International Journal of Mathematical Archive-10(1), 2019, 54-57
Available online through www.ijma.info ISSN 2229 – 5046

ALGEBRAIC CONSTRUCTION OF THE CANTOR SET

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(Received On: 12-11-18; Revised & Accepted On: 07-01-19)

ABSTRACT

The middle-third stages of the Cantor set and others were constructed step-by-step. In this paper, two algorithms together with the algebraic method, are derived as a faster way to construct the Cantor set.

Keywords: *Cantor set, general case algorithm, middle case algorithm, line segment, algebraic methods, direct substitution, unit interval, dimension moment, self-similarity, fractal dimension, Hausdorff dimension.*

1. INTRODUCTION

Georg F. L. P. Cantor (1845− 1918) in 1883 , formed the sequence $0, \frac{1}{3}, \frac{2}{3}, 1$ by dividing the unit interval $B_0 = [0,1]$ into three equal segments. He removed the open middle set, $\left(\frac{1}{3}, \frac{2}{3}\right)$ to form the union $B_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ of two closed sets which are similar to the unit interval. Cantor again, removed the open middle sets $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ to get another union $B_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$which is similar to B_1 [4,5]. Cantor continued the recursive process which led to the formation of the Cantor dust [4]. The set left after the recursive removal of middle open segments is the Cantor set because, Cantor first introduced it in 1883 after Henry J. S. Smith $(1826 - 1883)$ discovered it in 1874. It is called Cantor ternary set because, it consists of all real numbers in the unit interval that can be expressed as a ternary fraction using $0's$ and $2's$ only.

The Cantor set has deep and remarkable properties such as self-similarity. That is, it produces 2^n copies of itself at the n^{th} stage which exhibits self-similarity. The Cantor set has a fractal dimension and Hausdorff dimension of $\frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3} = 0.631$ and its dimension moment at the n^{th} stage is $\sum_{i=0}^{n} {}^{d_f} x_{2^i} = 1,\ x_{2^i} = \frac{1}{3^n}$, that is the conservation law [3,4]. It has topological and analytical properties as well. Generally, a Cantor set has a Hausdorff dimension of $\frac{n \ln 2}{n \ln j} = \frac{\ln 2}{\ln j} = \log_j 2$ where $ln(2)$ is because of the self-similarity property of Cantor sets. Dimension moment of a general Cantor set is given as $\sum_{i=0}^{n} {}^{d_f} x_{2^i} = 1,\ x_{2^i} = \frac{1}{j^n},$ where j is the number of segments on the unit interval [0,1].

Cantor sets appear in fields such as mechanics where quantum mechanics is defined on a Cantor set using a stochastic motion. They also appear in fields where sequential search methods are used to optimize objective functions. It is also useful in decision-making processes tied to collection development.

In the following section we used two algorithms with an algebraic approach to construct elements of the Cantor set based on the formulae [1, 2].

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2. THE CANTOR SET FORMULA

2.1 General Case Algorithm (GCA)

For predetermined values $2 \leq i \leq j-1, k = j+1$ we want to construct the $i^{th} - j^{th}$ stages of the Cantor set using the **General Case Algorithm** for k points in the interval [0,1].

Problem: To construct the $i^{th} - j^{th}$ stages of the Cantor set.

Step-1: Set

$$B_0 = [0,1], \qquad t = n_0^*$$

Step-2: Compute

$$x = \frac{i-1}{j}, \quad y = \frac{i}{j} \quad \text{and} \quad z = 1 - \frac{i}{j}$$

Step-3: Do until $n = t$

$$\{B_n\}_{n=1}^{\infty} = xB_{n-1} \cup [y + zB_{n-1}]$$

Exit Do

$$\{B_n\}_{n=1}^{t} = union\left(xB_{n-1},\ [y + zB_{n-1}]\right)$$

Loop

Step-4: Stop

It follows from the General ***Case Algorithm (GCA)*** that if

$$B_0 = [0,1],$$

Then

$$B_1 = xB_0 \cup [y + zB_0] = x[0,1] \cup \big[y + z[0,1]\big] = [0,x] \cup [y,(y+z) = 1] \tag{1}$$

$$B_2 = xB_1 \cup [y + zB_1] = x[[0,x] \cup [y,1]] \cup [y + z[[0,x] \cup [y,1]]]$$

$$B_2 = [0,x^2] \cup [xy,x] \cup [y,y+zx] \cup [y+zy,1] \tag{2}$$

$$B3 = xB_2 \cup [y + zB_2] = x\big[[0,x^2] \cup [xy,x] \cup [y,y+zx] \cup [y+zy,1]\big] \cup$$
$$[y + z[[0,x^2] \cup [xy,x] \cup [y,y+zx] \cup [y+zy,1]]]$$

$$B_3 = [0,x^3] \cup [x^2y,x^2] \cup [xy,xy+zx^2] \cup [xy+xyz,x] \cup [y,y+zx^2] \cup [y+xyz,y+$$
$$xz \cup y+yz, y+yz+xz2 \cup y+yz+yz2,1 \tag{3}$$

The process continues, culminating into

$$\{B_n\}_{n=4}^{\infty} = x(B_{(n-1)}) \cup [y + z(B_{(n-1)})],\ n \in \mathbb{N} \tag{4}$$

2.1.1 Construction By Direct Substitution

By substituting the values of x, y and z obtained at **Step 2** directly in to the equations (1), (2), (3) the general j^{th} sets can be constructed. That is:

$$(x,y,z) \to [(1),(2),(3)] = \{B_n\}_{n=1}^{3} = x(B_{(n-1)}) \cup [y + z(B_{(n-1)})],\ n \in \mathbb{N}$$

2.1.2 Example 1

Problem: To construct the $85^{th} - 999^{th}$ set for $t = 3$ (that is 3 iterations)

Step-1:

$$Set \quad B_0 = [0,1],\ i = 85,\ j = 999,\ t = 3 \text{ (tolerance)}$$

Step-2: Set

$$x = \frac{85-1}{999} = \frac{28}{333}, \quad y = \frac{85}{999} \quad \text{and} \quad z = 1 - \frac{85}{999} = \frac{914}{999}$$

Step-3: Since $n = 1 < t = 3$,

$$B_1 = \tfrac{28}{333}B_0 \cup \left[\tfrac{85}{999} + \tfrac{914}{999}B_0\right] = \tfrac{28}{333}[0,1] \cup \left[\tfrac{85}{999} + \tfrac{914}{999}[0,1]\right] = \left[0, \tfrac{28}{333}\right] \cup \left[\tfrac{85}{999}, 1\right]$$

Since $n = 2 < t = 3$,

$$B_2 = \frac{28}{333}\left[\left[0, \frac{28}{333}\right] \cup \left[\frac{85}{999}, 1\right]\right] \cup \left[\frac{85}{999} + \frac{914}{999}\left[\left[0, \frac{28}{333}\right] \cup \left[\frac{85}{999}, 1\right]\right]\right]$$

$$\Longrightarrow B_2 = \left[0, \frac{784}{110889}\right] \cup \left[\frac{2380}{332667}, \frac{28}{333}\right] \cup \left[\frac{85}{999}, \frac{53897}{332667}\right] \cup \left[\frac{162605}{332667}, 1\right]$$

Since $n = 3 = t$, we shall stop after this iteration.

$$B_3 = \frac{28}{333}B_2 \cup \left[\frac{85}{999} + \frac{914}{999}B_2\right]$$

$$\Longrightarrow B_3 = \left[0, \frac{21952}{36926037}\right] \cup \left[\frac{66640}{110778111}, \frac{784}{110889}\right] \cup \left[\frac{2380}{332667}, \frac{1509116}{110778111}\right] \cup \left[\frac{4552940}{332334333}, \frac{28}{333}\right] \cup \Longrightarrow B_3$$

$$\left[\frac{85}{999}, \frac{10142141}{110778111}\right] \cup \left[\frac{30452015}{332334333}, \frac{53897}{332667}\right] \cup \left[\frac{162605}{998001}, \frac{77538553}{332334333}\right] \cup \left[\frac{233451055}{997002999}, 1\right]$$

Step-4: Stop the operation since $n = t$.

2.1 Middle Case Algorithm (MCA)

We want to construct the middle-j^{th} stages of the Cantor set using the **Middle Case Algorithm (MCA)** for k points in the interval $[0,1]$.

Problem: To construct the middle$-j^{th}$ stages of the Cantor set.

Step-1: For a predetermined odd value j (the number of subintervals on the unit interval), set $k = j + 1$, $B_0 = [0,1]$, $\quad t = n_0^*$

Step-2: Compute

$$x = \frac{k-2}{2j} \quad \textit{and} \quad y = \frac{k}{2j}$$

Step-3: If j is even, go to **Step 4** and **use** the **GCA.**

Else,

Do until $n = t$

$$\{B_n\}_{n=1}^{\infty} = xB_{n-1} \cup [y + xB_{n-1}]$$

Exit Do

$$\{B_n\}_{n=1}^{t} = union\left(xB_{n-1},\ [y + xB_{n-1}]\right)$$

Loop

End if

Step-4: Stop

Again, if

$B_0 = [0,1]$,

$B_1 = xB0 \cup [y + xB0] = x[0,1] \cup [y + x[0,1]] = [0,x] \cup [y,1]$ (5)

$B_2 = xB_1 \cup [y + xB_1] = [0,x^2] \cup [xy,x] \cup [y,x^2 + y] \cup [xy + y,1]$ (6)

$B_3 = xB_2 \cup [y + xB_2]$

$B_3 = [0,x^3] \cup [x^2y,x^2] \cup [xy,x^3 + xy] \cup [x^2y + xy,x] \cup [y,x^3 + y] \cup [x^2 y + y,x^2 + y] \cup [xy + y,x^3 + xy + y] \cup [x^2 y + xy + y,1]$ (7)

This culminates into

$$\{B_n\}_{n=4}^{\infty} = xB_{n-1} \cup [y + xB_{n-1}],\ n \in \mathbb{N} \tag{8}$$

2.2.1 Construction by Direct Substitution

By substituting the values of x and y obtained at **Step 2** directly into the equations $(5), (6), (7)$ the middle$-j^{th}$ sets can be constructed. That is $(x,y) \rightarrow [(5),(6),(7)] = (8)$.

2.2.2 Example 2

Problem: To construct the middle -999^{th} set for $t = 3$ (that is 3 iterations)

Step-1: Set

$$B0 = [0,1], \quad j = 999, \quad k = 1000, \quad t = 3$$

Step-2: Set

$$x = \frac{1000-2}{1998} = \frac{499}{999} \quad and \quad y = \frac{1000}{1998} = \frac{500}{999}$$

Step-3: Since $n = 1 < t = 3$ and j is odd,

$$B_1 = \tfrac{499}{999}B_0 \cup \left[\tfrac{500}{999} + \tfrac{499}{999}B_0\right] = \tfrac{499}{999}[0,1] \cup \left[\tfrac{500}{999} + \tfrac{499}{999}[0,1]\right] = \left[0, \tfrac{499}{999}\right] \cup \left[\tfrac{500}{999}, 1\right]$$

Since $n = 2 < t = 3$,

$$B_2 = \tfrac{499}{999}\left[\left[0, \tfrac{499}{999}\right] \cup \left[\tfrac{500}{999}, 1\right]\right] \cup \left[\tfrac{500}{999} + \tfrac{499}{999}\left[\left[0, \tfrac{499}{999}\right] \cup \left[\tfrac{500}{999}, 1\right]\right]\right]$$

$$\implies B_2 = \left[0, \tfrac{249001}{998001}\right] \cup \left[\tfrac{249500}{998001}, \tfrac{499}{999}\right] \cup \left[\tfrac{500}{999}, \tfrac{748501}{998001}\right] \cup \left[\tfrac{749000}{998001}, 1\right]$$

Since $n = 3 = t$, we shall stop after this iteration.

$$B_3 = \tfrac{499}{999}B_2 \cup \left[\tfrac{500}{999} + \tfrac{499}{999}B_2\right]$$

$$\implies B_3 = \left[0, \tfrac{124251499}{997002999}\right] \cup \left[\tfrac{124500500}{997002999}, \tfrac{249001}{998001}\right] \cup \left[\tfrac{249500}{998001}, \tfrac{373501999}{997002999}\right] \cup \left[\tfrac{373751000}{997002999}, \tfrac{499}{999}\right] \cup$$

$$\left[\tfrac{500}{999}, \tfrac{623251999}{997002999}\right] \cup \left[\tfrac{623501000}{997002999}, \tfrac{748501}{998001}\right] \cup \left[\tfrac{749000}{998001}, \tfrac{872502499}{997002999}\right] \cup \left[\tfrac{872751500}{997002999}, 1\right]$$

Step-4: Stop the operation since $n = t$

3. CONCLUDING REMARKS

The Middle Case algorithm (MCA) and General case Algorithm (GCA) can be applied using algebraic methods to construct the Cantor Set. The MCA can be applied to delete open middle segments for any k points on $[0,1]$ while the GCA can be applied to delete any open subinterval, except the first and the last for k points in $[0,1]$. The approach used is one of the fastest way to construct the Cantor set. The two algorithms can be advanced in order to optimize accuracy (minimize inaccuracy or maximize accuracy) in construction due to multiple results obtained at the highest iteration n. One may be interested in removing more than one open segment from the unit interval $[0,1]$. We want to conclude with the following question. Do Cantor sets occur by fission? What kind of fission?

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Source of support: Nil, Conflict of interest: None Declared.