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A NEW TWO-PARAMETER EXPONENTIATED WEIBULL MODEL WITH PROPERTIES AND APPLICATIONS TO FAILURE AND SURVIVAL TIMES

MOHAMED K. A. REFAIE
Agami High Institute of Administrative Sciences, Alexandria, Egypt.

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ABSTRACT

A *new exponentiated Weibull lifetime model which exhibits many important hazard rate shapes has been introduced. The importance of the new exponentiated Weibull lifetime model is depended on the wider importance of the standard Weibull and the standard exponentiated Weibull and lifetime models. Empirically the flexibility of the new lifetime model has been illustrated via three types of lifetime data.*

Keywords: *Maximum Likelihood Method; Weibull Model; Exponentiated Weibull; Moments.*

1. INTRODUCTION

A random variable (r.v.) W is said to have the exponentiated Weibull distribution (EWD) if its probability density function (PDF) given by

$$h(w\,;\alpha\,,\beta)=\alpha\beta w^{\beta-1}\left[-exp\left(-w^{\beta}\right)+1\right]^{\alpha-1}exp\left(-w^{\beta}\right), \tag{1}$$

and cumulative distribution function (CDF)

$$H(w\,;\alpha\,,\beta)=\left[-exp\left(-w^{\beta}\right)+1\right]^{\alpha}, \tag{2}$$

respectively, for $w>0$, $\alpha>0$ and $\beta>0$. For $\alpha=1$ we get the well-known WD with only one parameter β. In this paper we shall introduce a new flexible version of the EWD with only two parameters via the Odd Lindley (OLi-G) family. The PDF and CDF of OLi-G family of distribution (see Silva et al. (2017)) are given by (with scale equal one)

$$f(x\,;\boldsymbol{\phi})=2^{-1}h(x\,;\boldsymbol{\phi})\overline{H}(x\,;\boldsymbol{\phi})^{-3}\,exp\left[-\frac{H(x\,;\boldsymbol{\phi})}{\overline{H}(x\,;\boldsymbol{\phi})}\right], \tag{3}$$

and

$$F(x\,;\boldsymbol{\phi})=1-2^{-1}\overline{H}(x\,;\boldsymbol{\phi})^{-1}\left[1+\overline{H}(x\,;\boldsymbol{\phi})\right]exp\left[-\frac{H(x\,;\boldsymbol{\phi})}{\overline{H}(x\,;\boldsymbol{\phi})}\right], \tag{4}$$

respectively. By using equations (1), (2) and (3) we obtain the two-parameters OLi-EWD density in Equation (5). A r.v. X is said to have the OLi-EWD if its PDF and CDF given as

$$\begin{aligned}f(x)&=2^{-1}\alpha\beta x^{\beta-1}\left\{-\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}+1\right\}^{-3}\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha-1}\\&\times exp\left(-x^{\beta}-\frac{\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}}{1-\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}}\right),x\geq 0,\end{aligned} \tag{5}$$

and

$$\begin{aligned}F(x)&=1-2^{-1}\left\{1-\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}\right\}^{-1}\left(a+\left\{1-\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}\right\}\right)\\&\times exp\left(-a\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}/\left\{1-\left[-exp\left(-x^{\beta}\right)+1\right]^{\alpha}\right\}\right),x\geq 0,\end{aligned} \tag{6}$$

respectively.

Let Y be a lifetime r.v. having the EW distribution with CDF $G(t\,;\alpha\,,\beta)$. The odds ratio (o.r.) that an individual (or component) following the lifetime Y will die (failure) at time x is

$$\left[-exp\left(-t^{\beta}\right)+1\right]^{\alpha}\div\left\{-\left[1-exp\left(-t^{\beta}\right)\right]^{\alpha}+1\right\}.$$

Consider that the variability of this o.r. of death is represented by the r.v. X and assume that it follows the Lindley model with scale $a=1$. We can write

$$Pr(Y\leq x)=Pr\left(X\leq\left(\left[-exp\left(-t^{\beta}\right)+1\right]^{\alpha}\div\left\{-\left[1-exp\left(-t^{\beta}\right)\right]^{\alpha}+1\right\}\right)\right)=F(x),$$

which is given by (6).

Corresponding Author: Mohamed K. A. Refaie

Let X be an arbitrary r.v. with CDF $F(x\,;\alpha\,,\beta)$. For any $W \in (0.1)$, the quantile function (QF) $Q(W)$ of the r.v. X is the solution of $W = F(Q(W))$ for all $Q(W) > 0$, from Equation (6), we get

$$-\left[\frac{-G\big(Q(W)\big)+2}{-G\big(Q(W)\big)+1}\right] exp\left\{-\frac{-G\big(Q(W)\big)+2}{-G\big(Q(W)\big)+1}\right\} = 2\,\frac{(W-1)}{exp(-1-a)},$$

where

$$-\left[\frac{-G\big(Q(W)\big)+2}{-G\big(Q(W)\big)+1}\right],$$

is the Lambert function $\mathbf{L}(\cdot)$ of the real argument $2\frac{(W-1)}{exp(-1-a)}$ and the Lambert $\mathbf{L}(\cdot)$ function is defined by

$$x = \mathbf{L}(x)\, exp[\mathbf{L}(x)],$$

from Silva et al. (2017), we can write the following equation for QF of the OLi-EWD

$$Q(W) = \sqrt[\beta]{-\log\left\{\sqrt[\alpha]{-\mathbf{L}(-2(-1+W)\, exp(1+a))}\right\}},$$

2. GRAPHICAL AND LINEAR PRESENTATIONS

Graphical presentation

We note that the PDF OLi-EWD model exhibits various important shapes for the PDF (see Figure 1). We also conclude that the HRF OLi-EWD exhibits the decreasing, the increasing and the bathtub (**U**) hazard rates (see Figure 2).

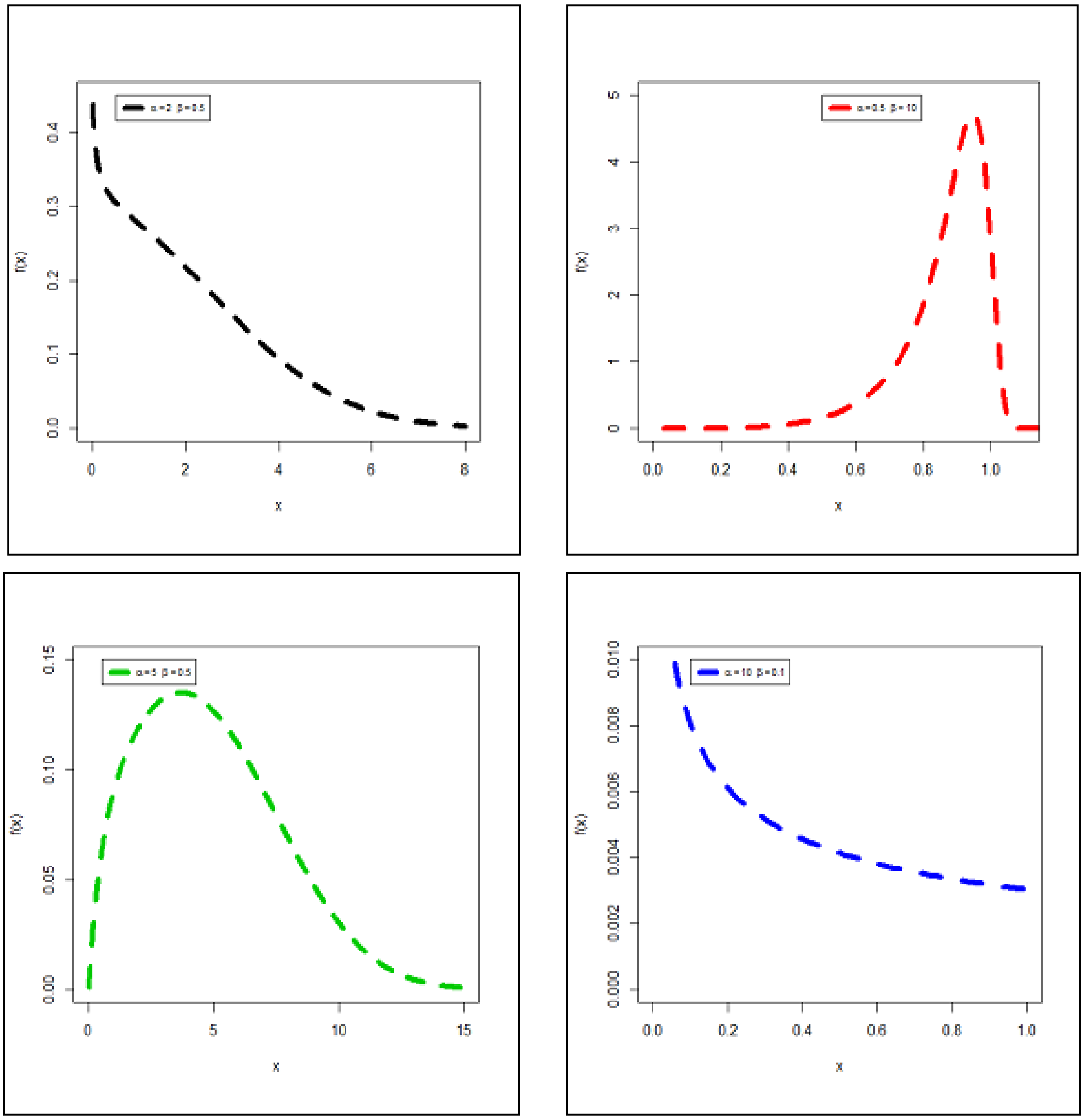


Figure 1: Plots of the OLi-EWD PDF.

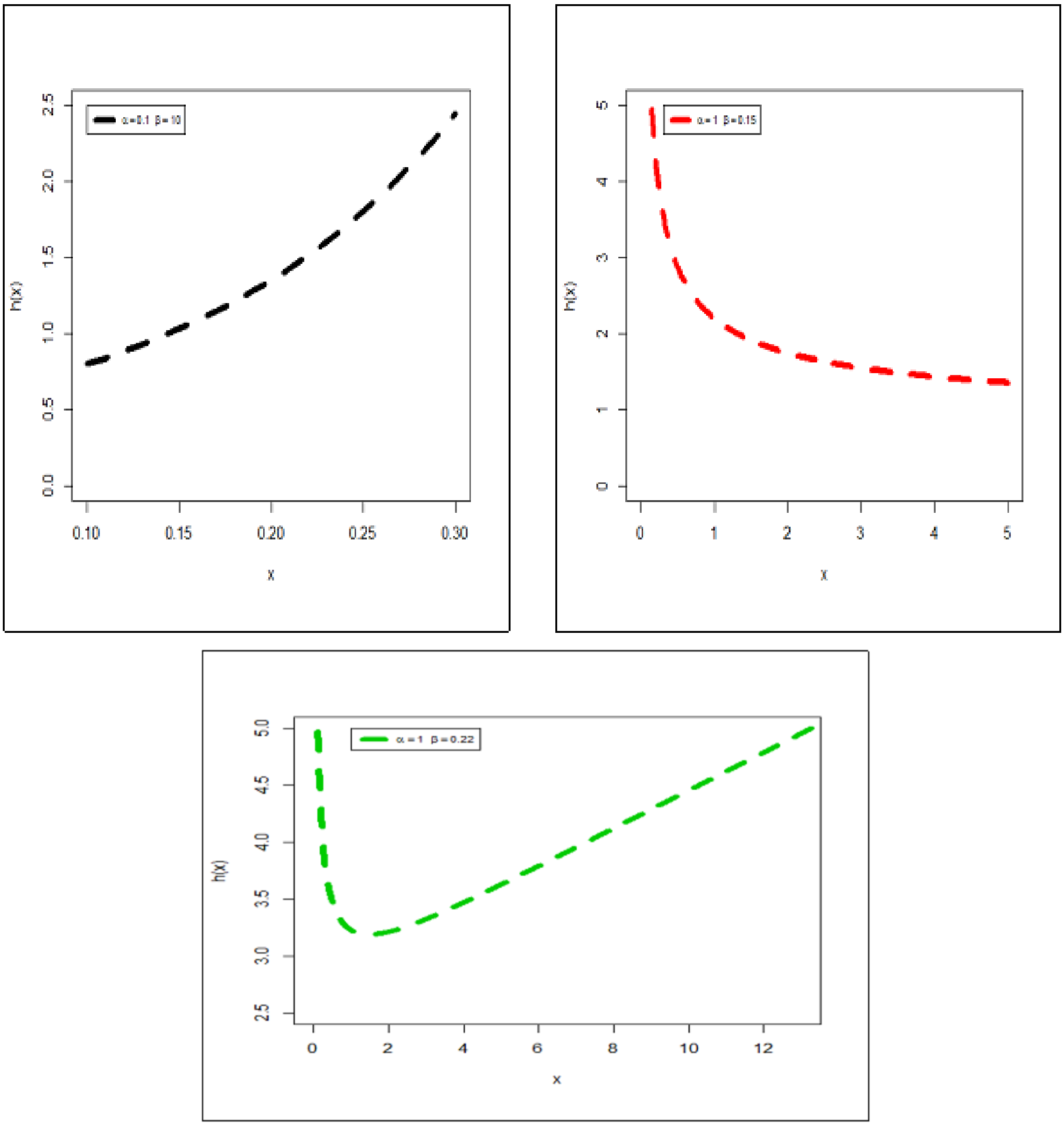


Figure 2: Plots of the OLi-EWD HRF.

Linear representation

The PDF of X in (5) can be easily expressed as

$$f(x) = \sum_{i,k=0}^{\infty} V_{i,k}\; h(x\,;[(2+i+k)\alpha]\,,\beta), \tag{7}$$

where

$$V_{i,k} = \frac{(-1)^k \Gamma(3+i+k)}{2i!\,\Gamma(3+k)(2+k+i)},$$

and

$$[(2+k+i)\alpha]\beta z^{\beta-1}\left[1-exp\left(-x^{\beta}\right)\right]^{-1+(2+k+i)\alpha} exp\left(-x^{\beta}\right) = h(x\,;[(2+k+i)\alpha]\,,\beta)$$

is the PDF of EWD model with parameters $[(2+i+k)\alpha] > 0$ and $\beta > 0$. The CDF of X can be given by integrating (7) as

$$F(x) = \sum_{i,k=0}^{\infty} \boldsymbol{V}_{i,k}\; H(x\,;(2+i+k)\,\alpha,\beta), \tag{8}$$

where

$$\left[1-exp\left(-t^{\beta}\right)\right]^{(2+i+k)\alpha} = H(x\,;[(2+i+k)\alpha]\,,\beta),$$

is the CDF of EWD model with parameters $[(2+i+k)\alpha] > 0$ and $\beta > 0$. For more details about the OLi-G family see Silva et al. (2017). See also Mudholkar and Srivastava (1993) and Nadarajah and Kotz (2006) for more information about the EWD.

3. STATISTICAL PROPERTIES

Moments

The q^{th} ordinary moment of X is given by

$$\mu_q^{'} = \int_0^\infty x^q f(x)dx = E(X^q).$$

Via (7), we get

$$\mu_q^{'} = \Gamma(1+\beta^{-1}q) \sum_{i,j,k=0}^{\infty} \boldsymbol{V}_{i,k}\, c_j^{\{[(2+i+k)\alpha],q\}} |_{(q>-\beta)}, \tag{9}$$

where

$$c_{a_3}^{(a_1,a_2)} = a_1(-1)^{a_3} \binom{a_1-1}{a_3} (1+a_3)^{-\frac{a_2+\beta}{\beta}},$$

$$\Gamma(1+v) = \prod_{m=0}^{v-1}(v-m)\,, v \in R^+,$$

and

$$\int_0^\infty exp(-x)\, x^{q-1} dx = \Gamma(q),$$

is the complete gamma (CGa) function. The q^{th} incomplete moment of X, say $I_q(w)$, is given by

$$I_q(w) = \int_0^w f(x) x^q\, dx.$$

Via (7), we have

$$I_q(w) = \gamma\left(1+\beta^{-1}q, w^{-\beta}\right) \sum_{i,j,k=0}^{\infty} \boldsymbol{V}_{i,k}\, c_j^{\{[(2+i+k)\alpha],q\}} |_{(q>-\beta)},$$

where

$$\gamma(\delta, z) = \int_0^z w^{\delta-1} exp(-w)\, dw = \sum_{\tau=0}^{\infty} \frac{(-1)^\tau}{\tau!\,(\delta+\tau)} z^{\delta+\tau},$$

is the lower incomplete gamma (LIGa) function.

Order statistics

Let $X_1, X_2, \dots, X_{n-1}, X_n$ be a random sample (RS) from the OLi-EWD model of distributions and let $X_{1\,:\,n}, X_{2\,:\,n}, \cdots, X_{n-1\,:\,n}, X_{n\,:\,n}$ be the corresponding order statistics. The PDF of the i^{th} order statistic, say $X_{i\,:\,n}$, can be expressed as

$$f_{i\,:\,n}(x) = \sum_{j=0}^{n-i} (-1)^j B^{-1}(i, n+1-i) f(x) F(x)^{j+i-1} \binom{n-i}{j}. \tag{10}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in Equation (10), we obtain

$$f_{i\,:\,n}(x) = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} \boldsymbol{A}_{m,p,j}\; h(x\,;(1+m+j+p)\,\alpha, \beta),$$

where

$$\boldsymbol{A}_{m,p,j} = \sum_{k=0}^{i-1} \frac{(-1)^{k+m} \left(\frac{1}{2}\right)^{1+j} B^{-1}(i, n+1-i)}{m!\,(1+m+j+p)} \binom{-1+k+n}{j} \binom{j+p+m}{j+m} \binom{i-1}{k}.$$

Then, the z^{th} moment of $X_{i\,:\,n}$ is given by

$$E(X_{i\,:\,n}^z) = \Gamma(1+\beta^{-1}z) \sum_{m,p,h=0}^{\infty} \sum_{j=0}^{k+n-i} \boldsymbol{A}_{m,p,j}\; c_h^{[(1+m+j+p)\alpha, z]} |_{(z>-\beta)}.$$

Moment of residual and reversed residual life (MRL & MRRL)

The τ^{th} MRL is given by

$$E[(X-t)^\tau |_{t>0, X>t,\ \tau=1,2,\dots}] = R_\tau(t).$$

So, the τ^{th} MRL of X can be given as

$$[1-F(t)]^{-1} \int_t^\infty (x-t)^\tau\, dF(x) = z_\tau(t),$$

subsequently we can write

$$R_\tau(t) = [1 - F(t)]^{-1} \sum_{i,k=0}^{\infty} \sum_{r=0}^{t} (-t)^{\tau-r} \binom{\tau}{r} \boldsymbol{V}_{i,k} \int_t^\infty x^\tau \; h(x\,;(2+i+k)\,\alpha,\beta)dx,$$

$$= \Gamma\left(1+\beta^{-1}\tau, t^{-\beta}\right)[1 - F(t)]^{-1} \sum_{i,j,k=0}^{\infty} \sum_{r=0}^{t} \boldsymbol{\delta}_{i,j,k,r}^{((2+i+k)\alpha,\tau)} \;|_{(\tau>-\beta)},$$

where

$$\boldsymbol{\delta}_{i,j,k,r}^{((2+i+k)\alpha,\tau)} = \boldsymbol{V}_{i,k} t^{\tau-r} (-1)^{i+\tau-r} [(2+i+k)\alpha](1+j)^{-(\tau+\beta)/\beta} \binom{i+k}{j}\binom{\tau}{r},$$

and

$$\Gamma(a,x)|_{(x>0)} = \int_x^\infty t^{a-1}\, exp(-t)\, dt,$$

is the upper incomplete gamma (UIGa) function and

$$\Gamma(a,x) + \gamma(a,x) = \Gamma(a).$$

The τ^{th} MRRL is given by

$$Z_\tau(t) = E\left[(t-X)^\tau \;|_{t>0\,,X\le t,\;\; \tau=1,2,\dots}\right],$$

uniquely determines $F(x)$. We have

$$F(t)^{-1} \int_0^t (t-x)^\tau \, dF(x) = Z_\tau(t).$$

Then, the τ^{th} moment of the reversed residual life of X becomes

$$Z_\tau(t) = F(t)^{-1} \sum_{i,k=0}^{\infty} \sum_{r=0}^{t} (-1)^r \binom{\tau}{r} t^{\tau-r} \boldsymbol{V}_{i,k} \int_0^t \; x^r \; h(x\,;(2+i+k)\,\alpha,\beta)dx,$$

$$= \gamma\left(1+\beta^{-1}\tau, t^{-\beta}\right) F(t)^{-1} \sum_{i,j,k=0}^{\infty} \sum_{r=0}^{t} \boldsymbol{b}_{i,j,k,r}^{((2+i+k)\alpha,\tau)} \;|_{(\tau>-\beta)},$$

where

$$\boldsymbol{b}_{i,j,k,r}^{((2+i+k)\alpha,\tau)} = \boldsymbol{V}_{i,k} t^{\tau-r} (-1)^{i+r} (j+1)^{-(\tau+\beta)/\beta} [(2+i+k)\alpha] \binom{i+k}{j}\binom{\tau}{r}.$$

4. MAXIMUM LIKELIHOOD METHOD

If $x_1, \dots, x_n$ be a RS of the new distribution with parameter vector $\mathbf{\Psi} = (\alpha,\beta)^T$. The log-likelihood function for $\boldsymbol{\Psi}$, say $\ell = \ell(\boldsymbol{\Psi})$, is given by

$$\ell = \ell(\mathbf{\Psi}) = -n\, log(2) + n\, log(\alpha) + n\, log(\beta) + (\beta-1) \sum_{i=1}^{n} log(x_i)$$

$$-3 \sum_{i=1}^{n} log(1-d_i^\alpha) + - \sum_{i=1}^{n} x_i^\beta \;(\alpha-1) \sum_{i=1}^{n} log(d_i) - \sum_{i=1}^{n} b_i, \tag{11}$$

where

$$d_i = \left[1 - exp\left(-x_i^\beta\right)\right] \text{ and } b_i = m_i^\alpha (1 - s_i^\alpha)^{-1},$$

Equation (11) can be maximized either via the different. The score vector elements are easily to be calculated

5. DATA ANALYSIS

We shall consider the Cramér-von-Mises (W^*) and the Anderson-Darling (A^*), wher

$$W^* = \left[(1/12n) + \sum_{j=1}^{n} \mathbf{c}_j\right](1 + 1/2n),$$

and

$$A^* = \left(n + \frac{1}{n}\sum_{j=1}^{n} \mathbf{q}_j\right)\left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right),$$

respectively, where

$$\mathbf{c}_j = [z_i - (2j-1)/2n]^2,$$

$$\mathbf{q}_j = (2j-1)\, log\left[\mathbf{z}_j\left(1 - \mathbf{z}_{n-j+1}\right)\right],$$

and $\mathbf{z}_j = F\left(y_j\right)$ the y_j 's values are the ordered observations. The calculated values for the MLEs and its corresponding standard errors (S.E.) (in parentheses) of the model parameters are listed in Tables 1, 3 and 5.

The numerical values of the statistics W^* and A^* are listed in Tables 2, 4 and 6. The estimated PDF, estimated HRF, total time test (TTT) plots for the three data sets are displayed in Figures 3, 4 and 5. The TTT plot for the three data sets are given in Figure 3, 4 and 5 and indicates that the empirical HRFs of data sets I, II and III are increasing. All required calculations are carried out via the R software.

Application-1: Modeling failure times

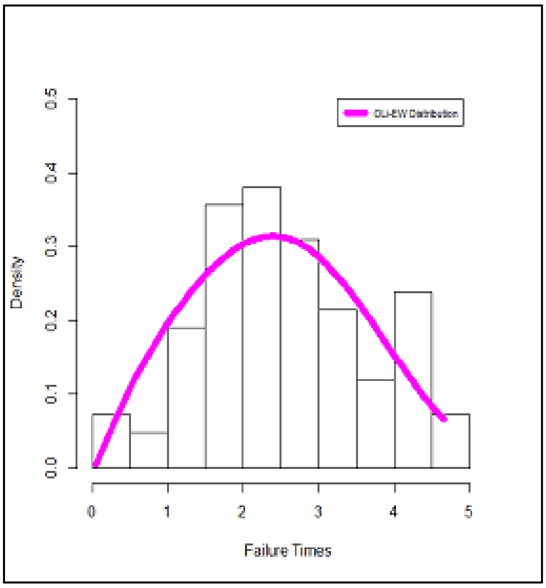
The data consist of 84 observations. The data are: 0.04, 1.86600, 3.1660, 2.632, 3.595, 1.0700, 1.9140, 2.6460, 3.699, 1.1240, 1.9810, 2.661, 2.890, 4.1210, 4.485, 2.385, 3.443, 0.301, 1.8760, 2.481, 3.467, 0.3090, 1.8990, 4.1670, 1.432, 4.3760, 1.615, 2.688, 2.934, 2.610, 3.4780, 0.5570, 1.911, 2.6250, 4.570, 1.303, 2.0890, 2.902, 3.7790, 3.9240, 1.2810, 4.0350, 1.281, 2.0850, 2.3000, 3.344, 4.602, 1.757, 3.5780, 0.9430, 1.912, 2.0380, 2.823, 2.2230, 3.114, 4.449, 1.619, 2.0970, 1.2480, 2.010, 2.2240, 3.1170, 1.6520, 1.6520, 2.229, 4.240, 1.4800, 2.135, 2.9620, 4.255, 1.5050, 2.154, 2.9640, 4.278, 1.5060, 2.190, 3.0000, 4.3050, 1.5680, 2.1940, 3.1030, 2.3240, 3.3760, 4.6630 Here, we shall compare the fits of the OLi-EWD distribution with those of other competitive models. The PDFs of these models are given in Appendix A. Based on the figures in Table2 we conclude that thehe OLi-EWD provides adequate fits as compared to other W models with small values for W^* and A^* . The proposed lifetime model is better than the MOE-WD, Ga-WD, BrXEWD, PTL-WD, Kw-WD, KwT-WD, MB-WD, Mc-WD, W-FrD, B-WD, TM-WD, and TExG-WD models, and a good ersatz to these models.

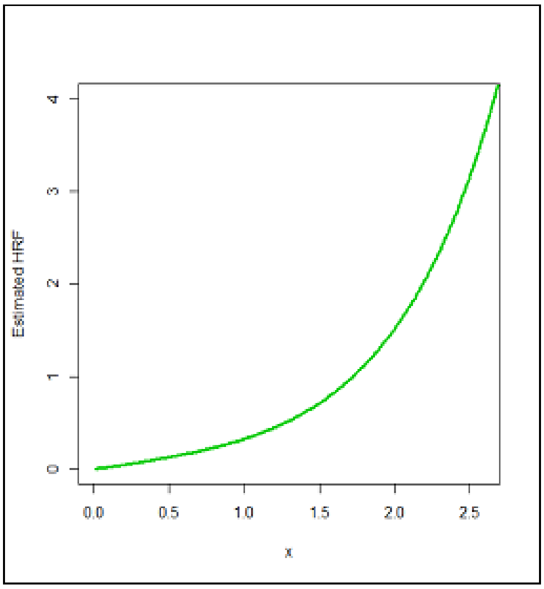
Table 1: MLEs (S.E. in parentheses) for data set I.

Distribution	Estimates
OLi-EWD(a, α, β)	3.816, 0.710
	(0.419), (0.0331)
PTL-WD(λ, α, b)	−5.782, 4.229, 0.658
	(1.395), (1.167), (0.039)
MOE-WD(γ, β, α)	488.899, 0.283, 1261.966
	(189.358), (0.013), (351.073)
Ga-WD(α, β, γ)	2.377, 0.848, 3.534
	(0.378), (0.001), (0.665)
Kw-WD(α, β, a, b)	14.433, 0.204, 34.659, 81.846
	(27.095), (0.042), (17.527), (52.014)
W-FrD(α, β, a, b)	630.938, 0.302, 416.097, 1.166
	(697.942), (0.032), (232.359), (0.357)
B-WD(α, β, a, b)	1.360, 0.298, 34.180, 11.496
	(1.002), (0.06), (14.838), (6.73)
TM-WD$(\alpha, \beta, \gamma, \lambda)$	0.272, 1, 4.6×10^{-6}, 0.469
	(0.014), (5.2×10^{-5}), (1.9×10^{-4}), (0.165)
KwT-W$(\alpha, \beta, \lambda, a, b)$	27.791, 0.178, 0.445, 29.525, 168.060
	(33.401), (0.017), (0.609), (9.792), (129.165)
MB-WD(α, β, a, b, c)	10.150, 0.163, 57.415, 19.386, 2.004
	(18.697), (0.019), (14.063), (10.019),(0.662)
Mc-WD(α, β, a, b, c)	1.940, 0.306, 17.686, 33.639, 16.721,
	(1.011), (0:045), (6.222), (19.994), (9.722)
TExG-WD$(\alpha, \beta, \lambda, a, b)$	4.257,0.153, 0.098, 5.231,1173.328
	(33.401), (0:017), (0.609),(9.792)

Table 2: the statistics W^* and A^* for data set I.

Distribution	W^*	A^*
OLi-EWD(a,α,β)	**0.0812**	**0.768**
PTL-WD(λ,α,b)	0.139	1.194
MOE-WD(γ,β,α)	0.399	4.448
Ga-WD(α,β,γ)	0.255	1.949
Kw-WD(α,β,a,b)	0.185	1.506
W-FrD(α,β,a,b)	0.254	1.957
B-WD(α,β,a,b)	0.465	3.219
TM-WD$(\alpha,\beta,\gamma,\lambda)$	0.806	11.205
KwT-WD$(\alpha,\beta,\lambda,a,b)$	0.164	1.363
MB-WD(α,β,a,b,c)	0.472	3.266
Mc-WD(α,β,a,b,c)	0.199	1.591
TExG-WD$(\alpha,\beta,\lambda,a,b)$	1.008	6.233





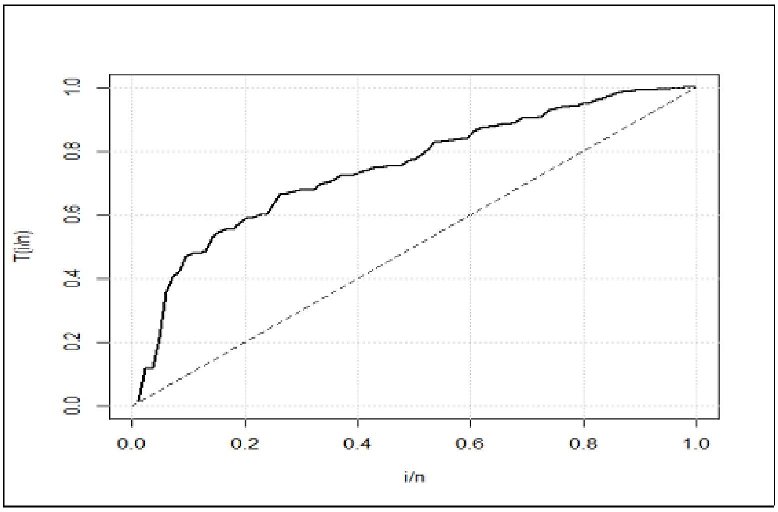


Figure 3: Estimated PDF, Estimated HRF and TTT plot for data set I.

Application-2: Modeling survival times

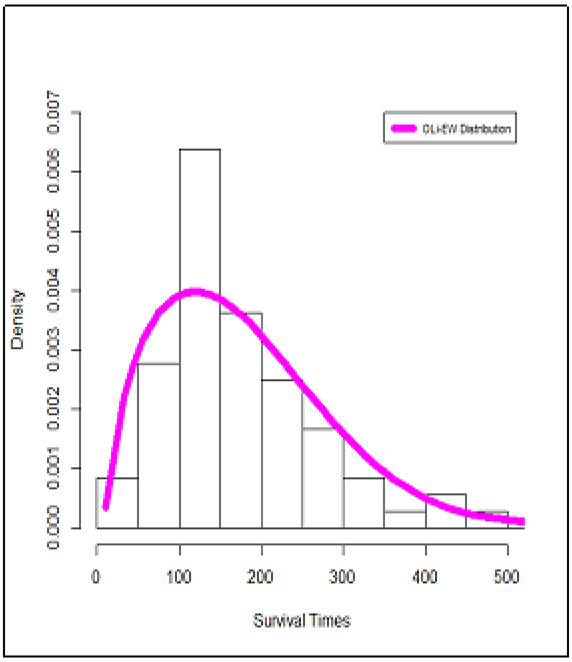
The second real data set corresponds to the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal (1960). We shall compare the fits of the OLi-EWD distribution with those of other competitive models. The PDFs of these models are given in Appendix B. Based on Table 4, we see that the OLi-EWD is better than the OW-WD, WLog-WD, GaE-ED.

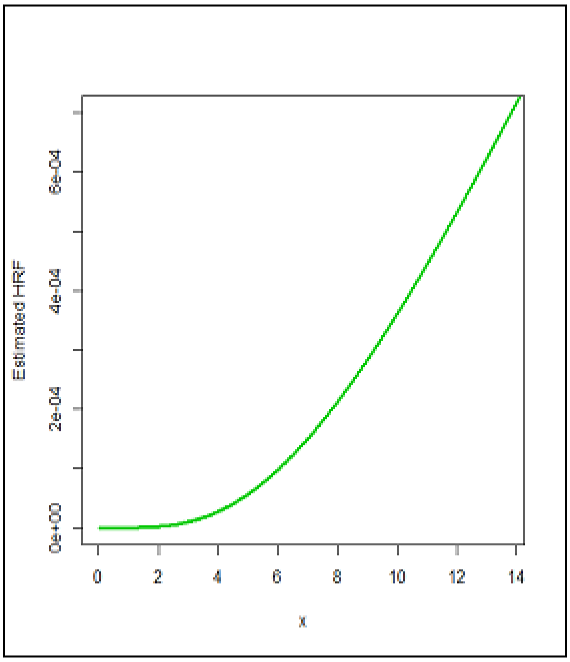
Table 3: MLEs (S.E. in parentheses) for data set II.

Distribution	Estimates
OLi-EWD(α, β)	38.151, 0.278 (8.447), (0.009)
OL-EWD(a, α, β)	0.002, 0.072, 0.281 (0.0004), (0.024), (0.009)
OW-WD(β, γ, λ)	11.157, 0.088, 0.457 (4.545) (0.036) (0.077)
WLog-WD(β, γ, λ)	1.787, 0.779, 0.025 (0.782), (0.333), (0.040)
GaE-ED$(\lambda, \alpha, \theta)$	2.114, 2.600, 0.008 (1.329), (0.559), (0.005)

Table 4: W^* and A^* for data set II.

Distribution	W^*	A^*
OLi-EWD(α, β)	**0.121**	**0.761**
OL-EWD(a, α, β)	0.252	1.475
OW-WD(β, γ, λ)	0.449	2.476
WLog-WD(β, γ, λ)	0.435	2.394
GaE-ED$(\lambda, \alpha, \theta)$	0.315	1.721





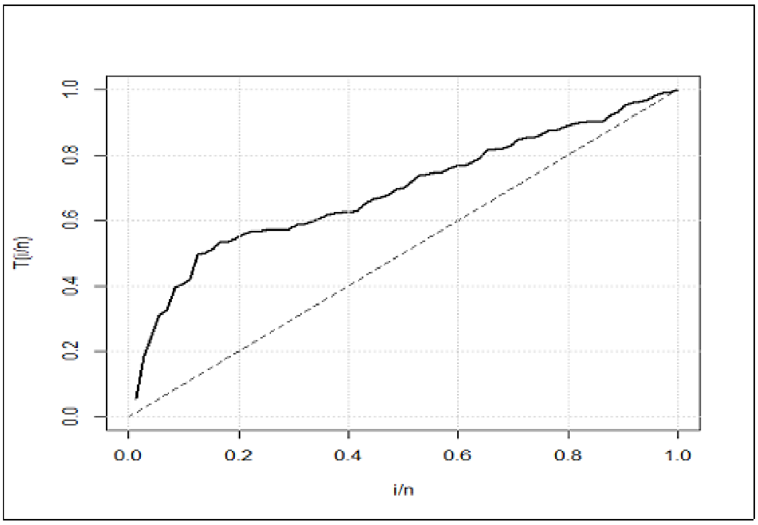


Figure 5: Estimated PDF, Estimated HRF and TTT plot for data set II.

Application 3: Modeling strengths data

This data consists of 63 observations (strengths of 1.5 cm glass fibers), the data originally obtained by workers at the UK National Physical Laboratory. The data are: 0.770, 0.81, .89, 1.490, 1.49, 1.500, 1.62, 1.36, 1.390, 0.550, 1.250, 1.270, 2.000, 1.630, 1.240, 1.69, 1.700, 1.70, 1.73, 1.76, 1.760, 1.77, 1.78, 1.810, 1.82, 1.84, 0.74, 0.840, 0.930, 1.04, 1.60, 1.601, 1.610, 1.670, 1.680, 1.680, 1.84, 1.11, 1.130, 1.280, 1.290, 1.30, 2.010, 1.610, 1.61, 1.620, 1.660, 1.42, 1.480, 1.480, 1.640, 1.66, 1.660, 1.500, 1.510, 1.520, 1.530, 1.540, 1.550, 1.550, 1.580, 1.590, 2.240. For this data set, we shall compare the fits of the OLi-EWD distribution with some competitive models. The PDFs of these models are given in Appendix C.

Table 5: MLEs (S.E. in parentheses) for data set III.

Distribution	Estimates
OLi-EWD(α,β)	4.321, 1.636
	(0.559), (0.0801)
E-WD(a,α,β)	0.671, 7.285,1.718
	(0.249), (1.707), (0.086)
T-WD(a,α,β)	−0.501, 5.149, 0.646
	(0.274), (0.666), (0.024)
OLL-WD(θ,α,β)	0.944, 6.026, 0.616
	(0.269), (1.348), (0.016)

Table 6: W^* and A^* for data set III.

Distribution	W^*	A^*
OLi-EWD(α,β)	**0.299**	**1.645**
E-WD(a,α,β)	0.636	3.484
T-WD(a,α,β)	1.036	2.169
OLL-WD(θ,α,β)	1.236	2.219

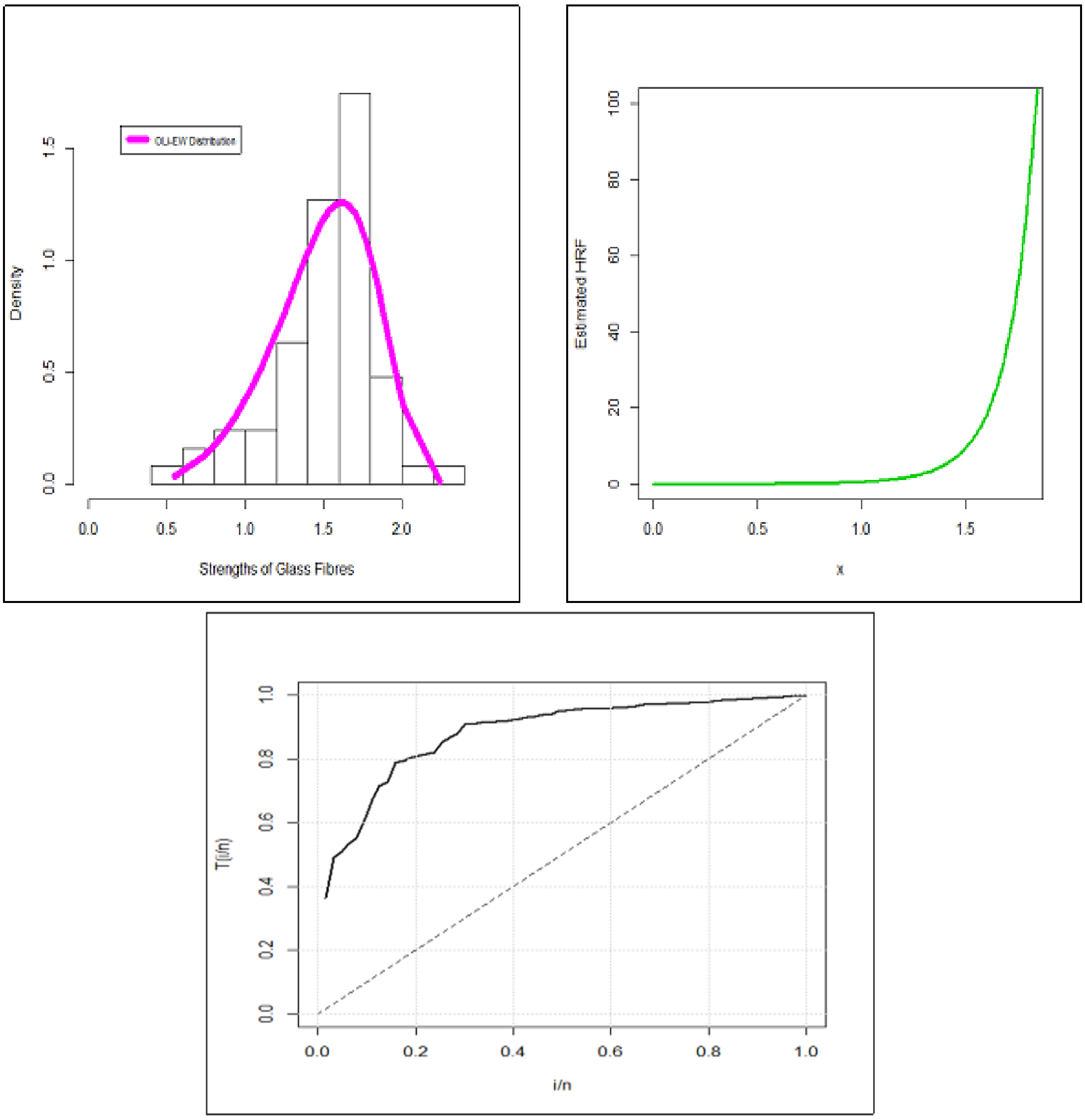


Figure 6: Estimated PDF, Estimated HRF and TTT plot for data set

6. CONCLUDING REMARKS

We introduced and studied a new extension of the exponentiated Weibull lifetime model called the Odd Lindley exponentiated Weibull Distribution (OLi-EWD) with only two-parameters. The OLi-EWD exhibits the increasing the decreasing and the bathtub (**U**) hazard rates. The OLi-EWD has been represented as a mixture of the exponentiated Weibull density. is better than many other versions of the exponentiated Weibull model existing in the literature.

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Appendix: A

The Beta-Weibull (B-WD) (Lee et al., 2007)

$$f(x)=\beta\alpha^{\beta}B^{-1}(a,b)x^{\beta-1}\{1-exp[-(\alpha x)^{\beta}]\}^{a-1}\,exp[-b(\alpha x)^{\beta}];$$

the Poisson Topp Leone-Weibull (PTL-WD) (New)

$$f(x)=2\lambda\alpha b a^{b}x^{b-1}[1-exp(-\lambda)]^{-1}\,exp(-2x^{b})\,[1-exp(-2x^{b})]^{\alpha-1}$$
$$\times\,exp\{-\lambda[1-exp(-2x^{b})]^{\alpha}\};$$

th Transmuted modified-Weibull (TM-WD) (Khan and King, 2013)

$$f(x)=\left(\alpha+\gamma\beta x^{\beta-1}\right)\left[1-\lambda+2\lambda\,exp\left(-\alpha x-\gamma x^{\beta}\right)\right]exp\left[-\alpha x-\gamma x^{\beta}\right],|\lambda|\leq 1;$$

Marshall Olkin extended-Weibull (MOE-W) (Ghitany et al., 2005)

$$f^{(MOE-W)}(x)=\alpha\beta\gamma^{\beta}\left[1-(1-\alpha)e^{-(\gamma x)^{\beta}}\right]^{-2}x^{\beta-1}\,exp[-(\gamma x)^{\beta}];$$

th transmuted exponentiated generalized Weibull (TExG-WD) (Yousof et al., 2015)

$$f(x)=ab\beta\alpha^{\beta}x^{\beta-1}\{1-exp[-a(\alpha x)^{\beta}]\}^{b-1}\,exp[-a(\alpha x)^{\beta}]$$
$$\times\left(1+\lambda-2\lambda\{1-exp[-a(\alpha x)^{\beta}]\}^{b}\right),|\lambda|\leq 1.$$

the Gamma-Weibull (Ga-WD) (Provost et al., 2011)

$$f(x)=\beta\alpha^{\gamma/\beta+1}\Gamma^{-1}(1+\gamma/\beta)x^{\beta+\gamma-1}\,exp[-\alpha x^{\beta}];$$

the Weibull-Fréchet (W-FrD) (Afify et al., 2016c)

$$f(x)=ab\beta\alpha^{\beta}x^{-(\beta+1)}\,exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-(b+1)}$$
$$\times\,exp\left\{-a\left[\frac{exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1-exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right]^{b}\right\},$$

the Kumaraswamy-Weibull (Kw-WD) (Cordeiro et al., 2010)

$$f^{(W-Fr)}(x)=ab\beta\alpha^{\beta}x^{\beta-1}\{1-exp[-(\alpha x)^{\beta}]\}^{a-1}\,exp[-(\alpha x)^{\beta}]\left\{1-\{1-exp[-(\alpha x)^{\beta}]\}^{a}\right\}^{b-1};$$

the Kumaraswamy transmuted-Weibull (KwT-WD) (Afify et al., 2016a)

$$f^{(KwT-W)}(x)=ab\beta\alpha^{\beta}x^{\beta-1}\left(1+\lambda-2\lambda\{1-exp[-(\alpha x)^{\beta}]\}\right)exp[-(\alpha x)^{\beta}]$$
$$\times\left[1-\left((1+\lambda)\{1-exp[-(\alpha x)^{\beta}]\}-\lambda\{1-exp[-(\alpha x)^{\beta}]\}^{2}\right)^{a}\right]^{b-1}$$
$$\times\left[\{1-exp[-(\alpha x)^{\beta}]\}\left(1+\lambda-\lambda\{1-exp[-(\alpha x)^{\beta}]\}\right)\right]^{a-1};$$

Modified beta-Weibull (MB-WD) (Khan, 2015)

$$f^{(MB-W)}(x)=\beta\gamma^{a}\alpha^{-\beta}B^{-1}(a,b)x^{\beta-1}\left\{1-exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{a-1}exp\left[-b\left(\frac{x}{\alpha}\right)^{\beta}\right]$$
$$\times\left(1-(1-\gamma)\left\{1-exp\left[-b\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}\right)^{-a-b};$$

the Mcdonald-Weibull (Mc-WD) (Cordeiro et al., 2014),

$f^{(Mc-W)}(x)=\beta c\alpha^{\beta}B^{-1}(a/c,b)x^{\beta-1\{1-exp[-(\alpha x)^{\beta}]\}^{a-1}}\,exp[-(\alpha x)^{\beta}]\left(1-\{1-exp[-(\alpha x)^{\beta}]\}^{c}\right)^{b-1}$ distributions, whose PDFs (for $x>0$). The parameters of the above densities are all positive real numbers except for the TM-W and TExG-W distributions.

Appendix: B

the three-parameters OL-EWD

$$f(x) = a^2(1+a)^{-1}\alpha\beta x^{\beta-1}\, exp\left(-x^{\beta}\right)$$
$$\times\left\{1-\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{-3}\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha-1}$$
$$\times\, exp\left(-a\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}/\left\{1-\left[1-exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}\right), x \geq 0$$

the Odd Weibull-Weibull (OW-WD) (Bourguignon et al., 2014)

$$f(x) = 1 - exp\left\{-\alpha\left[exp(\lambda x^{\gamma}) - 1\right]^{\beta}\right\};$$

the gamma exponentiated-exponential (GaE-ED) (Risti´c and Balakrishnan 2012)

$$f(x) = \frac{\alpha\theta}{\Gamma(\lambda)} exp(-\theta x)\,[1-exp(-\theta x)]^{\alpha-1}\{-\alpha\, log[1-exp(-\theta x)]\}^{\lambda-1};$$

distributions, whose PDFs (for $x > 0$).

Appendix: C

T-WD:

$$f(x) = \beta\alpha^{\beta}x^{\beta-1}\, exp\left[-(\alpha x)^{\beta}\right]\left(1+a-2a\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}\right),\ |a| \leq 1,$$

and OLL-WD:

$$f(x) = \theta\beta\alpha^{\beta}x^{\beta-1}\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{\theta-1} exp\left[-\theta(\alpha x)^{\beta}\right]$$
$$\left(\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{\theta} + exp\left[-\theta(\alpha x)^{\beta}\right]\right)^{-2}.$$

Some other extensions of the WD can also be used in this comparison, but are not limited to Yousof et al. (2015), Afify et al. (2016b, c), Yousof et al. (2016a,b), Cordeiro et al. (2017a, b), Yousof et al. (2017a,b,c,d,e), Brito et al. (2017), Korkmaz and Genç (2017), Korkmaz et al. (2018), Yousof et al. (2018a, b, c) and Hamedani et al. (2018a, b) among others.

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