow path connecting *w* and *v*.

Case-2: $\deg(w) = 3$

If $\deg(w) = 3$, then $w = u_i$ for some $i = 1, 2, \ldots, n$. Consider the following subcases:

Subcase-1: $\deg(v) = 3$

If $\deg(v) = 3$, then $v = u_j$ for some $j = 1, 2, \ldots, n$. If *i* and *j* have the same parity, then $(u_i, u_0, u_{j+1}, w_{j+1}, u_j)$ is a rainbow path connecting *w* and *v*, and if *i* and *j* have the different parity, then (u_i, u_0, u_j) is a rainbow path connecting *w* and *v*.

Subcase-3:. $\deg(v) = n$

If $\deg(v) = n$, then $v = u_0$. Note that (u_i, u_0) is a rainbow path connecting *w* and *v*.

Hence, f_n is a total rainbow 7-coloring of LC_n. Hence, $rtc(LC_n) \le 7$. Since the diameter of of LC_n is equal to 4, by
Remark 1, we must have, $rtc(LC_n) = 7$ if $n \ge 8$ and *n* is even.

For $n \ge 9$ and *n* is odd, Then we define $f_n : V(LC_n) \cup E(LC_n) \to \{1, 2, \ldots, 7\}$ as follows:

$$f_n(x) = \begin{cases} 1 & , \text{ if } x = w_i w_{i+1(\bmod n)} \text{ or } x = u_0 u_i \text{ with } i \text{ odd} \\ 2 & , \text{ if } x = u_0 u_i \text{ with } i \text{ even} \\ 3 & , \text{ if } x = u_i w_i \text{ or } x = u_i w_{i+1(\bmod n)}, \text{ with } i \text{ odd} \\ 4 & , \text{ if } x = u_i w_i \text{ or } x = u_i w_{i+1(\bmod n)}, \text{ with } i \text{ even} \\ 5 & , \text{ if } x = w_i \text{ or } x = u_{i+1(\bmod n)}, \text{ with } i \text{ odd} \\ 6 & , \text{ if } x = u_0 \\ 7 & , \text{ if } x = w_i \text{ or } x = u_{i+1}, \text{ with } i \text{ even} \end{cases}$$

It can also be shown that f_n is a total rainbow 7-coloring of LC_n. Hence, $rtc(LC_n) \le 7$. Since the diameter of of LC_n is equal to 4, by Remark 1, we must have, $rtc(LC_n) = 7$ if $n \ge 9$ and *n* is odd.

B. Total Rainbow Connection Number of Sunflower Graphs

This subsection gives the total rainbow connection number of sunflower graph. Theorem 3 is due to Sun (2013).

Theorem 3: If *G* is a connected graph, then

1. $rtc(G) = 1$ if and only *G* is a complete graph;
2. $rtc(G) \ne 2$;
3. $rtc(G) = m + n_2$ if and only if *G* is a tree.

Theorem 4: *Let* $W_n = (\{u_0\}, \varnothing) + [u_1, u_2, \ldots, u_n]$ *be a wheel of order* $n+1$ *and* SF_n *be the sunflower graph obtained by adding a vertex* w_i *joined by an edge to vertices* u_i *and* $u_{i+1(\bmod n)}$. *Then*

$$rtc(SF_n) = \begin{cases} 3 & , \text{ if } n = 3 \\ 6 & , \text{ if } n = 4, 5 \\ 7 & , \text{ if } n \ge 6\,. \end{cases}$$

Proof:

For $n = 3$, define $f_3 : V(SF_3) \cup E(SF_3) \to \{1, 2, 3\}$ as follows:

Table-1: *Images of the elements of* $V(SF_3)\cup E(SF_3)$

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	3	u_0u_1	2	u_3w_2	2
u_1	3	u_0u_2	2	u_3w_3	1
u_2	3	u_3u_1	2	u_1w_3	2
u_3	3	u_0u_3	2	u_1u_2	2
w_1	3	u_1w_1	1	u_2u_3	2
w_2	3	u_2w_1	2		
w_3	3	u_2w_2	1		

Then f_3 is a total rainbow 4-coloring of SF_3. Hence, $rtc(SF_3)\le 3$. By Theorem 2, we must have, $rtc(LC_3)=3$. For $n=4$, define $f_4: V(SF_4)\cup E(SF_4)\to\{1,2,3\}$ as follows:

Table-2: *Images of the elements of* $V(SF_4)\cup E(SF_4)$

v	$f(v)$	e	$f(e)$	e	$f(e)$
u_0	5	u_0u_1	1	u_4w_3	4
u_1	5	u_0u_2	1	u_4w_4	2
u_2	4	u_0u_3	2	u_1w_4	3
u_3	5	u_0u_4	3	u_1u_2	2
u_4	4	u_1w_1	1	u_2u_3	3
w_1	5	u_2w_1	2	u_3u_4	2
w_2	4	u_2w_2	1	u_4u_1	2
w_3	5	u_3w_2	3		
w_4	4	u_3w_3	1		

Then f_4 is a total rainbow 5-coloring of SF_4. Hence, $rtc(SF_4)\le 5$. Since the diameter of of SF_4 is equal to 3, by Remark 1, we must have, $rtc(SF_4)=5$.

For $n\ge 5$ and n is even, Then we define $f_n: V(SF_n)\cup E(SF_n)\to\{1,2,\ldots,7\}$ as follows:

$$f_n(x)=\begin{cases} 1 & , \text{ if } x=w_iw_{i+1(\bmod n)} \text{ or } x=u_0u_i \text{ with } i \text{ odd} \\ 2 & , \text{ if } x=u_0u_i \text{ with } i \text{ even} \\ 3 & , \text{ if } x=u_iw_i \text{ or } x=u_iw_{i+1(\bmod n)}, \text{ with } i \text{ odd} \\ 4 & , \text{ if } x=u_iw_i \text{ or } x=u_iw_{i+1(\bmod n)}, \text{ with } i \text{ even} \\ 5 & , \text{ if } x=w_i \text{ or } x=u_{i+1(\bmod n)}, \text{ with } i \text{ odd} \\ 6 & , \text{ if } x=u_0 \\ 7 & , \text{ if } x=w_i \text{ or } x=u_{i+1}, \text{ with } i \text{ even} \end{cases}$$

For $n\ge 9$ and n is odd, Then we define $f_n: V(LC_n)\cup E(LC_n)\to\{1,2,\ldots,7\}$ as follows:

$$f_n(x)=\begin{cases} 1 & , \text{ if } x=w_iw_{i+1(\bmod n)} \text{ or } x=u_0u_i \text{ with } i \text{ odd} \\ 2 & , \text{ if } x=u_0u_i \text{ with } i \text{ even} \\ 3 & , \text{ if } x=u_iw_i \text{ or } x=u_iw_{i+1(\bmod n)}, \text{ with } i \text{ odd} \\ 4 & , \text{ if } x=u_iw_i \text{ or } x=u_iw_{i+1(\bmod n)}, \text{ with } i \text{ even} \\ 5 & , \text{ if } x=w_i \text{ or } x=u_{i+1(\bmod n)}, \text{ with } i \text{ odd} \\ 6 & , \text{ if } x=u_0 \\ 7 & , \text{ if } x=w_i \text{ or } x=u_{i+1}, \text{ with } i \text{ even} \end{cases}$$

It can be shown that f_n is a total rainbow 7-coloring of SF_n and a total rainbow coloring of SF_7 cannot have a fewer than 7 colors. Hence, we must have, $rtc(SF_n)=7$ if $n\ge 5$.

IV. RECOMMENDATION

We recommend the rainbow total connection number of other cycle related graphs mentioned in Ponraj *et al.* (2015) be determined also.

V. ACKNOWLEDGEMENT

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