International Journal of Mathematical Archive-10(2), 2019, 24-31
Available online through www.ijma.info ISSN 2229 – 5046

IRREDUCIBLE ELEMENTS IN ALMOST LATTICES AND RELATIVELY COMPLEMENTED ALMOST LATTICES

G. NANAJI RAO[1] AND HABTAMU TIRUNEH ALEMU*[2]

[1,2]Department of Mathematics, Andhra University, Visakhapatnam, 530003, India.
[2]Addis Ababa Science and Technology University, Addis Ababa, Ethiopia.

(Received On: 28-12-18; Revised & Accepted On: 30-01-19)

ABSTRACT

The concepts of atom in an Almost Lattice(AL) L and atomic AL are introduced and proved that every finite AL with 0 is atomic AL. The concepts of meet(join) irreducible elements and meet(join) prime elements are introduced in an AL L and proved that if L is an AL with 0 satisfying minimum(maximum) condition, then every one of its element in L can be represented as the join(meet) of a finite number of join(meet) irreducible elements. Also, a necessary and sufficient condition for an element in an AL, $L_1 \times L_2$ of two ALs L_1 and L_2 to become join irreducible element is established and proved that every meet(join) prime element is meet(join) irreducible; but, the converse need not be true. The concepts of relatively complemented AL and sectionally complemented AL are introduced and proved that if L is finite and sectionally complemented, then every non zero element of L is a join of finitely many atoms. Further, the concepts of semicomplemented AL and weakly complemented AL are introduced and a necessary and sufficient condition for an AL L with 0 to become a weakly complemented AL is proved. Also, proved that every sectionally complemented AL is semicomplemented.

Key Words: *Lattice, Almost Lattice, Almost Lattice with zero, Ascending Chain, Descending Chain, Maximum Condition, Minimum Condition, Atom, Join Irreducible Element, Meet Irreducible Element, Meet Prime Element, Join Prime Element, Inner Element, Maximal Element, Relatively Complemented Almost Lattice, Sectionally Complemented Almost Lattice, Weakly Complemented Almost Lattice, SemiComplemented Almos Lattice*

AMS Mathematics Subject classification $(\mathbf{2000})$*:* $06D99$, $06D15$.

1. INTRODUCTION

The axiomatization of Boole's two valued propositional calculus lead to the concept of Boolean algebra, which is a complemented and bounded distributive lattice. M.H. Stone has proved that any Boolean algebra made into a Boolean ring (a ring with unity in which every element is an idempotent) and vice versa. The class of distributive lattices has occupied in major part of the present lattice theory, since lattices were abstracted from Boolean algebras through the class of distributive lattices and the class of distributive lattices has many interesting properties which lattices, in general, do not have. For this reason, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U.M. and Rao G.C. [4], as a common abstraction of existing lattice theoretic and ring theoretic generalizations of Boolean algebra. It was Garett Birkhoff's (1911 - 1996) work in the mid thirties that started the general development of the lattice theory. In a brilliant series of papers, he demonstrated the importance of the lattice theory and showed that it provides a unified frame work for unrelated developments in many mathematical disciplines. V. Glivenko, Karl Menger, John Van Neumann, Oystein Ore, George Gratzer, P. R. Halmos, E. T. Schmidt, G. Szasz, M. H. Stone, R. P. Dilworth and many others have developed enough of this field for making it attractive to the mathematicians and for its further progress. The traditional approach to lattice theory proceeds from partially ordered sets to general lattices, semimodular lattices, modular lattices and finally to distributive lattices. The concept of Almost Lattice (AL) was introduced by G. Nanaji Rao and Habtamu Tiruneh Alemu [1] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and lattices.

Corresponding Author: Habtamu Tiruneh Alemu*[2]
[2]Addis Ababa Science and Technology University, Addis Ababa, Ethiopia.

In this paper, we introduced the concepts of atom, atomic Almost Lattice(AL) and proved that every finite AL with 0 is an atomic AL. It is proved that if an AL L satisfies minimum condition, then L is atomic and we introduced the concepts of meet irreducible element and join irreducible element in an AL L. Also, proved that every atom in an AL L is join irreducible element and observed that converse is not true by means of example. It is proved that if L is an AL with 0 satisfying minimum(maximum) condition, then every one of its element can be represented as the join(meet) of a finite number of join(meet) irreducible elements and we derived a necessary and sufficient condition for an element in an AL $L_1 \times L_2$ where ALs L_1 and L_2 are two ALs to become join irreducible. We introduced the concepts of meet prime and join prime elements in an AL L which are generalizations of the concept of irreducible elements and proved that every meet(join) prime element in an AL L is meet(join) irreducible; converse need not be true. Further, the concept of a sub AL of an AL L is introduced and proved that every interval in an AL L is a sub AL. We introduced the concepts of relatively complemented AL and sectionally complemented AL and proved that if an AL L is a Boolean algebra, then L is relatively complemented lattice and hence is a sectionally complemented lattice. Also, proved that if L is finite and sectionally complemented, then every non zero element of L is a join of finitely many atoms. Finally, we introduced the concept of semicomplemented and weakly complemented ALs and we derived a necessary and sufficient condition that an AL L with 0 to become weakly complemented AL. It is also proved that, every weakly complemented AL is semicomplemented and every sectionally complemented AL is weakly complemented and hence every sectionally complemented AL is semicomplemented, which follows that, every relatively complemented AL is semicomplemented.

2. PRELIMINARIES

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1 *Let* $(P, \leq)$ *be a poset and* $a \in P$. *Then*

1. a is called the least element of P if $a \leq x$ for all $x \in P$.
2. a is called the greatest element of P if $x \leq a$ for all $x \in P$.

It can be easily observed that, if least (greatest) element exists in a poset, then it is unique.

Definition 2.2: *Let* $(P, \leq)$ *be a poset and* $a \in P$. *Then*

1. a is called a minimal element, if $x \leq a$ implies $x = a$ for all $x \in P$.
2. a is called maximal element, if $a \leq x$ implies $a = x$ for all $x \in P$.

It can be easily verified that least (greatest) element (if exists), then it is minimal (maximal) but, converse need not be true.

Definition 2.3: *Let* $(P, \leq)$ *be a poset and* $S \subseteq P$. *Then*

1. An element a in P is called a lower bound of S if $a \leq x$ for all $x \in S$.
2. An element a in P is called an upper bound of S if $x \leq a$ for all $x \in S$.
3. An element a in P is called the greatest lower bound (glb or infimum) of S if a is a lower bound of S and $b \in P$ such that b is a lower bound of S, then $b \leq a$.
4. An element a in P is called the least upper bound (lub or suprimum) of S if a is an upper bound of S and $b \in P$ such that b is a upper bound of S, then $a \leq b$.

Definition 2.4: ***(Zorn's Lemma)****:* If every chain of a partly ordered set $(\mathrm{P}, \leq)$ has an upper bound in P, then P has a maximal element.

Definition 2.5: *Let* $(P, \leq)$ *be a poset. If* P *has least element* 0 *and greatest element 1, then* P *is said to be a bounded poset.*

If $(P, \leq)$ is a bounded poset with bounds $0, 1$, then for any $x \in P$, we have $0 \leq x \leq 1$.

Definition 2.6: Let $(P, \leq)$ be a poset. Then

1. P is said to satisfy descending chain condition (dcc) if every descending chain inP is terminate. That is if $\ldots < x_n < x_{n-1} < \ldots < x_2 < x_1 < x_0$ is a descending chain in P, then there exists $n \in Z^+$such that $x_n = x_{n+1} = x_{n+2} = \ldots\ldots$
2. P is said to satisfy ascending condition (acc) if every ascending chain in Pisterminate. That is if $x_0 < x_1 < \ldots < x_{n-2} < x_{n-1} < x_n < \cdots$ is an ascending chain in P, then there exists $n \in Z^+$ such that $x_n = x_{n+1} = x_{n+2} = \cdots$
3. P is said to satisfy minimum(maximum) condition if every nonempty subset of P has a minimal(maximal) element.

Theorem 2.7: Let $(P, \leq)$ be a poset. Then we have the following.

1. P satisfies ascending chain condition(acc) if and only if P satisfies a minimum condition.
2. P satisfies descending chain condition(dcc) if and only if P satisfies a maximum condition.

Theorem 2.8: Let $(P, \leq)$ be a poset. If P satisfying minimum(maximum) condition. Then for any $x \in P$ there exists a minimal(maximal) element m in P such that $m \leq x(x \leq m)$.

Theorem 2.9: Every subchain of a partially ordered subset satisfying a minimum(maximum) condition has least(greatest) element.

Definition 2.10: Let $(P, \leq)$ be a poset. Then P is said to be lattice ordered set if for every pair $x, y \in P$, $l.u.b\{x, y\}$ and $g.l.b\{x, y\}$ exists.

Definition 2.11: *An algebra* $(L, \vee, \wedge)$ *of type* $(2, 2)$ *is called a lattice if it satisfies the following axioms. For any* $x, y, z \in L$,

1. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative Law)
2. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative Law)
3. $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. (Absorption Law)

It can be easily seen that in any lattice $(L, \vee, \wedge)$, $x \vee x = x$ and $x \wedge x = x$. (Idempotent Law)

Definition 2.12: *An algebra* $(L, \vee, \wedge)$ *of type (2, 2) is called an Almost Lattice if it satisfies the following axioms. For any* $a, b, c \in L$:

- A_1. $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
- A_2. $(a \vee b) \wedge c = (b \vee a) \wedge c$
- A_3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- A_4. $(a \vee b) \vee c = a \vee (b \vee c)$
- A_5. $a \wedge (a \vee b) = a$
- A_6. $a \vee (a \wedge b) = a$
- A_7. $(a \wedge b) \vee b = b$

Lemma 2.13: *Let L be an AL. Then for any* $a, b \in L$ *we have the following:*

1. $a \vee a = a$
2. $a \wedge a = a$
3. $a \wedge b = a$ if and only if $a \vee b = b$

Definition 2.14: *For any* $a, b \in L$*of an AL L, we say that* a *is less than or equal to* b *and write as* $a \leq b$ *if and only if* $a \wedge b = a$ *or, equivalently* $a \vee b = b$.

Theorem 2.15: *Let L be an AL such that* $a, b, c \in L$. *Then we have the following:*

1. *The relation* $\leq$ *is a partial ordering on L and hence* $(L, \leq)$ *is a poset.*
2. $a \leq b \Rightarrow a \wedge b = b \wedge a$
3. $a \leq a \vee b$
4. $a \wedge b \leq b$
5. $(a \vee b) \wedge a = a$
6. $(a \vee b) \wedge b = b$
7. $b \vee (a \wedge b) = b$
8. $a \wedge b = b \Leftrightarrow a \vee b = a$
9. $a \leq b \Rightarrow a \vee b = b \vee a$
10. $a \vee b = b \vee a \Rightarrow a \wedge b = b \wedge a$
11. If $a \leq candb \leq c, thena \wedge b \leq c$ and $a \vee b \leq c$
12. $(a \vee b) \vee b = a \vee b$
13. $(a \vee b) \vee a = a \vee b$
14. $a \vee (a \vee b) = a \vee b$
15. $a \wedge (a \wedge b) = a \wedge b$
16. $(a \wedge b) \wedge b = a \wedge b$
17. $b \wedge (a \wedge b) = a \wedge b$

Definition 2.16: *An AL L is said to be directed above if for any* $a, b \in L$ *there exists* $c \in L$ *such that* $a, b \leq c$.

Lemma 2.17: *Let L be an AL. Then the following are equivalent.*

1. L is directed above.
2. $\wedge$ is commutative.
3. $\vee$ is commutative.
4. L is a lattice.

Definition 2.18: *Let L be an AL. Then an element* $a \in L$ *is called maximal (minimal) if for any* $x \in L,\ a \leq x\ (x \leq a)$ *implies* $a = x (x = a)$.

Definition 2.19: An algebra $(L,\ \vee,\ \wedge, 0)$ of type $(2, 2, 0)$ is called an AL with 0 if it satisfies the the following axioms. For any $a, b, c \in L$:

- A_1. $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
- A_2. $(a \vee b) \wedge c = (b \vee a) \wedge c$
- A_3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- A_4. $(a \vee b) \vee c = a \vee (b \vee c)$
- A_5. $a \wedge (a \vee b) = a$
- A_6. $a \vee (a \wedge b) = a$
- A_7. $(a \wedge b) \vee b = b$
- 0_1. $0 \wedge a = 0$

Lemma 2.20: *Let* L *be an* AL *with* 0. *Then for any* $a, b \in L$, *we have the following:*

1. $a \wedge 0 = 0$.
2. $a \vee 0 = a$.
3. $0 \vee a = a$
4. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
5. $a \wedge b = b \wedge a$ *whenever* $a \wedge b = 0$

3. ATOMS AND IRREDUCIBLE ELEMENTS IN ALS

In this section, we introduce the concepts of atom, atomic Almost Lattice(AL) and prove that every finite AL with 0 is an atomic AL. Also, prove that if an AL L satis fies minimum condition, then L is atomic. We introduce the concepts of meet irreducible element and join irreducible element in an AL L and prove that every atom in an AL L is join irreducible element and observe that converse is not true by means of example. Also, prove that if L is an AL with 0 satisfying minimum(maximum) condition, then every one of its element can be represented as the join(meet) of a finite number of join(meet) irreducibleelements. Moreover,we derive a necessary and sufficient condition for an element in an AL $L_1 \times L_2$ where L_1 and L_2 are two ALs to become join irreducible. Finally, we introduce the concepts of meet prime and join prime elements in an AL L which are generalizations of the concept of irreducible elements and prove that every meet(join) prime element in an AL L is meet(join) irreducible; converse need not be true. First, we begin with the following definition.

Definition 3.1: Let L be an AL and $a, b \in L$ with $a \leq b$. Then we say that a is covered by b or, b covers a, write as $a \prec b$ if for any $c \in L, a \leq c \leq b$, implies either $a = c$ or $c = b$.

Definition 3.2: Let L be an AL with 0. Then an element $a(\neq 0) \in L$ is called an atom if 0 is covered by a.

Example 3.3: Let $L = \{0, a, b, c\}$ and define binary operations $\vee$ and $\wedge$ on L as follows:

∨	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

∧	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then clearly $(L,\ \vee,\ \wedge, 0)$ is an AL with 0. Also, we can observe that a is an atom but b and c are not atoms.

In the following we introduce the concept of atomic Almost Lattice.

Definition 3.4: An AL Lwith 0 is said to be atomic if for every non zero $a \in L$, there exists an atom $p \in L$ such that $p \leq a$.

Example 3.5: Let $L = \{0, a, b, c\}$. Define $\vee$ and $\wedge$ on L as follows.

∨	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

∧	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(L, \vee, \wedge, 0)$ is an atomic AL.

Now, we prove the following:

Theorem 3.6: Let L be a finite AL with 0 and $a(\neq 0) \in L$. Then there exists an atom $p \in L$ such that $p \leq a$.

Proof: Suppose L is finite AL and $a \in L$ such that $a \neq 0$. If a is an atom, then there is nothing to prove. Suppose a is not an atom, then there exists $a_1 \in L$ such that $a_1 < a$. If a_1 is an atom, then the result is clear. Otherwise continue the above process. Since L is finite, there exists $a_n \in L$ suchthat a_n is an atom and $a_n < a$.

Corollary 3.7: Let L be an AL with 0 which satisfies the minimum condition. Then for every non zero element a in L, there exists an atom p in L such that $p \leq a$.

Corollary 3.8: Let L be an AL which satisfies a minimum condition. Then L is atomic.
Now, we introduce the concept of meet irreducible and join irreducible elements in ALs.

Definition 3.9: Let L be an AL. Then an element $a \in L$ is said to be meet irreducible if $a = a_1 \wedge a_2 = a_2 \wedge a_1$ implies $a = a_1$ or $a = a_2$. Otherwise a is called meet reducible.

Definition 3.10: Let L be an AL. Then an element $a \in L$ is said to be join irreducible if $a = a_1 \vee a_2 = a_2 \vee a_1$ implies $a = a_1$ or $a = a_2$. Otherwise a is called join reducible.

Example 3.11: Every elements of an AL L defined in example 3.5 are meet as well as join irreducible.
Now, we prove the following.

Theorem 3.12: Let Lbe an AL with 0. Then every atom in L is join irreducible.

Proof: Suppose a is an atom and $a = b \vee c = c \vee b$ for some $b, c \in L$. Suppose $a \neq b$. Since $0 \leq b \leq b \vee c = a$ and $a \neq b$, $b = 0$. Hence $a = b \vee c = 0 \vee c = c$. Similarly, if $a \neq c$, we can prove that $a = b$. Therefore a is join irreducible.

But, the converse of the above theorem is not true. For, in example 3.3, b is join irreducible but not an atom. It can be easily verified that if L is an AL with 0, then 0 is join irreducible.

Theorem 3.13: Let L be an AL. If L is a chain, then every element in L is join as well as meet irreducible.

Proof: Suppose L is a chain. Let $a \in L$ such that $a = a_1 \vee a_2 = a_2 \vee a_1$ for some $a_1, a_2 \in L$. Now, since L is a chain, $a_1 \leq a_2$ *or* $a_2 \leq a_1$. It follows that, either $a = a_1$ or $a = a_2$. Therefore a is join irreducible. Similarly we can prove that every element in L is meet irreducible.

But, converse of the above theorem is not true. For, If L is a discrete AL, then clearly every element in L is meet as well as join irreducible. But, L is not a chain.

Theorem 3.14: Let L be an AL with 0 satisfying the minimum(maximum) condition. Then every one of an element in L can be represented as the join(meet) of a finite number of join(meet) irreducible elements.

Proof: Suppose L be an AL with 0 satisfying the minimum condition. Suppose H is the set of all elements in L which can not be represented as join of finite number of join irreducible elements. Now, we shall prove that H is empty. Clearly, H contains no join irreducible elements, since if a is such an element then $a = a \vee a$ or $a = a \vee 0 = 0 \vee a$ are easily found representations of the required form that contradicts our assumption for H. Suppose $H \neq \emptyset$. Since L satisfies minimum condition, H hasa minimal element say m. Clearly m is not join irreducible element. Then we can choose $m_1, m_2 \in L$ such that;
$m = m_1 \vee m_2 = m_2 \vee m_1$, $(m_1, m_2 < m)$

Since m is a minimal element of H, the elements $m_1, m_2 \notin H$. Hence m_1, m_2 can be represented as $m_1 = q_1 \vee q_2 \vee \ldots \vee q_s$ and $m_2 = r_1 \vee r_2 \vee \ldots \vee r_t$ where all the q_j and r_k are join irreducible. Therefore $m = \bigvee_{j=1}^{s} q_j \vee \bigvee_{k=1}^{t} r_k$, which is a join of finite number of join irreducible elements , a contradiction to $m \in H$. Thus $H = \emptyset$. Similarly, we can prove that if L is an AL with 0 satisfies the maximum condition, then every one of elements of L can be represented as the meet of a finite number of meet irreducible elements.

It can be easily verified that if L_1 and L_2 are two ALs, then $L_1 \times L_2$ is again an AL under point wise operations and is called direct product of L_1 and L_2. In the following, we give a necessary and sufficient condition for an element in $L_1 \times L_2$ to become join irreducible.

Theorem 3.15: Let L_1 and L_2 be two ALs and $L = L_1 \times L_2$. Then an element $p = (p_1, p_2) \in L$ is join irreducible in L if and only if $p_1 = 0$ and p_2 is join irreducible in L_2 or $p_2 = 0$ and p_1 is join irreducible in L_1.

Proof: Let $p = (p_1, p_2) \in L$. Then we have $p = (p_1, p_2) = (p_1, 0) \vee (0, p_2) = (0, p_2) \vee (p_1, 0)$. Suppose p is join irreducible in L. Then either $(p_1, p_2) = (p_1, 0)$or $(p_1, p_2) = = (0, p_2)$ and hence $p_1 = 0$ or $p_2 = 0$. Now, we shall prove that if $p_1 = 0$, then p_2 is join irreducible in L_2. Suppose $p_1 = 0$ and $p_2 = a_2 \vee b_2 = b_2 \vee a_2$, for some $a_2, b_2 \in L_2$. Now, $p = (p_1, p_2) = (0, p_2) = (0 \vee 0, a_2 \vee b_2) = (0, a_2) \vee (0, b_2) = (0, b_2) \vee (0, a_2)$. Therefore $(p_1, p_2) = (0, a_2)\ or\ (p_1, p_2) = (0, b_2)$. It follows that $p_2 = a_2$ or $p_2 = b_2$. Therefore p_2 is join irreducible in L_2. Similarly we can prove that if $p_2 = 0$, then p_1 is join irreducible in L_1. Conversely, assume the condition. Suppose $(p_1, p_2) = (a_1, a_2) \vee (b_1, b_2) = (b_1, b_2) \vee (a_1, a_2)$ where $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$. This implies $(a_1 \vee b_1, a_2 \vee b_2) = (b_1 \vee a_1, b_2 \vee a_2)$. Then $p_1 = a_1 \vee b_1 = b_1 \vee a_1$ and $p_2 = a_2 \vee b_2 = b_2 \vee a_2$. Now, if $p_1 = 0$ and p_2 is join irreducible in L_2, then $a_1 = 0 = b_1$and $p_2 = a_2$ or $p_2 = b_2$. Therefore $(p_1, p_2) = (0, a_2)$ or $(p_1, p_2) = (0, b_2)$. Therefore $p = (p_1, p_2)$ is join irreducible in L. Similarly, we can prove that if $p_2 = 0$ and p_1 is join irreducible in L_1, then $p = (p_1, p_2)$ is join irreducible in L.

In the following we introduce concepts of meet prime and join prime elements in an AL L which are generalization of the concept of irreducible elements.

Definition 3.16: An element a of an AL L is called meet prime if $a_1 \wedge a_2 = a_2 \wedge a_1 \leq a$ implies either $a_1 \leq a$ or $a_2 \leq a$.

Definition 3.17: An element a of an AL L is called join prime if $a \leq a_1 \vee a_2 = a_2 \vee a_1$ implies either $a \leq a_1$ or $a \leq a_2$.

Example 3.18: In an AL L of example 3.5, it is observed that a and b are meet prime and c is join prime.
It can be easily seen that, in a distributive lattice L an element is meet irreducibleifandonlyifitismeetprime. But,in thefollowing we prove every meet prime element is meet irreducible.

Theorem 3.19: Every meet prime element in an AL L is meet irreducible.

Proof: Suppose $a \in L$ is meet prime element. Let $a = a_1 \wedge a_2 = a_2 \wedge a_1$ for some $a_1, a_2 \in L$. Then we have $a \leq a_1, a_2$. On the other hand, we have $a_1 \wedge a_2 = a_2 \wedge a_1 \leq a$. It follows that, either $a_1 \leq a\ or\ a_2 \leq a$. Hence either $a = a_1$ or $a = a_2$. Therefore a is meet irreducible.

The converse of the above theorem is not true in general. For, suppose L_1 and L_2 are two discrete ALs with zero and each with at least three elements. Then clearly, $L = L_1 \times L_2$ is an AL under point wise operations. Choose $0 \neq p_1 \in L_1$ and $0 \neq p_2 \in L_2$. Put $p = (p_1, p_2)$. Now, let $(q_1, q_2), (r_1, r_2) \in L$ such that $p = (q_1, q_2) \wedge (r_1, r_2) = (r_1, r_2) \wedge (q_1, q_2)$. Then $(p_1, p_2) = (q_1 \wedge r_1, q_2 \wedge r_2) = (r_1 \wedge q_1, r_2 \wedge q_2)$ and hence $p_1 = q_1 \wedge r_1 = r_1 \wedge q_1$ and $p_2 = q_2 \wedge r_2 = r_2 \wedge q_2$. Since $p_1 \neq 0$ and $p_2 \neq 0$, it follows that, q_1, r_1, q_2, r_2 are non zero. Therefore $r_1 = q_1 \wedge r_1 = r_1 \wedge q_1 = q_1$ and $r_2 = q_2 \wedge r_2 = r_2 \wedge q_2 = q_2$. Therefore $(q_1, q_2) = (r_1, r_2)$. Hence $p = (p_1, p_2) = (q_1, q_2) = (r_1, r_2)$. Therefore p is meet irreducible. However, p is not meet prime. For, choose $q_i \in L_i - \{0, p_i\}\ \forall i = 1,2$. Then $(0, q_2) \wedge (q_1, 0) = (q_1, 0) \wedge (0, q_2) = (0,0) \leq (p_1, p_2) = p$. But $(0, q_2) \nleq p$ and $(q_1, 0) \nleq p$.
It can be easily seen that every join prime element in an AL L is join irreducible. But, converse is not true. For, Consider the following example.

Example 3.20: Let $L = \{0, a, b, c, 1\}$ and define binary operations on L as follows:

∨	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

∧	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

Then clearly $(L, \vee, \wedge, 0)$ is an AL with 0. In this AL the element a is join irreducible. But, it is not join prime, since $a \leq b \vee c = c \vee b$; but, $a \nleq b$ and $a \nleq c$.

4. RELATIVELY COMPLEMENTED ALS

n this section, we introduce the concept of a sub AL of an AL L and prove that every interval in an AL L is a sub AL. Also, we introduce the concepts of relatively complemented AL and sectionally complemented AL and prove that if an AL L is a Boolean algebra, then L is relatively complemented lattice and hence is a sectionally complemented lattice. Also, prove that if L is finite and sectionally complemented, then every non zero element of L is a join of finitely many atoms. We introduce the concept of semicomplemented and weakly complemented ALs and we derive a necessary and sufficient condition that an AL L with 0 to become weakly complemented AL. We prove that every weakly complemented AL is semicomplemented and every sectionally complemented AL is weakly complemented and hence every sectionally complemented AL is semicomplemented, which follows that, every relatively complemented AL is semicomplemented. Now, we begin with the following definitions.

Definition 4.1: Let L be an AL and a,b ∈ L with a ≤ b. Then [a,b] = {x ∈ L/a ≤ x ≤ b} is called an interval in L.

Definition 4.2: Let L be an AL. A nonempty subset S of L is called a sub AL of L if S is closed under the operations ∨ and ∧ in L. Now, we have the following lemma, whose proof is straightforward.

Lemma 4.3: Let L be an AL and let a,b ∈ L such that a ≤ b. Then we have the following:
1. $[a, b] = \{x \in L/a \leq x \leq a\}$ is a sub AL of L.
2. $(a] = \{x \in L/x \leq a\}$ is a sub AL of L.
3. $[a) = \{x \in L/a \leq x\}$ is a sub AL of L

Now, we introduce the concepts of relatively and sectionally complemented ALs.
Definition 4.4: An AL L is said to be relatively complemented if every interval $[a, b], a \leq b$ in L is a complemented lattice.

Definition 4.5: Let L be an AL with 0. Then L is said to be sectionally complemented if every intervals of the form $[0, a], a \in L$ is complemented lattice.

Next, we prove the following.
Theorem 4.6: Let L be an AL. Then the following implications hold.
1. If L is a Boolean algebra, then L is relatively complemented lattice.
2. If L has 0 and L is relatively complemented AL, then L is sectionally complemented.
3. If L is finite and sectionally complemented, then every non zero element of L is a join of finitely many atoms.

Proof: (1) Suppose L is a Boolean algebra and $a, b \in L$ with $a \leq b$. Let $x \in [a, b]$. Then $x \in L$. Since L is a Boolean algebra, x has complement say x' in L. Put $y = (x' \vee a) \wedge b$. We shall prove that y is the complement of x in $[a, b]$. Clearly $y \in [a, b]$. Now, $x \vee y = x \vee (x' \vee a) \wedge b = (x \vee (x' \vee a)) \wedge (x \vee b) = ((x \vee x') \vee a) \wedge (x \vee b) = (1 \vee a) \wedge b = 1 \vee b = b$. Again, $x \wedge y = x \wedge ((x' \vee a) \wedge b) = (x \wedge (x' \vee a)) \wedge b = ((x \wedge x') \vee (x \wedge a)) \wedge b = (0 \vee (x \wedge a)) \wedge b = (x \wedge a) \wedge b = a \wedge b = a$. Hence y is the complement of x in $[a, b]$. Thus every interval in L is a complemented lattice. Therefore L is relatively complemented AL. Proof of (2) is clear.
(3) Suppose L is finite and sectionally complemented. Let $a(\neq 0) \in L$ and let $p_1, p_2, p_3, \ldots, pn$ be atoms which are less than or equal to a. Put, $b = p_1 \vee p_2 \vee p_3 \vee \ldots \vee p_n$. Then $b \leq a$ since each $p_k \leq a \,\forall\, k$. Suppose $b < a$. Then we have $b \in [0, a]$. Since L is sectionally complemented, it has a complement c in $[0, a]$. Therefore $b \wedge c = 0$ and $b \vee c = a$. If $c = 0$, then $a = b \vee c = b \vee 0 = b$, a contradiction to $b < a$. So that $c \neq 0$. Then by theorem3.6, there exists an atom $p \in L$ such that $p \leq c$. Then we get $p \leq a$ since $c \leq a$. It follows that, $p = p_i$ for some $i, 1 \leq i \leq n$ and hence $p \leq b$. Therefore $p \leq b \wedge c = 0$. Hence $p = 0$, which is a contradiction to p is an atom. Therefore $a = b = p_1 \vee p_2 \vee p_3 \vee \ldots \vee p_n$. Thus every non zero element of L is a join of finitely many atoms.

Now, we introduce the concept of inner element and semicomplemented AL.

Definition 4.7: Let L be an AL with 0. Then an element $a \in L$ is said to be an inner element if $a \neq 0$ and a is not a maximal element.

Definition 4.8: Let L be an AL with 0 and $\upsilon \in L$. If there exists $x \in L$ such that $\upsilon \wedge x = 0$, then x is called a semicomplement of υ.
In an AL L we can easily observed that x is a semicomplement of y if and only if y is a semi complement of x (since $x \wedge y = 0 \Leftrightarrow y \wedge x = 0$). Also, it is clear that the set of all semicomplements (if exists) of an element $\upsilon \in L$ is a poset with the induced partial ordering on L. In this case the maximal element of this poset is called maximal semicomplement of υ. Also,we can easily seen that the set of all semicomplements of an element υ in L is an initial segment of L. Further, if L is an AL with 0, then 0 is the semicomplement of every element in L. Semicomplements other than 0 of any element in L are called proper semicomplements.

Definition 4.9: Let L be an AL with 0. Then L is said to be semicomplemented if every inner element in L has at least one proper semicomplement.

Now, we introduce the concept of weakly complemented AL and obtain a necessary and sufficientcondition for an AL with 0 to become weakly complemented AL.

Definition 4.10: Let L be an AL with 0. Then L is said to be weakly complemented if for any $a,\ b\ \in L,\ a < b,\ a$ has semicomplement which is not semicomplement of b. That is there exists $x \in L$ such that $a \wedge x = 0$; but, $b \wedge x \neq 0$.

Theorem 4.11: Let L be an AL with 0. Then L is weakly complemented if and only if for every pair of elements $a,\ b \in L$ with $a < b$, there exists $x \in L$ such that $(a \wedge b) \wedge x = 0$ and $(a \vee b) \wedge x \neq 0$.

Proof: Suppose L is a weaklycomplemented AL and $a,\ b \in L$ suchthat $a < b$. Then we have $a \wedge b \leq a \vee b$. Suppose $a \wedge b = a \vee b$. Then $a \vee (a \wedge b) = a \vee (a \vee b)$. It follows that, $a = a \vee b = b \vee a$ and hence $b \leq a$. Also, $a \wedge (a \wedge b) = a \wedge (a \vee b)$ which follows that, $a \wedge b = a$ and hence $a \leq b$. Hence $a = b$, which is a contradiction to $a < b$. Therefore $a \wedge b < a \vee b$. Now, since L is weakly complemented AL, there exists $x \in L$ such that $(a \wedge b) \wedge x = 0$ and $(a \vee b) \wedge x \neq 0$. Conversely, assume the condition. Let $a, b \in L$ such that $a < b$. Then by our assumption, there exists $x \in L$ such that $(a \wedge b) \wedge x = 0$ and $(a \vee b) \wedge x \neq 0$. We shall prove that $a \wedge x = 0$ and $b \wedge x \neq 0$. Consider, $a \wedge x = (a \wedge b) \wedge x = 0$ and $b \wedge x = (a \vee b) \wedge x \neq 0$. Therefore $a \wedge x = 0$ and $b \wedge x \neq 0$. Thus L is weakly complemented.

Theorem 4.12: Every weakly complemented AL is semicomplemented.

Proof: Suppose L is weakly complemented AL. Let $a \in L$ be an inner element. Then $a \neq 0$ and a is not maximal. Hence there exists $b \in L$ such that $a < b$. Now, by our assumption there exists $x \in L$ such that $a \wedge x = 0$ but $b \wedge x \neq 0$. Now, we shall prove that $x \neq 0$.

For, if $x = 0$, then $b \wedge x = b \wedge 0 = 0$, which is a contradiction. Hence $x \neq 0$ and $a \wedge x = 0$. Therefore L is semicomplemented AL.

Theorem 4.13: Every sectionally complemented AL is weakly complemented.

Proof: Suppose L is sectionally complemented AL. Let $a, b \in L$ suchthat $a < b$. Then $a \in [0, b]$. Now, since L is sectionally complemented, we have $[0, b]$ is complemented lattice. Hence there exists $x \in [0, b]$ such that $a \wedge x = 0$ and $a \vee x = b$. Now, suppose $b \wedge x = 0$. Then we have $x = (a \vee x) \wedge x = b \wedge x = 0$. Therefore $x = 0$. Now, $b = a \vee x = a \vee 0 = a$, a contradiction to $a < b$. Therefore $b \wedge x \neq 0$. Thus L is weakly complemented.

Corollary 4.14: Every relatively complemented AL with 0 is weakly complemented.

Corollary 4.15: Every sectionally complemented AL is semicomplemented.

Corollary 4.16: Every relatively complemented AL is semicomplemented.

REFERENCES

1. Nanaji Rao G. and Habtamu T. A.: Almost Lattices, J. Int. Math. Virt. Inst., 9(2019), 155–171.
2. Swamy U.M., and B. Venkateswarlu: irreducible elements in Algebraic lattices, Int. J. of Algebra and computation, 20(8), 969-975(2010).
3. Swamy U.M., Rao G.C.: Almost Distributive Lattice, Aust.Math.soc, (Series A), 31 (1981), 77-91.
4. Szasz, G.: Introduction to Lattice Theory, Academic press, New York and London, 1963.

Source of support: Nil, Conflict of interest: None Declared.